Multigroup populations, pair formation, and epidemic disease

K.P. Hadeler

Universität Tübingen
and Arizona State University

A tool: Homogeneous systems

$$f : \mathbb{R}^n_+ \to \mathbb{R}^n$$

$$f(\alpha x) = \alpha f(x), \quad \alpha \geq 0$$

$$\dot{x} = f(x)$$

No stationary points $x \neq 0$

Exponential solutions

$$x(t) = \hat{x} e^{\hat{\lambda} t}, \quad \hat{x} \in \mathbb{R}^n_+$$

Stability

Jacobian matrix

$$f'(\hat{x})$$

Eigenvalues

$$\lambda_1 = \hat{\lambda},\ \lambda_2, \ldots, \lambda_n$$

Stability condition

$$\Re\lambda_k < \lambda_1, \quad k = 2, \ldots, n$$

Projection

$$y = \frac{x}{e^T x}, \quad e^T = (1, \dots, 1)$$
$$\dot{y} = f(y) - e^T y \, y$$

on

$$\Delta = \{y \in \mathbb{R}^n_+ : \; e^T y = 1\}$$

Exponential solutions become stationary points.

Two-sex marriage problem

Keyfitz, Parlett, Yellin and Samuelson, ...

Hadeler/Waldstätter/ Wörz 1988

Iannelli/ Martcheva/ Milner (book 2005)

The standard model

x female single, y male single, p pair

$$\begin{aligned}\dot{x} &= \kappa_x p - \mu_x x + \mu_y p + \sigma p - \phi(x,y)\\ \dot{y} &= \kappa_y p - \mu_y y + \mu_x p + \sigma p - \phi(x,y)\\ \dot{p} &= -(\mu_x + \mu_y + \sigma)p + \phi(x,y)\end{aligned}$$

Pair formation function $\phi \geq 0$

$\phi(\alpha x, \alpha y) = \alpha\phi(x,y)$

$\phi(0,y) = \phi(x,0) = 0$

$\phi(x+u, y+v) \geq \phi(x,y)$, $u, v \geq 0$

Result:

One-sex solutions $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-\mu_x t}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-\mu_y t}$

Two-sex solution $\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{p} \end{pmatrix} e^{\hat{\lambda} t}$

If it exists then it is globally stable.

The two-sex solution exists when the female population is unstable against infection with males and conversely.

pair formation function

harmonic mean

$$\phi(x, y) = 2\rho \frac{xy}{x + y}$$

minimum

$$\phi(x, y) = \rho \min(x, y)$$

Which is “better”?

Age structure

$$x_t + x_a + \mu x + (\mu + \sigma)p - \phi = 0$$
$$y_t + y_b + \mu y + (\mu + \sigma)p - \phi = 0$$
$$p_t + p_a + p_b + p_c + (2\mu + \sigma)p = 0$$
$$x(t, 0) = \int B(a, b)pdadbdc$$
$$y(t, 0) = \int B(a, b)pdadbdc$$
$$p(t, a, b, 0) = \phi(x(\cdot), y(\cdot))(a, b)$$

What is ϕ?

posssible choice:

$$\phi(x(\cdot), y(\cdot))(a, b) = \frac{\rho(a, b)x(a)y(b)}{\int x(a)da + \int y(b)db}$$

generalized harmonic mean

Existence of exponential two-sex solution
Prüss, Zacher 2001

Application to sexually transmitted disease:

A faithful pair is immune

Dietz and KPH 1988

Jacquez, Koopman et al., ...

Pair formation in a constant population

$$x + p = \bar{x}, \quad y + p = \bar{y}$$

scalar equation

$$\dot{p} = -\sigma p + \phi(\bar{x} - p, \bar{y} - y)$$

Convergence to equilibrium

Now we generalize this concept.

Multitype (one-sex) model

$x = (x_i)$ vector of (single) types

$Y = (y_{ij})$ symmetric matrix of pairs

Dynamic pair formation

$$\begin{aligned} \dot{Y} &= \frac{XQX}{e^T x} - C * Y \\ \dot{x} &= (C * Y)e - \frac{XQx}{e^T x} \end{aligned}$$

$*$ Hadamard product

Invariant of motion

$$x + Ye = \bar{x}$$

The one-type two-sex model (with harmonic mean) is a special case:

$$Q = \begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \sigma & \sigma \\ \sigma & \sigma \end{pmatrix}$$

x_1 female, x_2 male, $y_{12} + y_{21}$ pair

should be true in general

Results:

For each $\bar{x} > 0$ there is at least one equilibrium.

If $\bar{x}$ has zeros then boundary equilibrium. Altogether $2^n - 1$ equilibria.

$n = 2$: unique and globally stable in the interior.

