The Crash-NIG Copula Model

Pricing of CDOs under changing market conditions

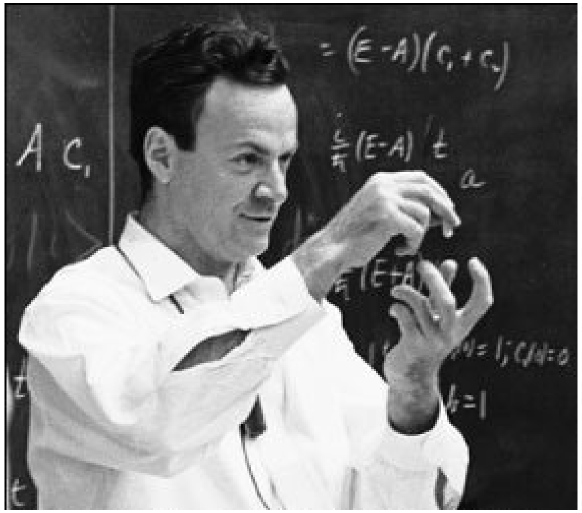
Prof. Dr. Rudi Zagst

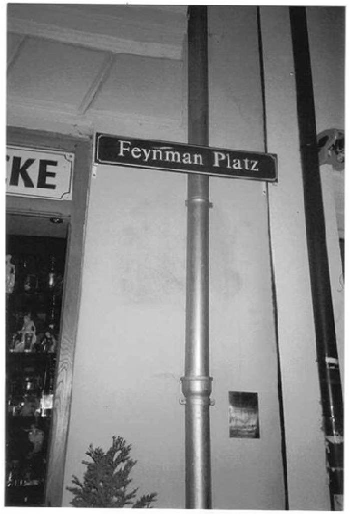
Chair of Mathematical Finance

Technische Universität München

Joint work with Anna Schlösser

The Crash-NIG Copula Model
Overview





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„When using a mathematical model careful attention must be given to uncertainties in the model“

Richard Feynman

Nobel price winner, 11.05.1918 - 15.02.1988

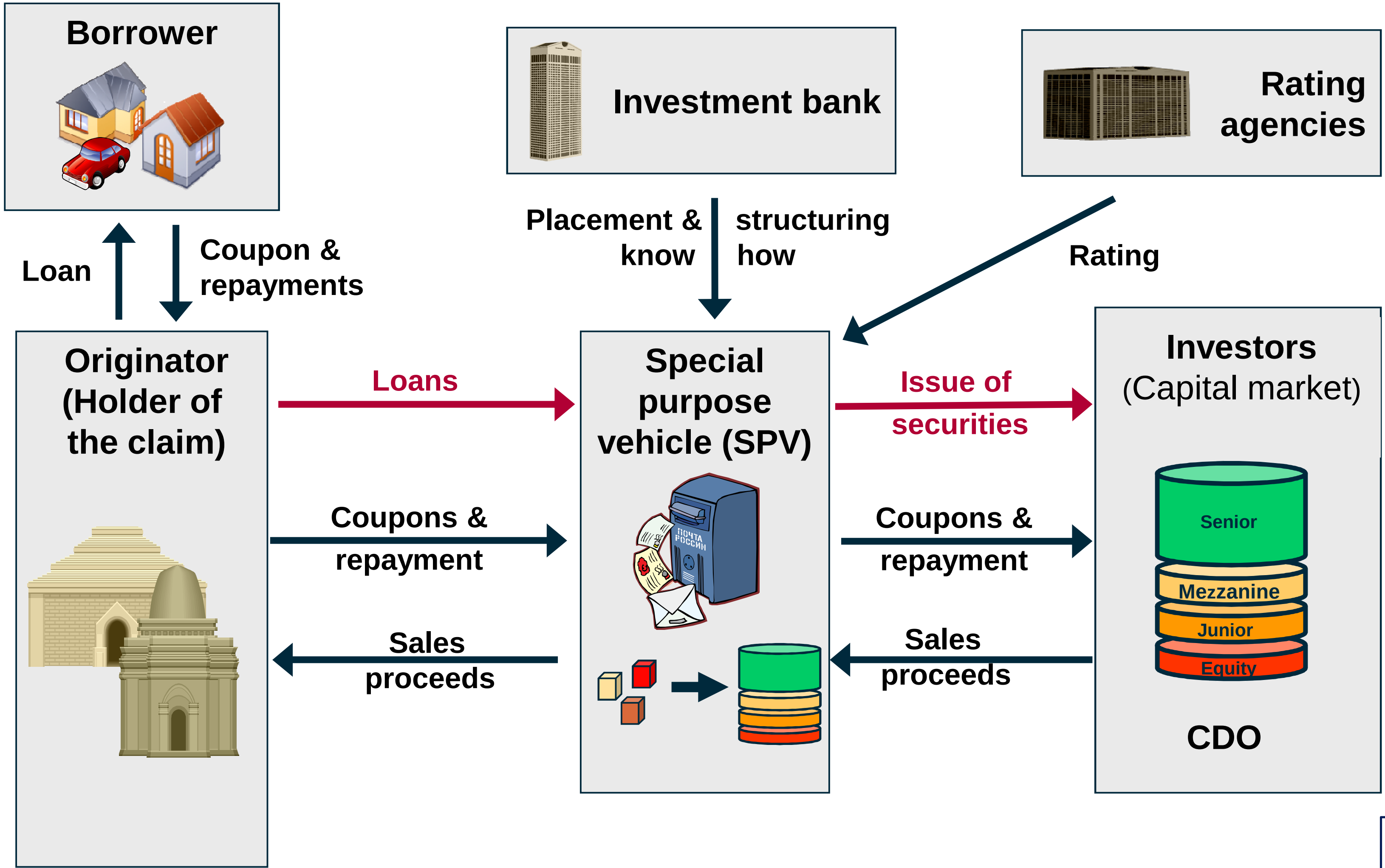
* Feynman-Platz close to the Isartor in Munich

