Integrable evolution equations on spaces of tensor densities: Hamiltonian and Lagrangian approches

Feride Tiglay

Fields Institute

1. Hamiltonian approach
   - A family of equations on *b* densities
   - μDP: Lax pair and bihamiltonian structure
   - Cauchy problem
   - μB: Lax pair and bihamiltonian structure
   - μBurgers' equation and the L^2-geometry of $\mathrm{Diff}^s \to \mathrm{Diff}^s/S^1$
2. Lagrangian Approach
   - Euler-Poincaré equations and orbit invariants
   - Generalized Euler-Poincaré equations
   - Euler-Poincaré equations on homogeneous spaces
   - Generalized Euler-Poincaré equations on homogeneous spaces
3. Applications

- J. Lenells, G. Misiołek and F. T., *Integrable evolution equations on spaces of tensor densities and their peakon solutions*, Commun. Math. Phys. 299, 129–161 (2010).

- F. T. and C. Vizman, *Generalized Euler-Poincaré Equations on Lie Groups and Homogeneous Spaces, Orbit Invariants and Applications*, Lett. Math. Phys., published online Feb. 6, 2011.

V. Arnold's approach

G Lie group 'configuration space'

$\mathfrak{g}$ its Lie algebra with an inner product $\langle\ ,\ \rangle$ 'kinetic energy'

$\langle\ ,\ \rangle$ equippes G with a right invariant metric and the motions of the system can be studied through:

- geodesic equations defined by the right invariant metric, or
- Hamiltonian reduction on the Lie algebra $\mathfrak{g}$.

$(\cdot,\cdot)$ a natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$
$A : \mathfrak{g} \to \mathfrak{g}^*$ the associated inertia operator s.t. $(Au, v) = \langle u, v\rangle$.
The Euler equation on $\mathfrak{g}^*$:

$$m_t = -\mathrm{ad}^*_{A^{-1}m} m, \qquad m = Au \in \mathfrak{g}^*, \tag{E}$$

$\mathrm{Diff}(S^1)$ and density modules—quadratic case

$G := \mathrm{Diff}(S^1)$, $\mathfrak{g} = \mathrm{vect}(S^1)$
"regular part" of the dual $\mathfrak{g}_r^* \simeq \mathcal{F}_2 = \{m(x)dx^2 : m \in C^\infty(S^1)\}$
with the pairing $(mdx^2, v\partial_x) = \int_0^1 m(x)v(x)\,dx$

$$\boxed{\begin{array}{c}\text{the coadjoint representation of}\\ \mathrm{vect}(S^1)\text{ on the regular part}\\ \text{of its dual space}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{the action of }\mathrm{vect}(S^1)\\ \text{on the space of}\\ \text{quadratic differentials}\end{array}}$$

$$\mathrm{ad}^*_{u\partial_x} mdx^2 = (um_x + 2u_x m)\,dx^2$$

and the Euler equation (E) on $\mathfrak{g}_r^*$ is

$$m_t = -\mathrm{ad}^*_{A^{-1}m} m = -um_x - 2u_x m, \qquad m = Au. \qquad (1)$$

On $\mathfrak{g}_r^*$:

$$m_t = -um_x - 2u_x m, \qquad m = Au. \tag{2}$$

On $\mathrm{vect}(S^1)$:

$$Au_t + 2u_x Au + uAu_x = 0 \tag{3}$$

with inertia operator

$$A = \begin{cases} 1 - \partial_x^2 & \text{for CH,} \\ \mu - \partial_x^2 & \text{for } \mu\text{CH,} \\ -\partial_x^2 & \text{for HS.} \end{cases} \tag{4}$$

$\mathrm{Diff}(S^1)$ and density modules—general set-up

- replace $\mathcal{F}_2$ by $\mathcal{F}_b$
  - a tensor density of weight $b \geq 0$ (respectively $b < 0$) on S^1 is a section of the bundle $\bigotimes^b T^*S^1$ (respectively $\bigotimes^{-b} TS^1$)

$$\mathcal{F}_b = \left\{ m(x)dx^b : m(x) \in C^\infty(S^1) \right\}.$$

- action of $\mathrm{Diff}(S^1)$ on each density module $\mathcal{F}_b$ is given by

$$\mathcal{F}_b \ni mdx^b \to m \circ \xi \, (\partial_x \xi)^b dx^b \in \mathcal{F}_b, \qquad \xi \in \mathrm{Diff}(S^1), \quad (5)$$

which generalizes $\mathrm{Ad}^* : \mathrm{Diff}(S^1) \to \mathrm{Aut}(\mathcal{F}_2)$

