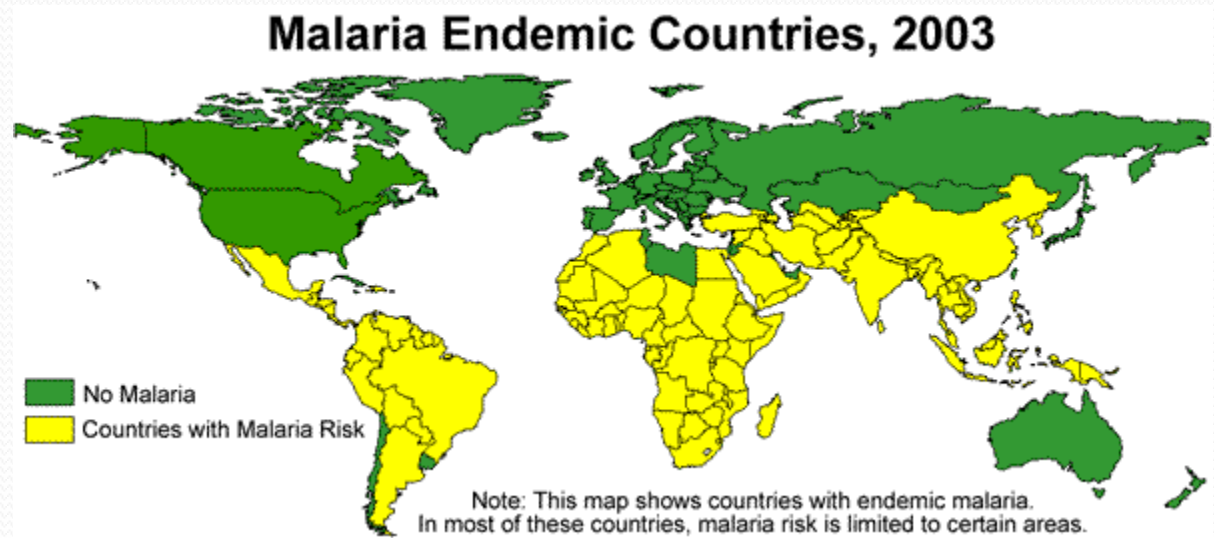


Traveling Waves in Diffusive Ross-Macdonald Type Host- Vector Models

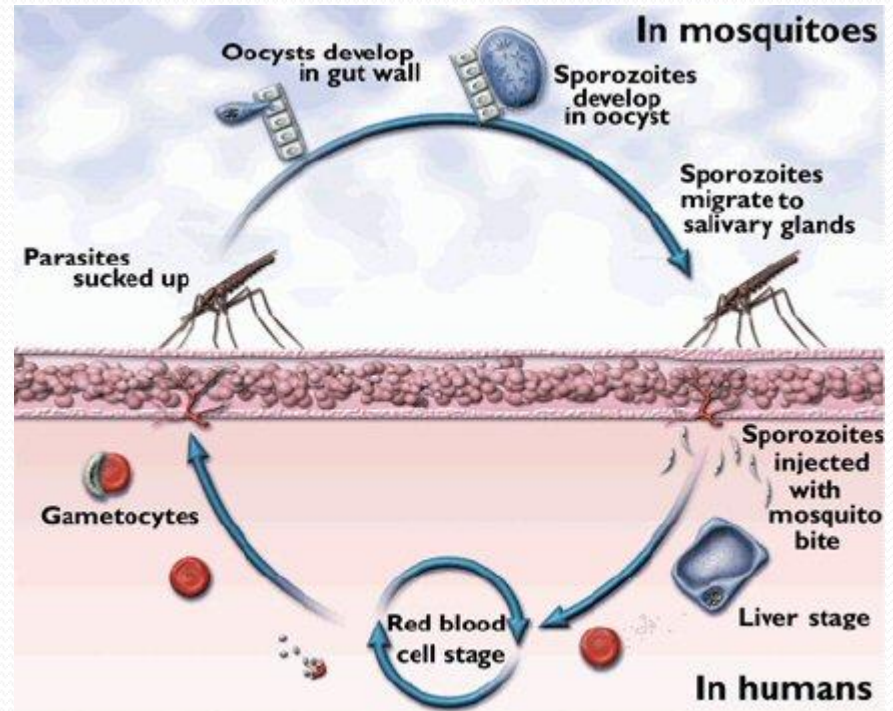
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Background