The Crash-NIG Copula Model
Overview

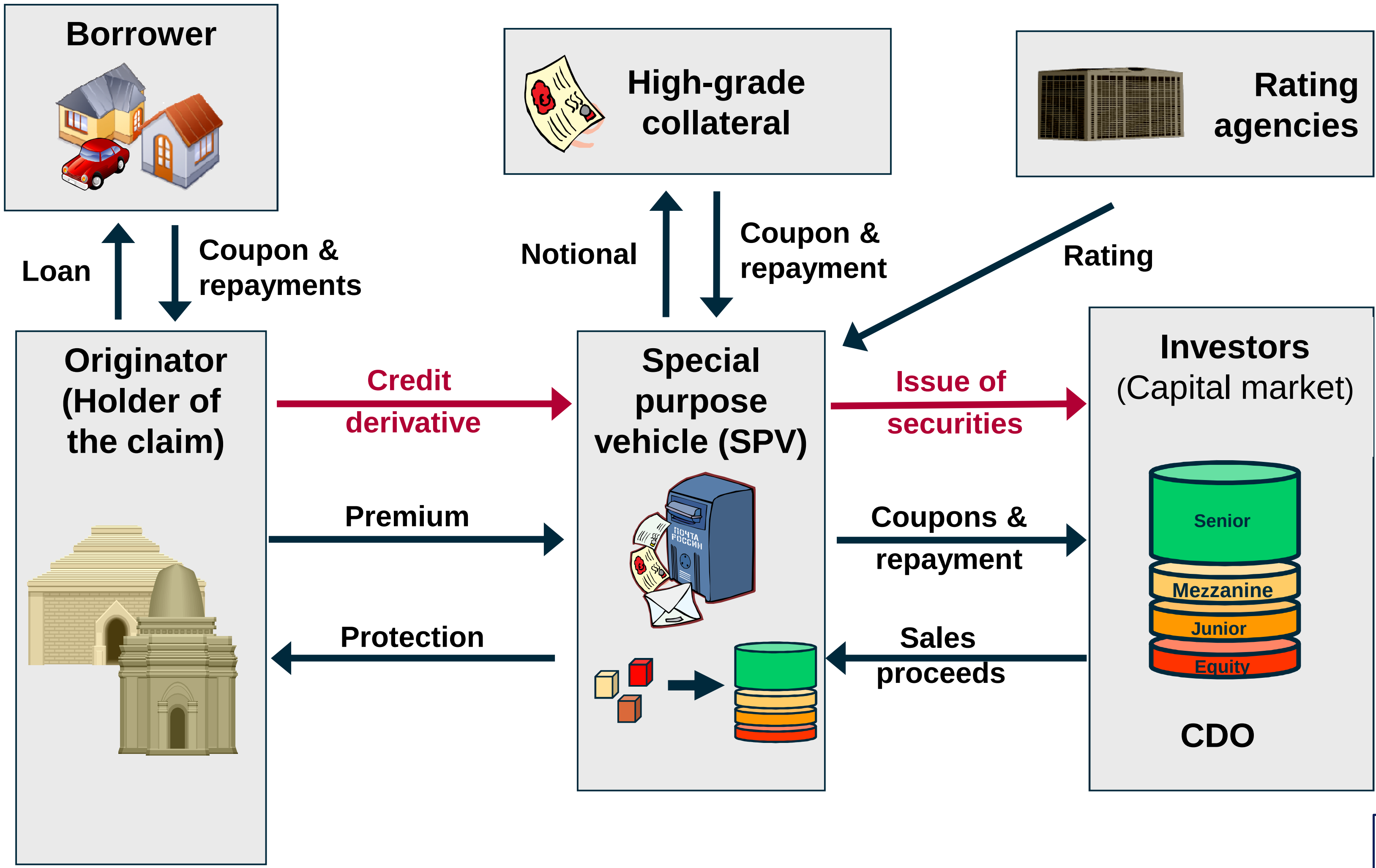
- **Collateralized Debt Obligations**
- **One-factor copula model for credit portfolios**
- **Model extensions**
- **Conclusion**

The Crash-NIG Copula Model

Collateralized Debt Obligations (True Sale CDO)



The Crash-NIG Copula Model
Collateralized Debt Obligations (Synthetic CDO)



The Crash-NIG Copula Model

Collateralized Debt Obligations (Synthetic CDO)

Reference portfolio of credit default swaps (CDS):

- Protection buyer pays a fixed periodic fee at previously fixed (equidistant) payment dates $t_1 < t_2 < \ldots < t_n$ to the protection seller as long as no credit event occurs.
- In exchange, the protection buyer receives a contingent payment from the protection seller triggered by a credit event.

No defaults occurs in the underlying portfolio:

- The investors receive their regular payments (plus notional at termination of the contract).

Default occurs in the underlying portfolio:

- CDO investors absorb all default related losses, starting with the most junior tranche.
- The SPV liquidates part of the high-grade collateral to make the conditional payments on the credit-derivative contract to the originator.
- The notional amount of the related tranche is reduced accordingly while the spread remains the same.

The Crash-NIG Copula Model
Collateralized Debt Obligations

- We want to price the tranche j that takes losses from the lower attachment point l_j to the upper attachment point u_j.
- The annualized spread for tranche j is denoted by s_j and the constant interest rate is denoted by r.
- Given the relative portfolio loss L(t), the loss affecting the tranche j is given by

$$L_j(t) = \min\{\max\{0, L(t) - l_j\}, u_j - l_j\},\ t \in [0,T].$$

- For tranche j the remaining nominal is given by $u_j - l_j - L_j(t)$, $t \in [0,T]$.
- The expected discounted premium (PL) and default leg (DL) are given by

$$PL_j = s_j \cdot \sum_{k=1}^{n} e^{-r \cdot t_k} \cdot \Delta t_k \cdot \left(u_j - l_j - \mathbb{E}[L_j(t_k)]\right)$$

and

$$DL_j = \sum_{k=1}^{n} e^{-r \cdot t_k} \cdot \left(\mathbb{E}[L_j(t_k)] - \mathbb{E}[L_j(t_{k-1})]\right)$$

- At issuance, the spread s_j is fixed so that $PL_j = DL_j$.

The Crash-NIG Copula Model
Collateralized Debt Obligations

Lemma 1:

Let F(t,x) denote the distribution function of the relative portfolio loss and R the recovery rate. Then, the expected loss of tranche j is given by

$$
\begin{aligned}
\mathbb{E}[L_j(t_k)] &= \int_0^1 \min\{(1-R)\cdot x, u_j\} - \min\{(1-R)\cdot x, l_j\}\, dF(t,x) \\
&= \int_{\frac{l_j}{1-R}}^{1} (1-R)\cdot x - l_j \, dF(t,x) - \int_{\frac{u_j}{1-R}}^{1} (1-R)\cdot x - u_j \, dF(t,x) \\
&= (1-R)\cdot \underbrace{\left(\int_{\frac{l_j}{1-R}}^{1} x - \tfrac{l_j}{1-R} \, dF(t,x) - \int_{\frac{u_j}{1-R}}^{1} x - \tfrac{u_j}{1-R} \, dF(t,x) \right)}_{R^0=0,\, l_j^0:=\frac{l_j}{1-R},\, u_j^0:=\frac{u_j}{1-R}}
\end{aligned}
$$

- **Problem:**
  Derivation of the distribution function of the relative portfolio loss.

The Crash-NIG Copula Model
Overview

- **Collateralized Debt Obligations**
- **One-factor copula model for credit portfolios**
- **Model extensions**
- **Conclusion**

One-factor copula models for credit portfolios

General distributions

Definition (one-factor copula model):

- Standardized asset return up to time t of the i-th issuer in the portfolio:

$$A_i(t) = a_i \cdot M(t) + \sqrt{1-a_i^2} \cdot X_i(t)$$

- M(t) and $X_i(t)$, i=1,...,m, independent random variables for m credit instruments.
- Known distribution functions: $F_M(t,\cdot)$ of M(t), $F_X(t,\cdot)$ of $X_i(t)$, and $F_A(t,\cdot)$ of $A_i(t)$.
- The variable $A_i(t)$ is mapped to default time τ_i of the i-th issuer using a percentile-to-percentile transformation, i.e. the issuer i defaults before time t when

$$A_i(t) \leq F_A^{-1}(t, Q(t)) = C(t)$$

- $Q_i(t)$=Q(t) (risk-neutral) probability of the issuer i=1,...,m to default before time t.
- Q(t) is estimated from the average CDS spread.

One-factor copula model for credit portfolios
General distributions: portfolio loss

Theorem 2 (One-factor copula model):

Loss distribution of a large homogeneous portfolio, i.e. identical portfolio weights, default probability, recovery, and correlation to the market factor, with the asset returns following the one-factor copula model, is given by

$$F_\infty(t,x) = 1 - F_M\left(t, \frac{F_A^{-1}(t,Q(t)) - \sqrt{1-a^2} \cdot F_X^{-1}(t,x)}{a} \right)$$

with $x \in [0,1]$ denoting the relative portfolio loss and Q(t) denoting the risk-neutral default probability of each issuer in the portfolio.