- the infinitesimal generator of the action in (5)

$$L^b_{u\partial_x}(mdx^b) = (um_x + bu_x m)\, dx^b \tag{6}$$

determines the action of $\mathrm{vect}(S^1)$ on $\mathcal{F}_b$

- the equation for the flow of the vector field defined by (6) is

$$m_t = -um_x - bu_x m \tag{7}$$

- substituting $m = Au$ transforms (7) into

$$Au_t + bu_x Au + uAu_x = 0 \tag{8}$$

- for $b = 3$ the inertia operator is

$$A = \begin{cases} 1 - \partial_x^2 & \text{for DP,} \\ \mu - \partial_x^2 & \text{for } \mu\text{DP,} \\ -\partial_x^2 & \text{for } \mu\text{B.} \end{cases} \tag{9}$$

μDP—Lax pair

$$\begin{cases} \psi_{xxx} = -\lambda m \psi, \\ \psi_t = -\frac{1}{\lambda}\psi_{xx} - u\psi_x + u_x\psi, \end{cases} \tag{10}$$

- $\lambda \in \mathbb{C}$ is a spectral parameter,
- $\psi(t,x)$ is a scalar eigenfunction and
- $m = \mu(u) - u_{xx}$

for

$$\mu(u_t) - u_{txx} + 3\mu(u)u_x - 3u_x u_{xx} - uu_{xxx} = 0. \qquad (\mu\mathrm{DP})$$

μDP—Bihamiltonian structure

$$m_t = J_0 \frac{\delta H_0}{\delta m} = J_2 \frac{\delta H_2}{\delta m},$$

with Hamiltonian functionals

$$H_0 = -\frac{9}{2}\int m\,dx \quad \text{and} \quad H_2 = -\int \left(\frac{3}{2}\mu(u)\left(A^{-1}\partial_x u\right)^2 + \frac{1}{6}u^3\right) dx$$

the operators J_0 and J_2 are given by

$$J_0 = -m^{2/3}\partial_x m^{1/3}\partial_x^{-3} m^{1/3}\partial_x m^{2/3} \qquad \text{and} \qquad J_2 = -\partial_x^3 A = \partial_x^5$$

and $m = \mu(u) - u_{xx}$.

μDP—Periodic Cauchy problem

$$\Lambda_\mu^{-2} v(x) = \left(\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12}\right) \int_0^1 v(x)\, dx + \left(x - \frac{1}{2}\right) \int_0^1 \int_0^x v(y)\, dydx - \int_0^x \int_0^y v(z)\, dzdy + \int_0^1 \int_0^x \int_0^y v(z)\, dzdydx.$$

is the inverse of the elliptic operator

$$\Lambda_\mu^2 : H^s(S^1) \to H^{s-2}(S^1), \qquad \Lambda_\mu^2 v = \mu(v) - v_{xx}.$$

We use Λ_μ^{-2} to rewrite μDP in the nonlocal form

$$u_t + uu_x + 3\mu(u)\, \partial_x \Lambda_\mu^{-2} u = 0$$

μDP—Local wellposedness and persistence

$$u_t + uu_x + 3\mu(u)\,\partial_x\Lambda_\mu^{-2}u = 0 \quad (11)$$
$$u(0,x) = u_0(x) \quad (12)$$

Theorem (Local wellposedness and persistence)

Assume $s > 3/2$. Then for any $u_0 \in H^s(\mathbb{T})$ there exists a $T > 0$ and a unique solution

$$u \in C\big((-T,T), H^s\big) \cap C^1\big((-T,T), H^{s-1}\big)$$

of the Cauchy problem (26)-(12) which depends continuously on the initial data u_0. Furthermore, the solution persists as long as $\|u(t,\cdot)\|_{C^1}$ stays bounded.

μDP—Blow-up

Theorem

Given any smooth periodic function u_0 with zero mean there exists $T_c > 0$ such that the corresponding solution of the μDP equation stays bounded for $t < T_c$ and satisfies $\|u_x(t)\|_\infty \nearrow \infty$ as $t \nearrow T_c$.

Proof.