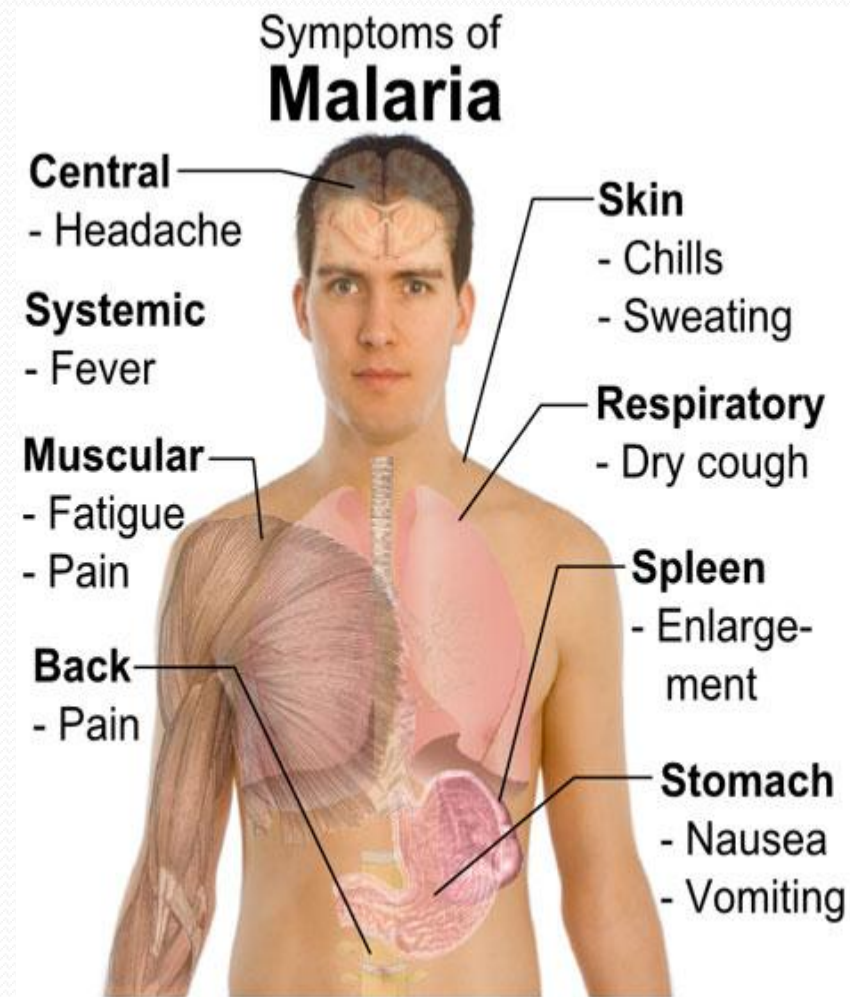
- Malaria is found throughout the tropical and subtropical regions of the world. According to WHO, in 2008, there were 247 million cases of malaria . Globally, it is one of the three most dangerous infectious diseases (the others are HIV/AIDS and tuberculosis).



The malaria parasite is a microscopic organism called a *Plasmodium*. The malaria parasite enters the human host when an infected female Anopheles mosquito takes a blood meal. Inside the human host, the parasite undergoes a series of changes as part of its complex life-cycle. Its various stages allow plasmodia to evade the immune system, infect the liver and red blood cells, and finally develop into a form that is able to infect a mosquito again when it bites an infected person.




- Malaria symptoms appear about 9 to 14 days after the infectious mosquito bite, although this varies with different plasmodium species. Typically, malaria produces fever, headache and vomiting. Several hours later, the fever subsides and a chill sets in. The cycle is repeated every 2 to 4 days. If drugs are not available for treatment or the parasites are resistant to them, the infection can progress rapidly to become life-threatening.



Model Assumptions

1. We consider a host-vector model for disease without immunity.
2. The spatial spread of the host and vector is modeled by diffusion terms.
3. The host population is assumed to be stable, that is, the birthrate is constant and equals to the mortality rate. The total vector population is also constant.