Remark (One-factor Gaussian copula model, Vasicek (1987,1991)):

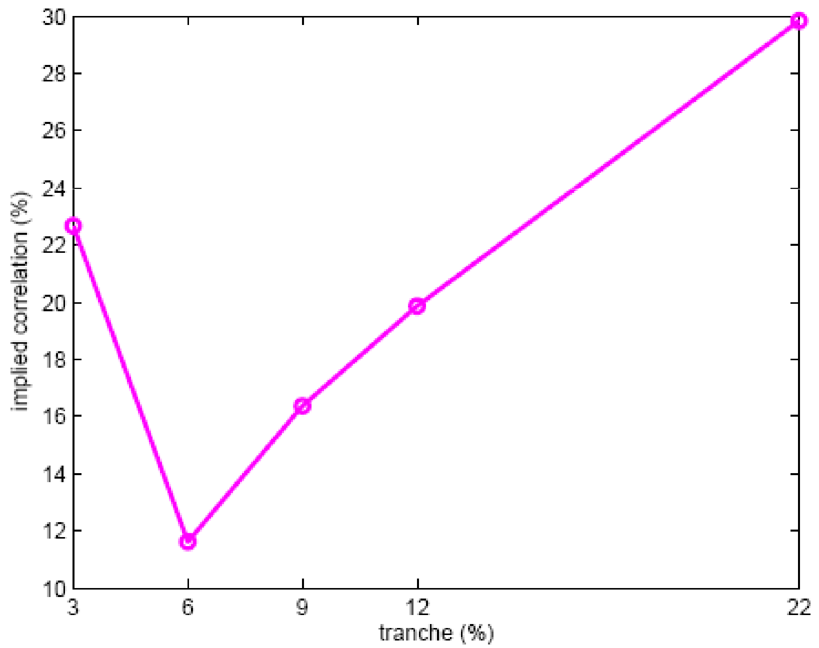
Gaussian distribution for all factors:

$$F_\infty(t,x) = \Phi\left(\frac{\sqrt{1-a^2} \cdot \Phi^{-1}(x) - \Phi^{-1}(Q(t))}{a} \right) = \Phi\left(\frac{\sqrt{1-a^2} \cdot \Phi^{-1}(x) - C(t)}{a} \right)$$

One-factor copula model for credit portfolios

The problems with the one-factor Gaussian copula model

- **Problem:** It is impossible to fit all tranches with the same implied correlation (correlation smile)



- Different implied correlations for tranches with different maturities.

One-factor copula models
Literature on solving the problem

Alternative distributions and copulas:
- O’Kane & Schloegel (2003): Student-t copula
- Schönbucher (2003): Archimedian copulas
- Hull & White (2004): Double-t copula
- Andersen & Sidenius (2005): Marshall-Olkin copula
- Moosbrucker (2005): Variance gamma distributions
- Aas & Haff (2006): Generalized hyperbolic skew Student's t-distribution
- Brunlid (2006): Generalized hyperbolic distribution
- Kalemanova et al (2007): NIG copula
- Albrecher et al (2007): Generalization of the model as Levy one-factor model

Additional stochastic factors:
- Andersen & Sidenius (2005): Random recovery and random correlation factor
- Hull et al (2005): Random correlation correlated with the market factor
- Trinh et al (2005): Idiosyncratic and systematic jumps to default

Empirical comparison: Burtschell et al (2009): Double-t copula

One-factor copula models
NIG distribution

- Density of the Normal Inverse Gaussian distribution:

$$f_{NIG}(x;\alpha,\beta,\mu,\delta)=\frac{\delta\cdot\alpha\cdot e^{\delta\cdot\gamma+\beta\cdot(x-\mu)}}{\pi\cdot\sqrt{\delta^2+(x-\mu)^2}}\cdot K_1\left(\alpha\cdot\sqrt{\delta^2+(x-\mu)^2}\right)$$

where $K_1(\omega)=\frac{1}{2}\cdot\int_0^\infty e^{-\frac{1}{2}\cdot\omega\cdot\left(t+t^{-1}\right)}dt$ is the modified Bessel function of the third kind.

- Scaling property:

$$X\sim NIG(\alpha,\beta,\mu,\delta)\Rightarrow \mathrm{c}\cdot X\sim NIG\left(\frac{\alpha}{c},\frac{\beta}{c},\mathrm{c}\cdot\mu,\mathrm{c}\cdot\delta\right),$$

- Closure under convolution for independent X and Y:

$$X_1\sim NIG(\alpha,\beta,\mu_1,\delta_1),\quad X_2\sim NIG(\alpha,\beta,\mu_2,\delta_2)\ \Rightarrow\ X_1+X_2\sim NIG(\alpha,\beta,\mu_1+\mu_2,\delta_1+\delta_2).$$

One-factor copula models

NIG copula model (Kalemanova, Schmid, Werner (2007))

Theorem 3 (One-factor NIG copula model):

Let

$$NIG_{(s)} = NIG\left(s\cdot\alpha, s\cdot\beta, -s\cdot\frac{\beta\cdot\gamma^2}{\alpha^2}, s\cdot\frac{\gamma^3}{\alpha^2}\right)$$

with corresponding distribution funtion $F_{NIG_{(s)}}(x)$. The standardized asset return up to time t of the i-th issuer in the portfolio, $A_i(t)$, is assumed to be of the form:

$$A_i(t) = a_i\cdot M(t) + \sqrt{1-a_i^2}\cdot X_i(t)$$

with independent random variables

$$M(t) \sim NIG_{(1)},\ X_i(t) \sim NIG_{\left(\frac{\sqrt{1-a^2}}{a}\right)}$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$. Then, $A_i(t) \sim NIG_{\left(\frac{1}{a}\right)}$ and issuer i defaults before time t if

$$A_i(t) \le F^{-1}_{NIG_{\left(\frac{1}{a}\right)}}(t, Q(t)) = C(t).$$

One-factor copula models
NIG copula model (Kalemanova, Schmid, Werner (2007))

Theorem 3 (One-factor NIG copula model, continued):

Furthermore, the distribution function of the portfolio loss at time t is given by

$$F_\infty(t,x) = 1 - F_{NIG_{(1)}}\left(\frac{F^{-1}_{NIG_{\left(\frac{1}{a}\right)}}(Q(t)) - \sqrt{1-a^2} \cdot F^{-1}_{NIG_{\left(\frac{\sqrt{1-a^2}}{a}\right)}}(x)}{a} \right).$$

Remark:

If the random variable X follows a $NIG_{(s)}$ distribution, then

$$\mathbb{E}[X] = 0, \ \mathbb{V}ar[X] = 1, \ Skewness \ \ \mathbb{S}[X] = 3 \cdot \frac{\beta}{s \cdot \gamma^2},$$

$$Kurtosis \ \ \mathbb{K}[X] = 3 \cdot \left(1 + \frac{\alpha^2}{s^2 \cdot \gamma^4} \left(1 + 4 \cdot \left(\frac{\beta}{\alpha} \right)^2 \right) \right)$$

One-factor copula models

Empirical comparison of the Gaussian and NIG copula models

- Calibration of the iTraxx tranches on 12.04.2006

	Market	Gaussian	t(4)-t(4)	t(3)- t(3)	NIG(1)	NIG(2)
0-3%	23,53%	23,53%	23,53%	23,53%	23,53%	23,53%
3-6%	62,75 bp	140,46 bp	73,3 bp	53,88 bp	62,75 bp	62,75 bp
6-9%	18 bp	29,91 bp	28,01 bp	23,94 bp	27,9 bp	27,76 bp
9-12%	9,25 bp	7,41 bp	16,53 bp	15,96 bp	17,64 bp	17,42 bp
12-22%	3,75 bp	0,8 bp	8,68 bp	9,94 bp	9,79 bp	9,6 bp
absolute error		94,41 bp	32,82 bp	27,82 bp	24,34 bp	23,77 bp
correlation		15,72%	19,83%	18,81%	16,21%	15,94%
α					0,4794	0,6020
β					0	-0,1605
comp. time		0,5 s	12.6 s	11 s	1,5 s	1,6 s

- t(n)-t(n) denotes a double-t distribution with n degrees of freedom
- NIG(1) denotes the NIG model with β=0 and thus only one parameter α
- NIG(2) denotes the NIG model with two parameters, α and β.