The case without separation: $C = 0$

$$\dot{Y} = \frac{XQX}{e^T x}, \quad \dot{x} = -\frac{XQx}{e^T x}$$

$$Y(0) = 0, \quad x(0) = \bar{x}$$

For $t \to \infty$: $Y(t) \to A$, $x(t) \to 0.$

$$Ae = \bar{x}$$

Equivalent formulation

$$\dot{Y} = \frac{XQX}{e^T x}$$

$$\dot{x} = \bar{x} - x - \frac{XQx}{e^T x}$$

open problem: formula connecting A to Q?

Complete pair formation

as opposed to dynamic pair formation

Assume $A = A^T$ with $Ae = \bar{x}$ is given. Each individual must form a pair. No individual can be in two different pairs.

Problem: find all suitable matrices A, find explicit formulas, make biologically relevant choices

Normalized problem of complete pair formation

Assume that $\bar{x} = p$ is normalized by $e^T p = 1$.

Complete pair formation is nothing else than a symmetric matrix $A \geq 0$ with the property

$$Ae = p,$$

Question: Do we want all matrices A for a given p or do we want a matrix function $p \mapsto A(p)$ with certain additional properties?

The representation formula

For given p the set of all A is a compact convex polyhedron in matrix space.

The formula of Busenberg and Castillo-Chavez 1991, Blythe et al.

$$A(p) = P\left(\Phi + \frac{(e - \Phi p)(e^T - p^T\Phi)}{1 - p^T\Phi p}\right)P$$

where Φ is any symmetric matrix with

$$0 \leq \Phi < ee^T$$

What is the meaning of this formula?

Choosing $\Phi = 0$ gives "random pair formation"

$$A = pp^T$$

Try to understand the formula in biological terms!

Ben Morin 2010

Substochastic matrices

$\Psi = (\psi_{ij})$

not necessarily symmetric

$\psi_{ij} \geq 0$, $\sum_i \psi_{ij} \leq 1$

$$\Psi \geq 0, \quad e^T \Psi \leq e^T$$

Complement the matrix to a stochastic matrix

$$\mathcal{S} = \{A \geq \Psi, \quad e^T A = e^T\}$$

Proposition:

The matrices $A \in \mathcal{S}$ can be represented as

$$A = \Psi + X\mathrm{diag}[e - \Psi^T e]$$

where X is an arbitrary (column) stochastic matrix. The elements of $A = (a_{ij})$ are

$$a_{ij} = \psi_{ij} + x_{ij}(1 - \sum_k \psi_{kj}).$$

Probabilistic interpretation:

Given Ψ, find A such that the transition probability from state j to state i respects the lower bound ψ_{ij}.
For each state j the number $1 - \sum_i \psi_{ij}$ is the probability mass not allocated by the matrix Ψ. The column $x^{(j)} = (x_{ij})_{i=1}^{n}$ of the stochastic matrix X is the probability distribution according to which the remaining probability mass of state j is distributed to the states i.

Special choices for the matrix X

1) Allocate all remaining mass for the state j to the state j. Then $X = I$ and

$$A = \Psi + D, \quad D = (d_i \delta_{ij}), \quad d_j = 1 - \sum_i \psi_{ij}$$

2) Use the same distribution $x \in \Delta$ for all states j. Then

$$A = \Psi + x(e^T - e^T \Psi)$$

The perturbation has rank 1 unless Ψ is already stochastic.

The family (depending on $x \in \Delta$) of stochastic matrices

$$A = \Psi + x(e^T - e^T\Psi)$$

is distinguished by a **probabilistic** property

Additional requirement

Can the matrix Ψ be complemented to a stochastic matrix A with given stationary distribution p?

$$\mathcal{S}_p = \{A \geq \Psi : \; e^T A = e^T, \; Ap = p\}$$

Assume that Ψ is not stochastic and does not have p as an eigenvector.

Proposition: Suppose $p \in \Delta$, $e^T\Psi \leq e^T$, $e^T\Psi \neq e^T$, $\Psi p \leq p$, $\Psi p \neq p$. Then

$$A = \Psi + \frac{(p - \Psi p)(e^T - e^T\Psi)}{1 - e^T\Psi p} \in \mathcal{S}_p.$$

Other elements of $\mathcal{S}_p$?

A theory of preferences

n groups with sizes p_i such that $\sum_i p_i = 1$.
An individual in group i has preference
$\phi_{ij} \geq 0$ for j.
ϕ_{ij} are not necessarily symmetric.
Assume $\phi_{ij} < 1$.
The product $p_i \phi_{ij}$ is the total preference in group i for group j. Can we find a stochastic matrix A that respects the preferences and also respects the group size distribution?

matrix formulation

$\Phi = (\phi_{ij}) \geq 0$ matrix of preferences

$$\Phi \geq 0, \quad \Phi < ee^T$$

$p = (p_i)$ group sizes, and $P = (p_i \delta_{ij})$.
The matrix $P\Phi$ is substochastic,

$$e^T P\Phi < e^T, \quad P\Phi p \leq p, \quad P\Phi p \neq p$$

Proposition: $P\Phi$ can be complemented to a stochastic matrix $A \geq P\Phi$ such that $Ap = p$.
There is a rank 1 perturbation of $P\Phi$ with this property,

$$A = P\Phi + \frac{(p - P\Phi p)(e^T - p^T\Phi)}{1 - p^T\Phi p}$$

Compare to $A = pe^T$!