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4. For transmission of the disease, it is assumed
 - a susceptible host can receive the infection only by contacting with infected vectors
 - a susceptible vector can receive the infection only from the infected host.

Parameters

H := the total population of hosts

M := the total population of vectors

$U(t, x)$:= the numbers of infected hosts at time t and location x

$V(t, x)$:= the number of infected vectors at time t and location x

a := the rate of biting on host by a single mosquito (number of bites per unit time)

b := the proportion of infected bites on host that produce an infection

c := the transmission efficiency from infected hosts to vectors

r := the per capita rate of recovery in hosts

μ := the per capita rate of mortality in vectors

d_1 := the diffusion rate of hosts

d_2 := the diffusion rate of vectors

Thus we have the equations for the rate of change in the number of infected hosts and vectors:

$$\frac{\partial U}{\partial t} = d_1 \Delta U - rU(t, x) + \left(\frac{a}{H} \right) b [H - U(t, x)] V(t, x)$$

$$\frac{\partial V}{\partial t} = d_2 \Delta V - \mu V(t, x) + \left(\frac{a}{H} \right) c U(t, x) [M - V(t, x)]$$

Rescale

Define

$$u(t, x) = \frac{U(t, x)}{H}, \quad v(t, x) = \frac{V(t, x)}{M}, \quad m = \frac{M}{H}$$

Then we obtain the following diffusive Ross-Macdonald model:

$$\frac{\partial u}{\partial t} = d_1 \Delta u - ru(t, x) + abm[1 - u(t, x)]v(t, x)$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v - \mu v(t, x) + acu(t, x)[1 - v(t, x)]$$

Basic Reproduction Number

Define the basic reproduction number $R_0 = \frac{a^2bcm}{r\mu}$

Lemma. The diffusive system has at most two spatially homogeneous steady states.

- (i) if $R_0 \leq 1$, then system has a unique trivial (disease-free) equilibrium $(0,0)$;
- (ii) if $R_0 > 1$, then system has two equilibria, the trivial equilibrium $(0,0)$ and the positive (endemic) equilibrium (u^*, v^*) , where

$$(u^*, v^*) = \left(\frac{a^2bcm - r\mu}{ac(abm + r)}, \frac{a^2bcm - r\mu}{abm(ac + \mu)} \right) = \left(\frac{R_0 - 1}{R_0 + \frac{ac}{\mu}}, \frac{R_0 - 1}{R_0 + \frac{abm}{r}} \right)$$

Existence of Traveling Waves

Assume $x \in (-\infty, \infty)$. Rewrite the system as follows:

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} u_{xx} \\ v_{xx} \end{bmatrix} + F\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)$$

where

$$F\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} -ru + abm(1-u)v \\ -\mu v + acu(1-v) \end{bmatrix}$$

- A traveling wave solution with speed C is a solution of the form

$$\begin{bmatrix} u(t, x) \\ v(t, x) \end{bmatrix} = \begin{bmatrix} U(x - Ct) \\ V(x - Ct) \end{bmatrix} = \begin{bmatrix} U(z) \\ V(z) \end{bmatrix},$$

which connects the trivial equilibrium and the endemic equilibrium, i. e.

$$\lim_{z \rightarrow \infty} \begin{bmatrix} U(z) \\ V(z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \lim_{z \rightarrow -\infty} \begin{bmatrix} U(z) \\ V(z) \end{bmatrix} = \begin{bmatrix} u^* \\ v^* \end{bmatrix}$$

- The corresponding ODE system

$$\begin{aligned}d_1 U'' + CU' - rU + abm(1 - U)V &= 0, \\d_2 V'' + CV' - \mu V + acU(1 - V) &= 0,\end{aligned}$$

We use Theorem 4.2 of Li, Weinberger & Lewis (2005) to establish the existence of traveling wave solutions of the system.