The Crash-NIG Copula Model
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- **Model extensions**
- **Conclusion**

Model extensions
Overview (Schlösser, Zagst (2010))

1. **Term structure**

   → Consistent modeling of portfolio loss distributions over different time horizons

2. **Large Homogeneous Cells** (Desclee et al (2006) for the Gaussian copula model)

   → Relaxing the assumption of a large homogeneous portfolio and modeling different ratings and rating transitions

3. **Regime-switching correlation**

   → Allow several correlation regimes but still keep the model semi-analytically tractable and fast

Model extensions
Term structure

- The appropriate process for the factors with NIG-distributed increments is given by $N_{(s)}(t)$ with a scaling factor s, independent increments

$$dN_{(s)}(t) \sim NIG\left(s\cdot\alpha, s\cdot\beta, -s\cdot\frac{\beta\cdot\gamma^2}{\alpha^2}dt, s\cdot\frac{\gamma^3}{\alpha^2}dt\right) =: \mathrm{NIG}_{(s)}(\mathrm{dt}),\quad \gamma=\sqrt{\alpha^2-\beta^2},$$

and the following properties:

1. The increments have zero mean and variance dt.
2. The process has zero mean, variance t, skewness $3\cdot\frac{\beta}{s\cdot\gamma^2\cdot\sqrt{t}}$

   and kurtosis $3+3\cdot\left(1+4\cdot\left(\frac{\beta}{\alpha}\right)^2\right)\cdot\frac{\alpha^2}{s^2\cdot\gamma^4\cdot t}$.

3. $N_{(s)}(t) \sim NIG\left(s\cdot\alpha, s\cdot\beta, -s\cdot\frac{\beta\cdot\gamma^2}{\alpha^2}\cdot t, s\cdot\frac{\gamma^3}{\alpha^2}\cdot t\right) =: \mathrm{NIG}_{(s)}(\mathrm{t})$.

Model extensions
Term structure

Theorem 4 (Term-structure one-factor NIG copula model):

The asset return up to time t of the i-th issuer in the portfolio, $A_i(t)$, is assumed to be of the form:

$$A_i(t) = a \cdot M(t) + \sqrt{1-a^2} \cdot X_i(t)$$

with independent stochastic processes:

$$M(t) \sim NIG_{(1)}(t),\ \ X_i(t) \sim NIG_{\left(\frac{\sqrt{1-a^2}}{a}\right)}(t)$$

Then, $A_i(t) \sim NIG_{\left(\frac{1}{a}\right)}(t)$ and the distribution function of the portfolio loss at time t is given by

$$F_\infty(t,x) = 1 - F_{NIG_{(1)}(t)}\left(\frac{F^{-1}_{NIG_{\left(\frac{1}{a}\right)}(t)}(Q(t)) - \sqrt{1-a^2} \cdot F^{-1}_{NIG_{\left(\frac{\sqrt{1-a^2}}{a}\right)}(t)}(x)}{a} \right).$$

Model extensions
Large homogeneous cells

- Proposed by Desclee et al (2006) for the Gaussian copula model.
- Portfolio consists of J rating cells j=1,…,J, each containing a large number of credits with similar properties, i.e.

  - the same weight of all issuers in one cell,

  - the same correlation coefficient a_j to the market factor for the rating cell j,

  - the same risk-neutral default probability $Q_j(t)$ in cell j
- The weight of cell j in the portfolio is denoted by w_j and sums up to 1.
- Within each rating cell, the Large Homogeneous Portfolio model is applied.
- Thus, the portfolio loss, conditional on the realization of the market factor M, is given by

$$L_t(M(t)) = \sum_{j=1}^{J} (1-R) \cdot \omega_j \cdot F_{NIG\left(\frac{\sqrt{1-a_j^2}}{a_j}\right)(t)} \left(\frac{F^{-1}_{NIG\left(\frac{1}{a_j}\right)}(Q_j(t)) - a_j \cdot M(t)}{\sqrt{1-a_j^2}} \right)$$

Model extensions
Large homogeneous cells

Lemma 5:

The loss distribution of an infinitely large homogeneous cell portfolio with asset returns following a term-structure one-factor NIG copula model is given by

$$F_{\infty}^{LHC}(t,x) = 1 - F_{NIG_{(1)}(t)}\left(L_t^{-1}(x)\right).$$

with $x \in [0,1]$ denoting the percentage portfolio loss. The inverse function $L_t^{-1}(x)$ must be computed numerically.

Model extensions

Large homogeneous cells

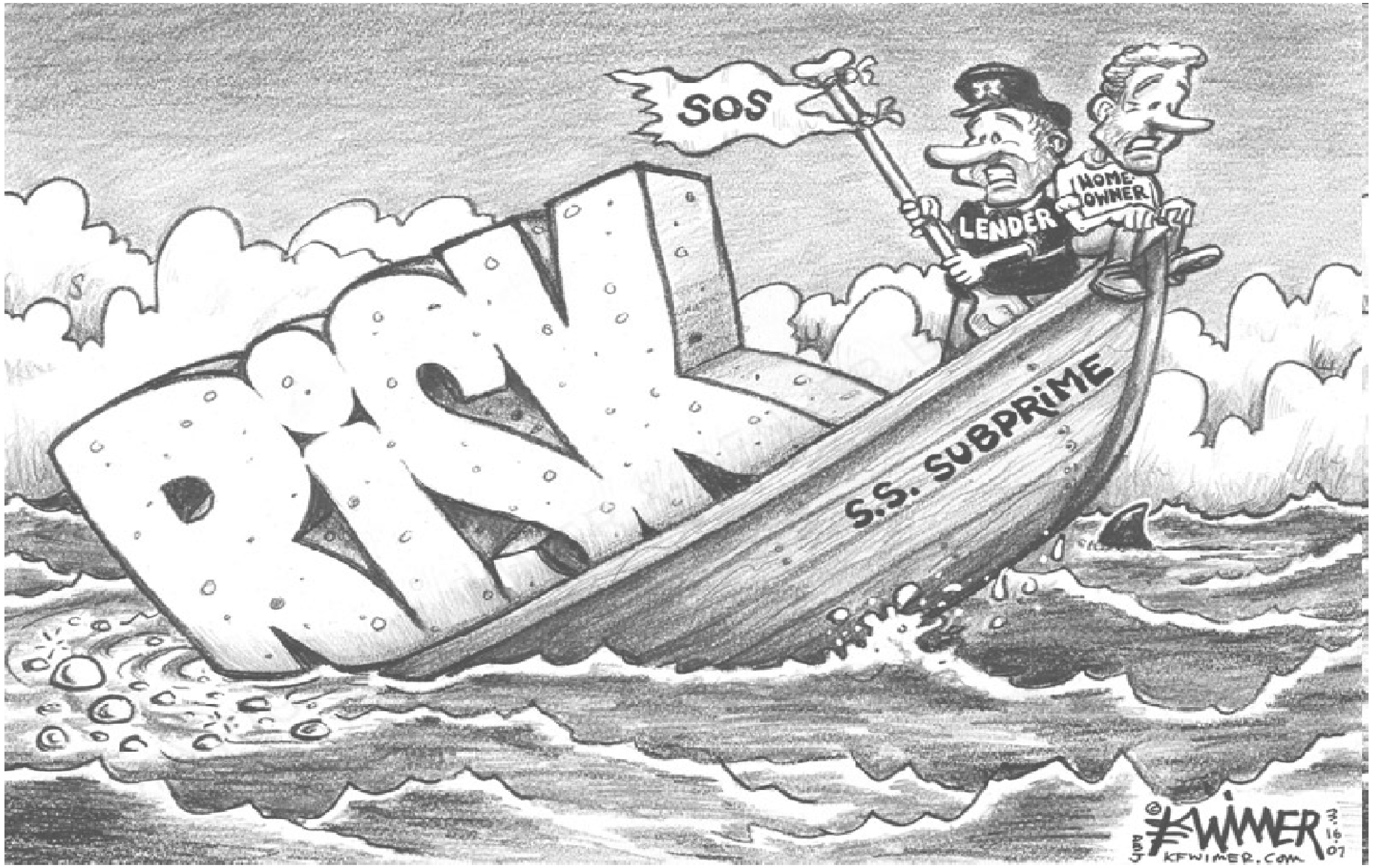
- Calibration to iTraxx data for the 12th of April 2006:

