Asymptotic Stability of the Toda *m*-solitons; Workshop on the Short Pulse Equation

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Abstract

We prove that multi-soliton solutions of the Toda lattice are both linearly and nonlinearly stable. Our proof uses neither the inverse spectral method nor the Lax pair of the model but instead studies the linearization of the Bäcklund transformation which links the (m-1)-soliton solution to the m-soliton solution. We use this to construct a conjugation between the Toda flow linearized about an m-solition solution and the Toda flow linearized about the zero solution, whose stability properties can be determined by explicit calculation.

This is joint work with Nick Benes of the Boston University and Aaron Hoffman of Olin College.

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Introduction

I want to describe a general method for studying the stability of pulse-like solutions in dispersive equations:

1. Has been used mostly in integrable systems with a Bäcklund transform which relates different soliton like solutions either to each other, or to the zero state (e.g. KdV, Toda, NLS, ...)
2. Can sometimes serve as the basis for a perturbative argument for stability in nearly integrable systems which do not admit Bäcklund transforms (e.g. the FPU model).

History

1. Ideas similar to those I'll describe have been used non-rigorously for some time.
2. First reference I have found is E. Mann; *The perturbed Korteweg-de Vries equation considered anew*, J. Math. Phys. **38** p. 3772 (1997), who studies perturbations of the KdV equation and uses the Bäcklund transform to help compute the Green's function for the KdV equation linearized about the 1-soliton.
3. Closer in spirit to our work is the paper of Tsigaridis, et al, *Evolution of near-soliton initial conditions...*, Chaos, Solitons and Fractals, **23** p. 1841 (2005).

History

G. Tsigaridas et al. / Chaos, Solitons and Fractals 23 (2005) 1841–1854

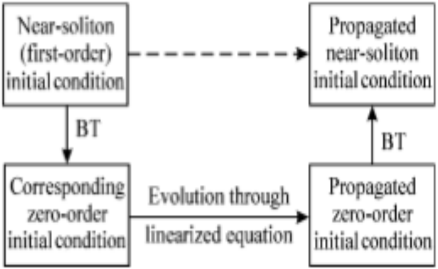

Fig. 1. The outline of the proposed technique.

Overview

1. The first rigorous use of this idea (that I'm aware of) was my Merle and Vega, *L^2 stability of solitons for the KdV equation*, IMRN p. 735 (2003), who used a Miura transformation to establish the stability of the KdV soliton by mapping to a kink solution for the mKdV equation (whose stability was already known).
2. Mizumachi and Pego *Asymptotic stability of Toda lattice solitons*, Nonlinearity, **21**, p. 2099 (2008), used this idea in the Toda model.
3. We'll apply a rigorous version of this same idea to compute the stability of the m-soliton solution of the Toda model.

The Toda Model

The Toda model is model of a system of point masses, which interact with their nearest neighbors through an exponential force law:

$$\ddot{Q}_j = e^{-(Q_j - Q_{j-1})} - e^{-(Q_{j+1} - Q_j)}$$

The Toda Model

1. Was first investigated in an attempt to better understand the Fermi-Pasta-Ulam experiments.
2. Was found to be completely integrable.
3. In particular, there exist m-soliton solutions for any $m = 1, 2, 3, \ldots$.
4. These soliton solutions can be constructed via a Bäcklund transformation.

Soliton Solutions

1. The one-soliton solution has the explicit form:

$$Q_n^1(t;\kappa,\gamma) = \log \frac{\cosh(\kappa n - t\sinh\kappa + \gamma)}{\cosh(\kappa(n+1) - t\sinh\kappa + \gamma)} - \kappa$$

2. Note the following important properties:
   (a) There is a two-parameter family of such solutions parameterized by γ and κ.
   (b) $\lim_{n\to-\infty} Q_n^1(t;\kappa,\gamma) = 0$.
   (c) $\lim_{n\to\infty} Q_n^1(t;\kappa,\gamma) = -2\kappa$.

m-solitons

1. There are also families of m-soliton solutions characterized by $2m$ parameters, γ_k, κ_k, $k = 1, \ldots, m$.
2. As time goes to infinity (or minus infinity) these solutions "decompose" into a sum of m, one-soliton solutions.