The system satisfies the assumptions of Theorem 4.2 in Li et al. (2005)

1. The function F has two zeroes.
2. The system is cooperative since

$$\begin{aligned}\frac{\partial F_1}{\partial v} &= abm[1 - u(x, t)] \geq 0, \\ \frac{\partial F_2}{\partial u} &= ab[1 - v(x, t)] \geq 0,\end{aligned}$$

3. The function F does not depend explicitly on x or t .
4. F is continuous, and has uniformly bounded continuous first partial derivatives for $(0, 0) \leq (u, v) \leq (u^*, v^*)$ and is differentiable at $(0, 0)$. The Jacobian matrix J at $(0, 0)$ has non-negative off-diagonal entries, and has a positive eigenvalue $\lambda_+ = \frac{1}{2}[-(r + \mu) + \sqrt{(r + \mu)^2 + 4r\mu(R_0 - 1)}] > 0$ whose eigenvector has positive components.
5. The diagonal matrix of the diffusion coefficients is constant and positive.

Theorem

There is a minimal wave speed c_0 such that for every $C \geq c_0$, the nonlinear system has a non-increasing traveling wave solution $(u(t, x), v(t, x)) = (U(x - Ct), V(x - Ct))$ which satisfies

$$\lim_{(x-Ct) \rightarrow -\infty} (U, V) = (u^*, v^*), \quad \lim_{(x-Ct) \rightarrow \infty} (U, V) = (0, 0)$$

If $C < c_0$, then system does not have such traveling wave solution.

Moreover, the minimal wave speed c_0 for the nonlinear system is equal to the spread rate c^* of the system; that is, the system has a single spreading speed.

Spread Rate Analysis

- We now consider how to calculate the spread rate c^* for the nonlinear system. Note that the nonlinear system satisfies the *subtangential condition*, i.e.

$$F\left(\rho \begin{bmatrix} u \\ v \end{bmatrix}\right) \leq \rho DF(0) \begin{bmatrix} u \\ v \end{bmatrix}$$

for any positive ρ , then by Theorem 4.2 of Weinberger et al. (2002), the spread rate c^* for the nonlinear system is linearly determinate, i.e. $c^* = \bar{c}$ where \bar{c} is the spread rate for the linearized system

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} u_{xx} \\ v_{xx} \end{bmatrix} + J \begin{bmatrix} u \\ v \end{bmatrix}$$

with $J = DF(0,0) = \begin{bmatrix} -r & abm \\ ac & -\mu \end{bmatrix}$

Theorem

(i) The spread rate c^* of the nonlinear system and the spread rate \bar{c} of the linearized system both exist and $c^* = \bar{c}$.

(ii) The spread rate \bar{c} of the linearized system is given by $\bar{c} = \inf_{\lambda > 0} \sigma_1(\lambda)$

where $\sigma_1(\lambda)$ is the largest eigenvalue of the matrix $B_\lambda = \frac{J + \lambda^2 D}{\lambda}$

Let

$$\theta = \text{tr}(J) = -(\mu + r) < 0$$

$$d = \det(J) = r\mu - a^2bcm < 0$$

Then the characteristic polynomial of \mathbf{B}_λ is

$$p(\sigma, \lambda) = \sigma^2 - \sigma \frac{\theta + \lambda^2(d_1 + d_2)}{\lambda} + \lambda^2 d_1 d_2 - \mu d_1 - r d_2 + \frac{d}{\lambda^2}$$

We denote the larger root $\sigma_1(\lambda)$, and the other one $\sigma_2(\lambda)$

- This is well defined, and analogue analysis to Hadeler& Lewis (2002) we have the following lemmas:

Lemma 1. Both $\sigma_1(\lambda)$ and $\sigma_2(\lambda)$ are real.

Lemma 2. $\sigma_1(\lambda)$ is positive for all values of λ and achieves its minimum on the interval $0 < \lambda < \infty$ for finite values of λ .