	Maturity	0-3%	3-6%	6-9%	9-12%	12-22%	error		parameter
Market	5	23,53%	62,75 bp	18 bp	9,25 bp	3,75 bp			
	7	36,875%	189 bp	57 bp	26,25 bp	7,88 bp			
	10	48,75%	475 bp	124 bp	56,5 bp	19,5 bp			
Gaussian LHC	5	28,85%	92,02 bp	32,70 bp	13,74 bp	2,76 bp	49,44 bp	a_{AAA}	0,6052
	7	53,43%	198,81 bp	71,91 bp	32,88 bp	7,88 bp	30,85 bp	a_{AA}	0,0004
	10	63,19%	445,90 bp	133,39 bp	65,30 bp	18,42 bp	48,37 bp	a_{A}	0,7211
							128,66 bp	a_{BBB}	0,0005
term-structure NIG(1) LHC	5	24,92%	58,42 bp	23,4 bp	14,25 bp	7,59 bp	18,61 bp	a_{AAA}	0,4217
	7	48,19%	202,08 bp	53,31 bp	27,08 bp	12,05 bp	22,27 bp	a_{AA}	0,5139
	10	56,09%	475,00 bp	124,00 bp	51,93 bp	18,87 bp	5,20 bp	a_{A}	0,4522
							46,09 bp	a_{BBB}	0,2598
								α	0,2269

	AAA	AA	A	BBB
portfolio weight	0,8 %	10,4%	42,4%	46,4 %

Model extensions

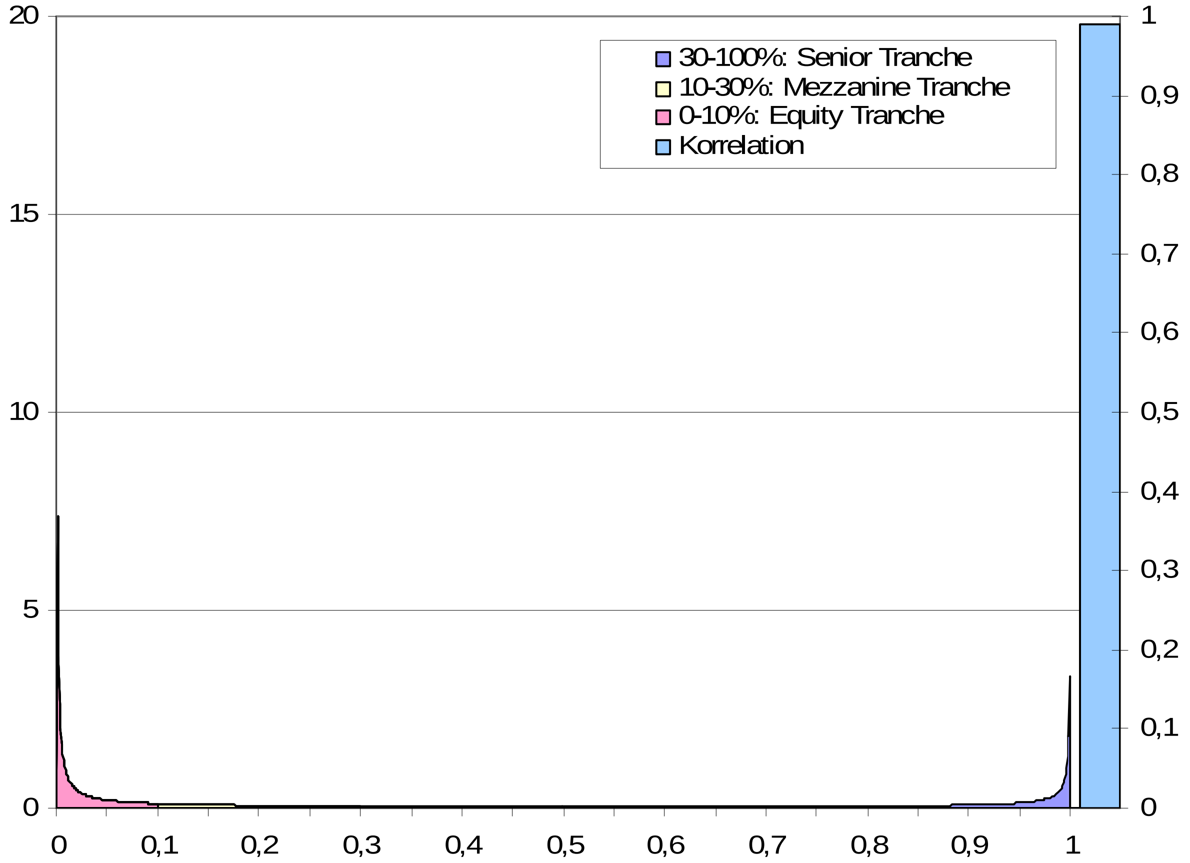
Regime-switching correlation: motivation and model requirements



Model extensions

Regime-switching correlation: motivation and model requirements

- **Experiment:** One-factor Gaussian copula model, 10% default probability, different correlations.



Model extensions

Regime-switching correlation: motivation and model requirements

- One correlation regime is enough for pricing purposes ↔ unrealistic for simulating future scenarios.
- Correlation as a stochastic process does not fit to the concept of the factor copula models.
- Several correlation regimes are sufficient for a simulation-based risk measurement framework.
- The **Crash-NIG model** having two different correlation states has to satisfy the following requirements:
    1. The distribution of the market factor does not depend on the correlation.
    2. The distributions of the factors in both states have zero mean.
    3. The distributions of both factors in different states are stable under convolution.
    4. The asset return has the same distribution in both states to ensure an easy derivation of the default thresholds.