$$\lim_{t \to \pm\infty} \left| Q_n^m(t) - \sum_{j=1}^{m} Q_n^1(t; \kappa_j, \gamma_j^{\pm}) \right| = 0$$

The Hamiltonian formalism

Instead of writing the equations of motion in terms of the displacements Q_n, we can use the relative displacements $R_n = Q_{n+1} - R_n$. If we then set $U = (R, P)$, we can rewrite the Toda equations as the Hamiltonian system

$$\dot{U} = JH'(U)$$

where

$$H = \sum_n \left(\frac{1}{2}P_n^2 + V(R_n)\right)$$

with potential energy function $V(R) = e^{-R} - 1 + R$, and the symplectic operator

$$J = \begin{pmatrix} 0 & S - I \\ I - S^{-1} & 0 \end{pmatrix}$$

Function Spaces and Stability

If we are to have stability in this Hamiltonian system we must choose the function spaces in which we work appropriately.

1. One the basis of numerical experiments, general solutions resolve themselves into finitely many solitons, plus a small, dispersive tail that trails behind the solitary waves, which become increasingly well separated and distinct.
2. All solitary waves (or components of an m-soliton) travel with speed greater than $c = 1$.
3. Small dispersive disturbances propagate according to the linearized equation and travel with a speed less than or equal to one.

Function Spaces and Stability

This suggests that we work in weighted sequence spaces which concentrate the norm on parts of the solution moving with the solitons and with this in mind we define the weighted norm

$$\|x\|_a = \|x\|^2_{\ell^2_a} = \sum_{n \in \mathbf{Z}} e^{2an} x_n^2$$

and let ℓ^2_a denote the associated weighted Hilbert Space. Note that we will use $\|x\|$ to denote the standard ℓ^2 norm.

Zero eigenvalues

Can we expect solitons to be asymptotically stable in these spaces w.r.t. all perturbations? No - there are zero eigenvalues associated motion along the $2m$-dimensional manifold of m-solitons corresponding to changes in amplitude and phase.
However, we can compute an explicit basis for the zero eigenspace:

1. $\partial_{\gamma_i} U^m$ - $i = 1, \ldots, m$, eigenfunction corresponding to changes in phase.
2. $\partial_{\kappa_i} U^m$ - $i = 1, \ldots, m$, generalized eigenfunction corresponding to change in amplitude.

Function Spaces

With this in mind, we expect stability (if at all) in spaces of the following type:

$$X_m(t) := \{u \in \ell^2_a \times \ell^2_a | \langle u, J^{-1}\partial_{\kappa_i} U^m(t)\rangle = \langle u, J^{-1}\partial_{\gamma_i} U^m(t)\rangle = 0,\ i = 1, \dots, m\}.$$

Our main result is that in such spaces, the *m*-soliton solution of the Toda-equation is actually stable.

Theorem

Let $\gamma_i \in \mathbb{R}$ and $\kappa_i > 0$ be given for $i = 1, \cdots, m$. Define $\kappa_{min} = \min\{\kappa_i | i = 1, \ldots, m\}$. Let $c > 1$, let $a \in (0, 2\kappa_{min})$, and let $\beta := ca - 2\sinh(a/2)$. Let $\Phi(t, x)$ be the evolution operator for the linearization of the Toda equations about an m-soliton solution with parameters κ_i and γ_i, $i = 1, \ldots, m$.
Then there exists a constant $K > 0$ such that for any $u_0 \in X_m(s)$ and for all $t \geqslant s$,

$$||e^{a(n-ct-T)}\Phi(t,s)u_0|| \leqslant Ke^{-\beta(t-s)}||e^{a(n-cs-T)}u_0||.$$

Remarks about the Theorem

1. The condition $c > 1$ insures that the weight in the norm moves with a speed greater than that of dispersive tails in the solution. The conditions on a and β insure that it moves more slowly than the slowest soliton in the m-soliton solution.
2. Given this theorem it is relatively easy to show that the m-soliton solutions are nonlinearly stable by using a variation of constants approach, and allowing the phase and speed of the solitons to vary in such a way to insure that the orthogonality conditions are satisfied at all times.