Pair formation

(Szenario of B + C-C 1991)

n groups with sizes p_i such that $\sum_i p_i = 1$. An individual in group i makes c_i contacts per time unit, and $\gamma_{ij} \geq 0$ is the probability that such contact is with an individual of group j, hence $\sum_j \gamma_{ij} = 1$. The total number of contacts of individuals from group i with individuals from group j is $p_i c_i \gamma_{ij}$.

Assume that contacts are symmetric! Need a balance law

$$p_i c_i \gamma_{ij} = p_j c_j \gamma_{ji}$$

Introduce matrices $\Gamma = (\gamma_{ij})$, $C = (c_i \delta_{ij})$, $P = (p_i \delta_{ij})$.

Required properties

$$PC\Gamma = \Gamma^T CP, \quad \Gamma \geq 0, \quad \Gamma e = e.$$

The case $C = I$:

Find row stochastic matrices Γ such that

$$P\Gamma = \Gamma^T P$$

The set of these matrices is a compact and convex polyhedron in matrix space.
Find such matrices in the form $\Gamma = MP$.
Then the conditions on M become
$PMP = PM^T P$, $MP \geq 0$, $Mp = e$. These conditions are satisfied if

$$M = M^T, \quad M \geq 0, \quad Mp = e.$$

Pair formation preferences

$$\Phi < ee^T$$

Look for M with

$$M = M^T, \quad M \geq \Phi, \quad Mp = e,$$

One candidate is

$$M = \Phi + \frac{(e - \Phi p)(e^T - p^T \Phi)}{1 - p^T \Phi p}$$

while ee^T is another.

Proposition:

Suppose a symmetric preference matrix $\Phi \geq 0$, $\Phi < ee^T$ is given and a vector of group sizes $p \in \Delta$. Then the matrix

$$\Gamma = \left(\Phi + \frac{(e - \Phi p)(e^T - p^T \Phi)}{1 - p^T \Phi p} \right) P.$$

satisfies the conditions $\Gamma \geq \Phi P$, $P\Gamma = \Gamma^T P$, $\Gamma e = e$.

This Γ is one candidate. Another is ep^T.

The pair formation problem is the special case of the preference problem when the matrix Φ is symmetric.

Multitype epidemic model

n types or social groups

x susceptible, y infected, z recovered

$$x + y + z = p \in \Delta$$

$$\begin{aligned}
\dot{x} &= -XBy \\
\dot{y} &= XBy - Dy \\
\dot{z} &= Dy
\end{aligned}$$

$$x(0) = p, \quad y(0) \approx 0, \quad z(0) = 0$$

Basic reproduction number

,

$$R(p) = \rho(PBD^{-1}), \quad B = B(p)$$

Invariant of motion

$$BD^{-1}y + BD^{-1}x - \log x$$

Final size equation

$$BD^{-1}x - \log x = BD^{-1}p - \log p$$

Unique solution $\bar{x}$.

We take the view that B and D are given while p is subject to variation.
What is the worst possible case for $R(p)$?

$$R_{\max} = \max_{p \in \Delta} \rho(PA)$$

Connection to max algebra. Ongoing work with L. Elsner.

Different strains

$I_1.I_2$ normal strain, untreated and treated
I_3, I_4 resistant strain, untreated and treated

$$\begin{aligned}
\dot{S} &= \mu - S(\beta_1 I_1 + \beta_2 I_2 + \beta_3 I_3 + \beta_3 I_4) - \mu S \\
\dot{I}_1 &= S(\beta_1 I_1 + \beta_2 I_2) - \alpha_1 I_1 - \mu I_1 - \kappa I_1 \\
\dot{I}_2 &= \kappa I_1 - \alpha_2 I_2 - \mu I_2 \\
\dot{I}_3 &= S(\beta_3 I_3 + \beta_4 I_4) - \alpha_3 I_3 - \mu I_3 - \kappa I_3 \\
\dot{I}_4 &= \kappa I_3 - \alpha_4 I_4 - \mu I_4 \\
\dot{R} &= \alpha I_1 + \alpha_2 I_2 + \alpha_3 I_3 + \alpha_4 I_4 - \mu R
\end{aligned}$$

Kermack-McKendrick model + competitive exclusion model

$$R_k = \frac{\beta_k}{\alpha_k + \mu}, \quad k = 1, 2, 3, 4$$

$$R_1 > 1 > R_2$$

$$R_3 < R_1$$

$$R_4 > R_3, \quad R_4 > 1$$

I there a "window" for κ such that both strains are eliminated?

Maybe, if $R_3 < 1$