Model extensions

Regime-switching correlation: The Crash-NIG copula model

Theorem 6 (Crash-NIG copula model):

The Crash-NIG model is given by the asset return up to time t of the i-th issuer in cell j=1,…,J, $A_{ij}(t)$, of the form:

$$dA_{ij}(t) = a_j \cdot dM(t) + \sqrt{1-a_j^2} \cdot dX_{ij}(t)$$

with independent stochastic processes:

$$dM(t) \sim NIG_{(1)}\left(\Lambda_t^2\, dt\right),\ dX_{ij}(t) \sim NIG_{\left(\frac{\sqrt{1-a_j^2}}{a_j}\right)}\left(\frac{1-\Lambda_t^2 \cdot a_j^2}{1-a_j^2}\, dt\right),$$

where Λ_t is a Markov process with state space $\{1,\lambda\}$, initial distribution π, and transition function $\{P(h)\}_{h\geq 0}$. The distribution of the increment of the asset return is

$$dA_{ij}(t) \sim NIG_{\left(\frac{1}{a_j}\right)}(dt).$$

Model extensions

Regime-switching correlation: The Crash-NIG copula model

Remarks:

- In the first correlation regime, the variance of all factor changes is dt.
- The variance of the factors in the second regime is

$$\mathbb{V}ar(dM) = \lambda^2\, dt\,,\ \ \mathbb{V}ar(dX_{ij}) = \frac{1-\lambda^2 \cdot a_j^2}{1-a_j^2}\, dt\,.$$

- The correlation of the asset returns of an issuer i_1 from the rating cell j_1 and an issuer i_2 from the rating cell j_2 in the second regime is

$$\mathbb{C}orr\left[dA_{i_1 j_1}(t), dA_{i_2 j_2}(t)\right] = \frac{a_{j_1} \cdot a_{j_2} \cdot \mathbb{V}ar\left[dM(t)\right]}{dt} = a_{j_1} \cdot a_{j_2} \cdot \lambda^2\,.$$

- The model is straightforward for more than two regimes.
- There is no analytical expression for the unconditional distributions of the factors, but they can be approximated by moment matching with NIG distribution.

Model extensions

Regime-switching correlation: The Crash-NIG copula model

Theorem 7:

Let the assumptions of Theorem 6 be satisfied and assume that $\Lambda_0 = z \in \{1,\lambda\}$ at time 0. Furthermore, let $T^z(t) := \left(T_1^z(t), T_\lambda^z(t)\right)'$ be the stochastic process indicating the duration of the stay in state 1 resp. λ from the starting state z at time t=0 up to time t and $\hat{T}^z(t) := T_1^z(t) + \lambda^2 \cdot T_\lambda^z(t)$. Then, the distributions of M(t) and $X_{ij}(t)$, conditional on the realization of $T^z(t)$, are NIG-distributed:

$$
\begin{aligned}
M(t)\big|T^z(t) \quad &\sim \quad NIG_{(1)}\left(\hat{T}^z(t)\right) \\
X_{ij}(t)\big|T^z(t) \quad &\sim \quad NIG_{\left(\frac{\sqrt{1-a_j^2}}{a_j}\right)}\left(\frac{t - a_j^2 \cdot \hat{T}^z(t)}{1-a_j^2}\right)
\end{aligned}
$$

The distribution of the asset return is

$$
A_{ij}(t) \sim NIG_{\left(\frac{1}{a_j}\right)}(t).
$$

Model extensions
Regime-switching correlation: The Crash-NIG copula model

Theorem 8:

Let the assumptions of Theorem 7 be satisfied. Furthermore, let

$$\hat{M}(t) \sim NIG_{(1)}\left(\mathbb{E}[\hat{T}^z(t)]\right)$$
$$\hat{X}_{ij}(t) \sim NIG_{\left(\frac{\sqrt{1-a_j^2}}{a_j}\right)}\left(\frac{t-a_j^2\cdot\mathbb{E}[\hat{T}^z(t)]}{1-a_j^2}\right)$$

Then, the distribution of $\hat{M}(t)$ and $\hat{X}_{ij}(t)$ fits the first two “moments” of the distributions of M(t) and $X_{ij}(t)$. Furthermore, the third and fourth “moment” of the approximate distributions are not higher than those of the exact distributions.

In the special case of β=0, the skewness of both distributions is zero.

Remark:

The approximation of Theorem 8 will be applied in the sequel.

Model extensions
Regime-switching correlation: The Crash-NIG copula model

Theorem 9:

Given the approximate model of Theorem 8 and the LHC assumption, the loss distribution of the infinitely large homogeneous cell portfolio with asset returns following a Crash-NIG copula model is given by

$$F_{\infty}^{LHC}(t,x) = 1 - F_{NIG_{(1)}\left(\mathbb{E}[\hat{T}^{z}(t)]\right)}\left(L_t^{-1}(x)\right)$$

with $x \in [0,1]$ denoting the percentage portfolio loss and

$$L_t(M(t)) = \sum_{j=1}^{J}(1-R)\cdot\omega_j\cdot F_{NIG_{\left(\frac{\sqrt{1-a_j^2}}{a_j}\right)}\left(\frac{t-a_j^2\cdot\mathbb{E}[\hat{T}^{z}(t)]}{1-a_j^2}\right)}\left(\frac{F^{-1}_{NIG_{\left(\frac{1}{a_j}\right)}}(Q_j(t)) - a_j\cdot M(t)}{\sqrt{1-a_j^2}}\right).$$

Model extensions

The Crash-NIG copula model: calibration procedure

Data:

- The history of the iTraxx Europe tranched index since its origination on the 21th of June 2004 until the 6th of May 2008 (from “Morgan Markets”).
- Rating composition of the portfolio on a daily basis (calculated from “Markit”).
- Rating-average spreads of the iTraxx portfolio constituents (computed from single-CDS).

Set-up:

- **Three-state model**: the two crisis (market turbulences after the downgrade of Ford and General Motors in Mai 2005 and the credit crisis starting in July 2007) have different character.
- The iTraxx tranches market is more liquid than the single-CDS market
  → **liquidity indicators** l_r, with r=1,2,3 the current state:
    - $(1-l_r)\cdot s$ liquidity spread part
    - $l_r \cdot s$ credit spread part.

Model extensions

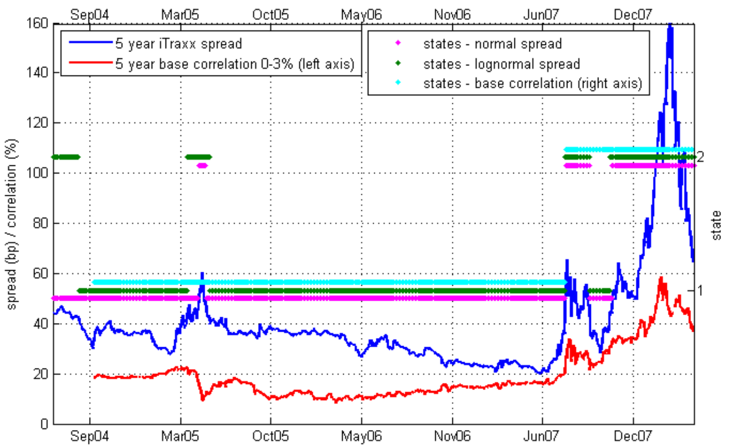
The Crash-NIG copula model: calibration procedure

Two-step calibration:

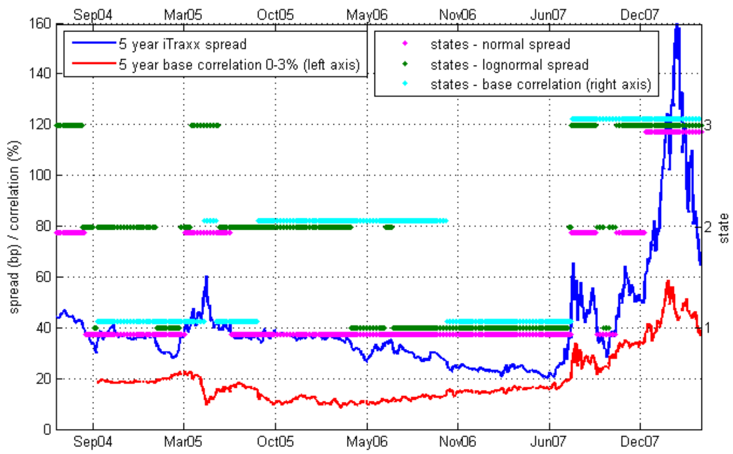
1. Hidden Markov Model (HMM) is estimated separately using the 5 year iTraxx index spreads under the assumption of a normal distribution:

   - Transition matrix estimated with Baum-Welch algorithm

   - Most likely states estimated with Viterbi algorithm.