The Bäcklund Transform

The proof is based on the Bäcklund Transform:

$$P + e^{-(Q'-Q-\kappa_m)} + e^{-(Q-Q'_{-}+\kappa_m)} = 2\cosh\kappa_m$$
$$P' + e^{-(Q'-Q-\kappa_m)} + e^{-(Q_{+}-Q'+\kappa_m)} = 2\cosh\kappa_m.$$

One can show that it (P,Q) is a solution of the Toda-equation, and if (P',Q') is related to (P,Q) via the Bäcklund transform, then (P',Q') is also a solution of the Toda-equation.
To crucial special cases of this are:

1. If (P,Q) is an m-soliton ($m > 1$) then (P',Q') is an $m-1$-soliton.
2. If (P,Q) is a 1-soliton, then (P',Q') is the zero solution.

Outline of the proof:

There are three main steps in the proof:

(i) that the linearized Bäcklund transformation commutes with the linearized Toda flow,

(ii) that the linearized Bäcklund transformation and the linearized Toda flow preserves orthogonality with the neutral modes of the linearized Toda system, and

(iii) that the linearized Bäcklund transformation is an isomorphism between the spaces $X_{m-1}(t)$ and $X_m(t)$ defined above.

Outline of the proof:

Pictorially, we want to show that the following diagram commutes:

$$\begin{array}{ccc}
u_m(s) \in X_m(s) & \xrightarrow[\Phi_m(t,s)]{} & u_m(t) \in X_m(t) \\
\uparrow {\scriptstyle B_m(s)} & & \uparrow {\scriptstyle B_m(t)} \\
u_{m-1}(s) \in X_{m-1}(s) & \xrightarrow[\Phi_{m-1}(t,s)]{} & u_{m-1}(t) \in X_{m-1}(t)
\end{array}$$

Figure: Commuting diagram used in the induction step

Linearized BT commutes with linearized Toda Flow

From now on, (Q', P') denotes the $m-1$-soliton and (Q, P) denotes the n-soliton.
Linearize the BT:

$$\begin{aligned} p_n + (\alpha_n - \beta_n)q_n + (\beta_n q'_{n-1} - \alpha_n q'_n) &= 0 \\ p'_n + (\alpha_n q_n - \beta_{n+1} q_{n+1}) + (\beta_{n+1} - \alpha_n) q'_n &= 0 \,, \end{aligned}$$

where we have introduced the new variables:

$$\alpha_n := e^{-(Q'_n - Q_n - \kappa_m)} \,, \quad \beta_n := e^{-(Q_n - Q'_{n-1} + \kappa_m)}$$

Linearized BT commutes with linearized Toda Flow

The fact that the linearized BT commutes with the linearized TF can now be proven by writing out the evolution equation for the difference between the $(q'(t), p'(t))$ and $(q(t), p(t))$ and showing these satisfy a linear evolution equation - hence if their difference is originally zero, it will remain so for all time.

Linearize BT and linearized TF preserve orthogonality

This insures that the linearized BT maps from and to the functions spaces we have defined, and that the linearized TF preserves these spaces. The proof consists of two steps, both of which are straightforward computations:

1. First rewrite the orthogonality conditions for $u = (q, p)$ in terms of the components: i.e. if ∂U represents the derivative of U w.r.t. either κ_i or γ_i, we show that

$$\langle u, J\partial U\rangle = \langle p, \partial Q\rangle - \langle q, \partial P\rangle$$

2. Next show, using the linearized BT that

$$\langle p, \partial Q\rangle = \langle q', \partial P'\rangle - \langle p', \partial Q'\rangle + \langle q, \partial P\rangle$$

which by the preceding remark is equivalent to $\langle u', J\partial U'\rangle = -\langle u, J\partial U\rangle = 0$

Linearize BT and linearized TF preserve orthogonality

To see how the last point is proven, note that we can rewrite the linearized BT as:

$$
\begin{aligned}
Cq' &= Lq + p \\
p' &= \hat{C}q + Mq'.
\end{aligned}
$$

where the linear operators $C = \alpha - \beta S^{-1}$, $\hat{C} = \alpha - S\beta = \alpha - \beta_+ S$, $L = \alpha - \beta$, and $M = \alpha - \beta_+$. (Here the subscript " $+$ " means that the index on that term is shifted by 1 -i.e. $n \to n+1$, which a similar convention for a " $-$ " subscript.