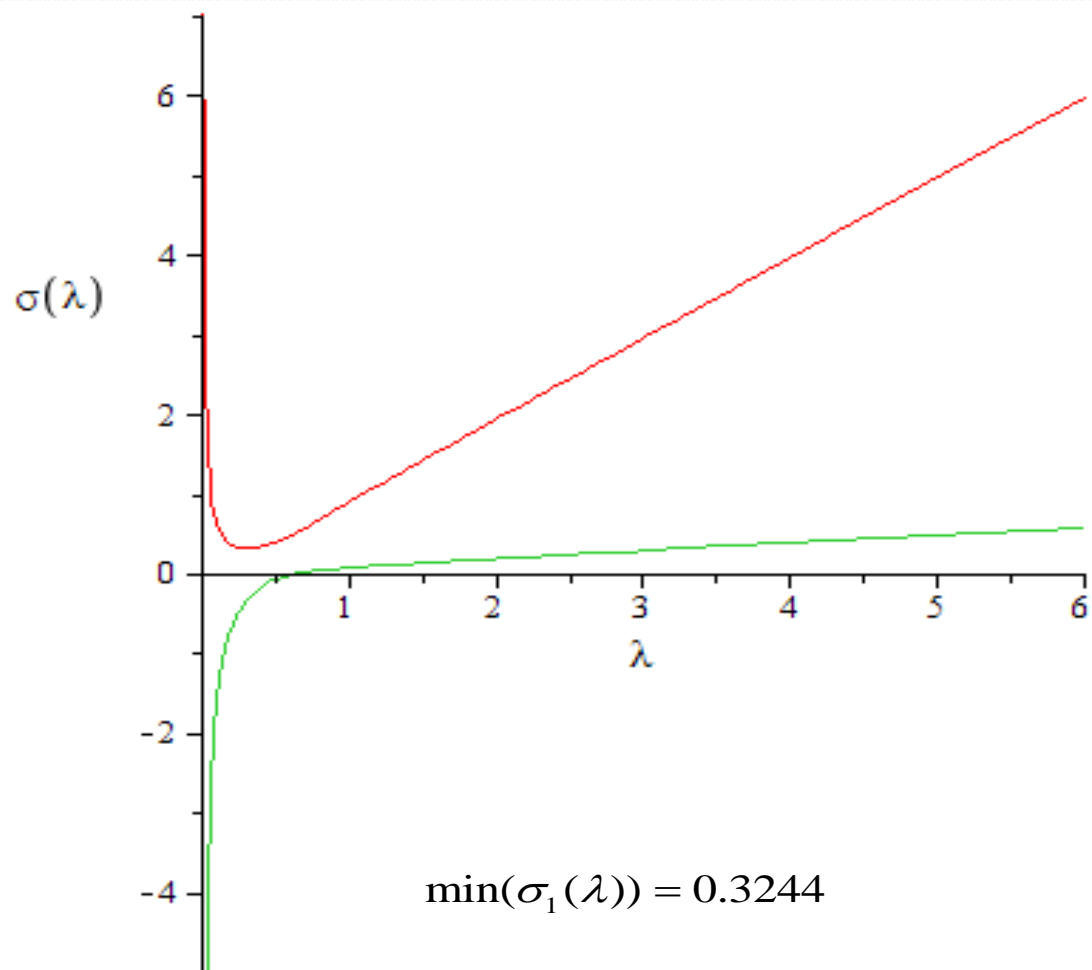
Lemma 3. The function $\sigma_1(\lambda)$ has a unique minimum

$$\overline{\sigma_1} = \min_{\lambda > 0} \sigma_1(\overline{\lambda_1}) > 0$$

Table 1: Variables and Parameters

<i>Parameters and Variables</i>		<i>Values</i>	References
Dependent Variables			
$u(t, x)$	proportion of infected humans		
$v(t, x)$	proportion of infected mosquitoes		
Parameters and Constants			
d_1	diffusion rate of hosts		
d_2	diffusion rate of mosquitos		
m	ratio of mosquitos to hosts	2	[1,2]
a	biting rate on a host per mosquito	$0.2 - 0.5/day$	[3,4,8]
b	infected mosquito to host transmission efficiency	0.5	[5, 6,8]
c	infected host to mosquito transmission efficiency	0.5	[5, 6,8]
r	per capita host recovery rate	$0.01 - 0.05/day$	[3,6,7,8]
μ	per capita mortality rate of mosquitos	$0.05 - 0.5/day$	[3,6,7]

References: [1]=Harada et al. (1998), [2]=Ishikawa et al. (2003), [3]=Macdonald (1957), [4]=Dietz et al. (1974), [5]=Gu et al. (2003a), [6]=Le Menach et al. (2005), [7]=Aron and May (1978), [8]=Smith et al. (2004).



Future Work

- A recent published paper by Lou & Zhao provided a thoroughly analysis on the periodic Ross–Macdonald model with diffusion and advection.
- Due to the incubation period of the several forms of parasites, it is necessary to consider a delay model.



Thank you!