Calibration of the HMM with two states



Calibration of the HMM with three states

Model extensions

The Crash-NIG copula model: calibration procedure

Two-step calibration:

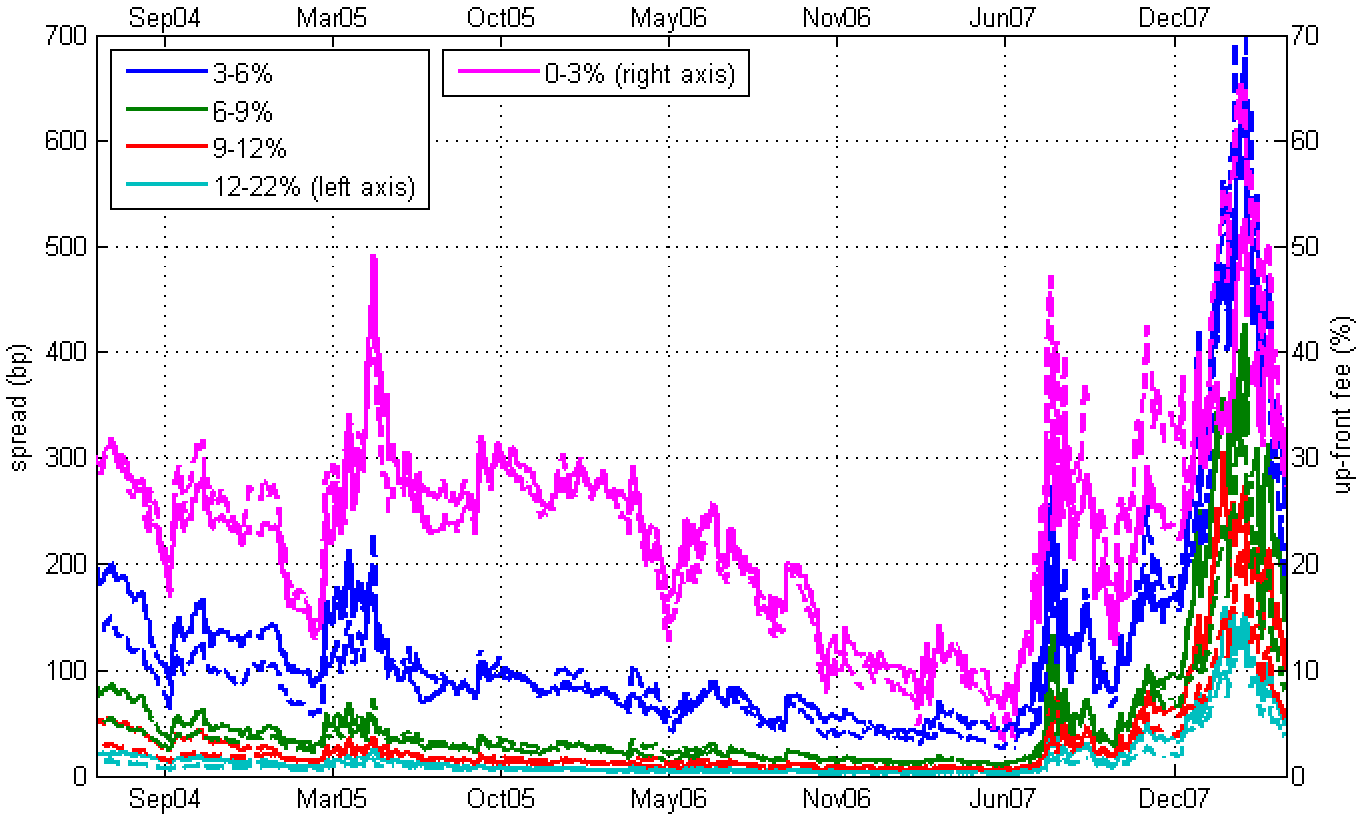
2. Other NIG model parameters (a_j, λ, α and l_r) are calibrated to the tranches' history:

Parameter	Crash-NIG model (3 states, liquidity)	Crash-NIG model (2 states, liquidity)	NIG model (1 state, liquidity)	Crash-NIG model (2 states, no liquidity)
alpha	0,3274	0,3957	0,3615	0.1717
a1	0,2562	0,1680	0,2476	0,3869
a2	0,5437	0,4275	0,9607	0,4493
a3	0,3429	0,4275	0,4975	0,4494
a4	0,2130	0,1767	0,3256	0,2820
a5	0,0828	0,1705	0,1161	0,2541
lambda_1	0,2353	2,2220		2,1141
lambda_2	1,7443			
l_1	0,9679	0,9439	0,9562	
l_2	0,8827	0,7330		
l_3	0,7361			
Aver. Error (%)	14,8	18,13	23,98	25,76

Model extensions

The Crash-NIG copula model: calibration results

- Calibration results of the 5-year iTraxx tranches:



The Crash-NIG Copula Model
Overview

- **Collateralized Debt Obligations**
- **One-factor copula model for credit portfolios**
- **Model extensions**
- **Conclusion**

The Crash-NIG Copula Model
Conclusion

- The NIG-factor copula model was extended to a term-structure model which allows to consistently model CDO tranches with different maturities.
- The Large Homogeneous Cell (LHC) setting of Desclee et al (2006) was applied to the term-structure NIG model demonstrating a much better fit to market quotes than the Gaussian LHC model.
- The model was extended to the Cash-NIG copula model which allows for different correlation regimes such as a correlation break down.
- A liquidity premium was introduced which allowed for the explanation of the liquidity dry-out during the financial crisis.
- The Crash-NIG model can be applied to a joint Monte Carlo simulation with other risk factors and thus allows for the simulation of a portfolio of traditional and structured credit instruments.
- The Crash-NIG model thus allows for the scenario-based optimization of balanced portfolios including structured credit instruments.