Linearize BT and linearized TF preserve orthogonality

Then

$$\begin{aligned}
\langle p, \partial Q \rangle &= \langle Cq' - Lq, \partial Q \rangle \\
&= \langle q', \hat{C}\partial Q \rangle - \langle q, L\partial Q \rangle \\
&= \langle q', \partial P' - M\partial Q' \rangle - \langle q, C\partial Q' - \partial P \rangle \\
&= \langle q', \partial P' \rangle - (\langle q', M\partial Q' \rangle + \langle q, C\partial Q' \rangle) + \langle q, \partial P \rangle \\
&= \langle q', \partial P' \rangle - \langle p', \partial Q' \rangle + \langle q, \partial P \rangle \qquad (1) \\
& \qquad (2)
\end{aligned}$$

A similar calculation shows that the linearized TF preserves the orthogonality conditions.

The linearized BT is an isomorphism

The final (and hardest) step in the proof is to show that the linearized BT is an isomorphism. Basically, we must show two things:

1. Given (q', p'), solve $\hat{C}q = p' - Mq'$ for q and then set $p = Cq' - Lq$, and
2. given (q, p), solve $Cq' = Lq + p$ for q' and then let $p' = \hat{C}q + Mq'$.

Obviously, we need to understand the solvability conditions associated with the operators C and $\hat{C}$.

The linearized BT is an isomorphism

These two operators just define first order linear difference equations and by more or less explicit computation of their solutions, and using the formulas for the m-solitons.

For example, one finds that if q and q are in ℓ^2_a, then $(Lq+p) \in ran(C)$ if and only if $\langle u, J^{-1}\partial_{\gamma_m} U\rangle = 0$.

In like fashion, the fact that the inverse transformation is well defined depends on the second orthogonality condition, $\langle u, J^{-1}\partial_{\kappa_m} U\rangle = 0$.

These calculations also show that the linearized BT and its inverse are uniformly bounded in time.

Outline of the proof (redux):

The three steps in the proof:

(i) that the linearized Bäcklund transformation commutes with the linearized Toda flow,

(ii) that the linearized Bäcklund transformation and the linearized Toda flow preserves orthogonality with the neutral modes of the linearized Toda system, and

(iii) that the linearized Bäcklund transformation is an isomorphism between the spaces $X_{m-1}(t)$ and $X_m(t)$ defined above.

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\begin{array}{ccc}
u_m(s) \in X_m(s) & \xrightarrow[\Phi_m(t,s)]{} & u_m(t) \in X_m(t) \\
\uparrow {\scriptstyle B_m(s)} & & \uparrow {\scriptstyle B_m(t)} \\
u_{m-1}(s) \in X_{m-1}(s) & \xrightarrow[\Phi_{m-1}(t,s)]{} & u_{m-1}(t) \in X_{m-1}(t)
\end{array}
$$

Linearized BT as a general strategy for proving stability

There are by now a number of examples showing that the linearized BT transforms provide a powerful tool to prove stability of localized traveling waves in dispersive systems.

1. The Toda model: Mizumachi & Pego; Benes, Hoffman & Wayne
2. The KdV and gKdV equation: Merle & Vega; Mizumachi
3. The gKP-II equation: Mizumzchi & Tzvetkov
4. The NLS equation: Mizumachi & Pelinovsky
5. The FPU model: Hoffman & Wayne; Mizumachi

Future work (and a connection to this workshop!)

Hoffman and I are currently studying the stability of kink type solutions in the Sine-Gordon equation (which are linked to the zero solution via an auto-Bäcklund transformation) and plan to use those results to then study the stability of the soliton solutions of the short pulse equation which can be obtained from the Sine-Gordon kinks via a second Bäcklund transform.