The Crash-NIG Copula Model
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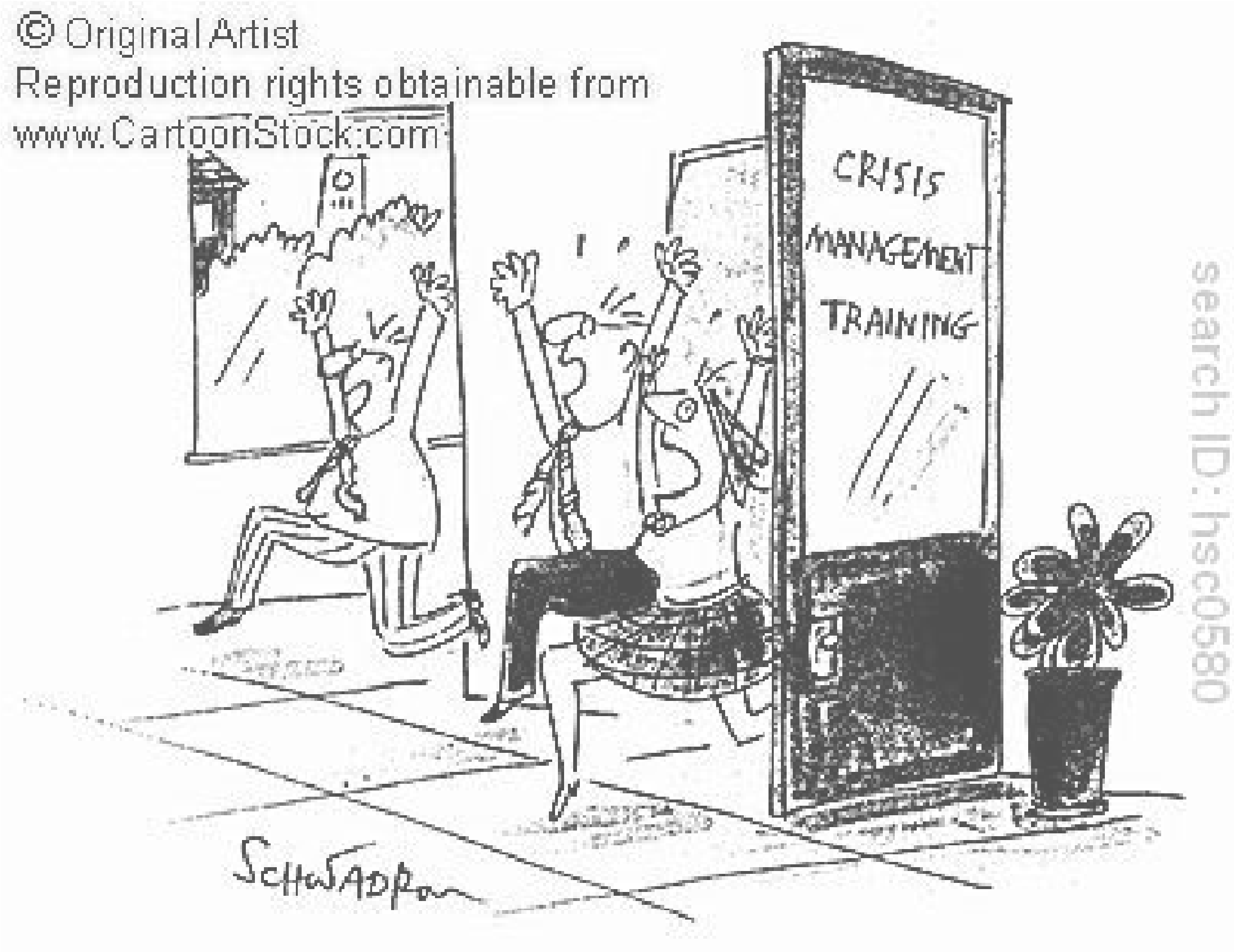
The Crash-NIG Copula Model
Conclusion



Appendix
NIG process

Definition: The Normal Inverse Gaussian process can be defined as

$$Z(t) = \mu \cdot t + D^{(\beta)}\left(T^{(\delta,\gamma)}(t)\right), \text{ for each } t \geq 0,$$

where $D^{(\beta)}(t) = \beta \cdot t + B(t)$ is a Brownian motion with drift and $T^{(\delta,\gamma)}(t)$ is an inverse Gaussian subordinator such that

$$\begin{aligned} T^{(\delta,\gamma)}(t) &= \inf\{ s > 0 : D^{(\gamma)}(s) = \delta \cdot t \}, \\ D^{(\gamma)}(t) &= \gamma \cdot t + \hat{B}(t), \end{aligned}$$

and $\beta \in \mathbb{R}$, $\gamma = \sqrt{\alpha^2 - \beta^2}$, $\alpha \in \mathbb{R}$ with $\alpha^2 \geq \beta^2$ and independent standard Brownian motions B(t) and $\hat{B}(t)$.

- The NIG process is by construction a time changed Brownian motion and is a Lèvy process.
- The process $N_{(s)}(t)$ is a special case of the general NIG process with parameters chosen in the way to have zero mean and variance t.

Appendix
The Crash-NIG copula model: Moments

Theorem:

Let the assumptions of Theorem 7 be satisfied. Then, the "moments" of the unconditional distributions of M(t) and X_{ij}(t) are given by

$$\mathbb{E}[M(t)] = 0, \ \mathbb{V}ar[M(t)] = \mathbb{E}[\hat{T}^z(t)], \ \mathbb{S}[\mathrm{M(t)}] = \frac{3\cdot\beta}{\gamma^2}\cdot\mathbb{E}\left[\frac{1}{\sqrt{\hat{T}^z(t)}}\right],$$

$$\mathbb{K}[\mathrm{M(t)}] = 3\cdot\left(1+\left(1+4\cdot\left(\frac{\beta}{\alpha}\right)^2\right)\cdot\frac{\alpha^2}{\gamma^4}\cdot\mathbb{E}\left[\frac{1}{\hat{T}^z(t)}\right]\right)$$

and

$$\mathbb{E}[X_{ij}(t)] = 0, \ \mathbb{V}ar[X_{ij}(t)] = \frac{t - a_j^2\cdot\mathbb{E}[\hat{T}^z(t)]}{1-a_j^2}, \ \mathbb{S}[X_{ij}(\mathrm{t})] = \frac{3\cdot\beta\cdot a_j}{\gamma^2}\cdot\mathbb{E}\left[\frac{1}{\sqrt{t-a_j^2\cdot\hat{T}^z(t)}}\right],$$

$$\mathbb{K}[X_{ij}(\mathrm{t})] = 3\cdot\left(1+\left(1+4\cdot\left(\frac{\beta}{\alpha}\right)^2\right)\cdot\frac{\alpha^2\cdot a_j^2}{\gamma^4}\cdot\mathbb{E}\left[\frac{1}{t-a_j^2\cdot\hat{T}^z(t)}\right]\right)$$

Appendix
The Crash-NIG copula model: Moments

Theorem:

Let the assumptions of Theorem 7 be satisfied. Furthermore, let β=0. Then, the NIG-distribution that fits the first four “moments” of M(t) is given by

$$NIG\big(\alpha_M(t), \beta_M(t), \mu_M(t), \delta_M(t)\big),\ \gamma_M(t) = \sqrt{\alpha_M^2(t) - \beta_M^2(t)}$$

with

$$\beta_M(t) = 0,\ \text{i.e. } \gamma_M(t) = \alpha_M(t),\ \mu_M(t) = 0,\ \text{and}$$

$$\alpha_M(t) = \frac{\alpha}{\sqrt{\mathbb{E}[\hat{T}^z(t)] \cdot \mathbb{E}\left[\frac{1}{\hat{T}^z(t)}\right]}},\ \delta_M(t) = \alpha \cdot \sqrt{\frac{\mathbb{E}[\hat{T}^z(t)]}{\mathbb{E}\left[\frac{1}{\hat{T}^z(t)}\right]}}$$

Appendix
The Crash-NIG copula model: Moments

Theorem 7 (continued):

Correspondingly, the NIG-distribution that fits the first four "moments" of $X_{ij}(t)$ is given by

$$NIG\left(\alpha_{X_{ij}}(t), \beta_{X_{ij}}(t), \mu_{X_{ij}}(t), \delta_{X_{ij}}(t)\right),\ \gamma_{X_{ij}}(t) = \sqrt{\alpha^2_{X_{ij}}(t) - \beta^2_{X_{ij}}(t)}$$

with

$$\beta_{X_{ij}}(t) = 0\,,\ \text{i.e. }\ \gamma_{X_{ij}}(t) = \alpha_{X_{ij}}(t)\,,\ \mu_{X_{ij}}(t) = 0\,,$$

$$\alpha_{X_{ij}}(t) = \frac{\alpha \cdot \sqrt{1-a_j^2}}{a_j \cdot \sqrt{\left(t - a_j^2 \cdot \mathbb{E}[\hat{T}^z(t)]\right) \cdot \mathbb{E}\left[\frac{1}{t - a_j^2 \cdot \hat{T}^z(t)}\right]}}\,,$$

$$\delta_{X_{ij}}(t) = \frac{\alpha}{a_j \cdot \sqrt{1-a_j^2}} \cdot \sqrt{\frac{t - a_j^2 \cdot \mathbb{E}[\hat{T}^z(t)]}{\mathbb{E}\left[\frac{1}{t - a_j^2 \cdot \hat{T}^z(t)}\right]}}$$