$\dot{\xi} = u \circ \xi$ implies $\partial_x \dot{\xi} = (u_x \circ \xi)\, \partial_x \xi$ and setting $w = \partial_x \dot{\xi} / \partial_x \xi$ we find that

$$w(t,x) = \frac{1}{t + \left(1/u_{0x}(x)\right)}.$$

$\square$

μDP—Global existence

Theorem

Let $s > 3$. Assume that $u_0 \in H^s(S^1)$ has non-zero mean and satisfies the condition

$$\Lambda_\mu^2 u_0 \geq 0 \quad (\text{or } \leq 0).$$

Then the Cauchy problem for μDP has a unique global solution u in $C(\mathbb{R}, H^s(S^1)) \cap C^1(\mathbb{R}, H^{s-1}(S^1))$.

μBurgers—Lax pair

$$\begin{cases} \psi_{xxx} = -\lambda m \psi, \\ \psi_t = -\frac{1}{\lambda}\psi_{xx} - u\psi_x + u_x\psi, \end{cases} \tag{13}$$

- $\lambda \in \mathbb{C}$ is a spectral parameter,
- $\psi(t, x)$ is a scalar eigenfunction and
- $m = -u_{xx}$

for

$$-u_{txx} - 3u_x u_{xx} - uu_{xxx} = 0. \tag{μB}$$

μBurgers—Bihamiltonian structure

$$m_t = J_0 \frac{\delta H_0}{\delta m} = J_2 \frac{\delta H_2}{\delta m},$$

with Hamiltonian functionals

$$H_0 = -\frac{9}{2} \int m \, dx \quad \text{and} \quad H_2 = -\frac{1}{6} \int u^3 dx$$

the operators J_0 and J_2 are given by

$$J_0 = -m^{2/3} \partial_x m^{1/3} \partial_x^{-3} m^{1/3} \partial_x m^{2/3} \qquad \text{and} \qquad J_2 = \partial_x^5$$

and $m = -u_{xx}$.

Burgers' equation

$$u_t + uu_x = 0 \tag{B}$$

- $\mathrm{Diff}^s(S^1)$ circle diffeomorphisms of Sobolev class H^s
- L^2 inner product on $T_\eta \mathrm{Diff}^s(S^1)$ induces a weak Riemannian metric on $\mathrm{Diff}^s(S^1)$
- a geodesics $\eta(t)$ in $\mathrm{Diff}^s(S^1)$ satisfy the equation

$$\nabla_{\dot\eta}\dot\eta = \ddot\eta = \left(u_t + uu_x\right) \circ \eta = 0 \tag{14}$$

and hence correspond to (classical) solutions of the Burgers equation. Here $\eta(t)$ is the flow of $u(t,x)$, i.e.

$$\dot\eta(t,x) = u(t, \eta(t,x))$$

L^2-geometry of $\mathrm{Diff}^s \to \mathrm{Diff}^s/S^1$

- homogeneous space $\mathrm{Diff}_0^s = \mathrm{Diff}^s/S^1$ is a smooth Hilbert manifold for $s > 3/2$ with $T_{[e]}\mathrm{Diff}_0^s = H_0^s(S^1)$.
- $\pi : \mathrm{Diff}^s \to \mathrm{Diff}_0^s$ is a Riemannian submersion with each tangent space decomposing as

$$T_\xi \mathrm{Diff}^s = P_\xi(T_\xi \mathrm{Diff}^s) \oplus_{L^2} Q_\xi(T_\xi \mathrm{Diff}^s).$$

- the two orthogonal projections $P_\xi : T_\xi \mathrm{Diff}^s \to T_{\pi(\xi)}\mathrm{Diff}_0^s$ and $Q_\xi : T_\xi \mathrm{Diff}^s \to \mathbb{R}$ are given explicitly by the formulas $P_\xi(W) = W - \int_0^1 W(x)\,dx$ and $Q_\xi(W) = \int_0^1 W(x)\,dx$.
- a curve $\eta(t)$ in $\mathrm{Diff}^s(S^1)$ is an L^2 geodesic (and hence correspond to a solution of Burgers' equation) $\Longleftrightarrow$

$$P_\eta \nabla_{\dot\eta}\dot\eta = Q_\eta \nabla_{\dot\eta}\dot\eta = 0.$$

Theorem

A smooth function $u = u(t,x)$ is a solution of the μB equation

$$u_{txx} + 3u_x u_{xx} + uu_{xxx} = 0$$

if and only if the horizontal component of the acceleration of the associated flow $\eta(t)$ in $\mathrm{Diff}^s(S^1)$ is zero i.e. $P_\eta \nabla_{\dot\eta}\dot\eta = 0$. In fact, given any $u_0 \in H^s(S^1)$ the flow of u has the form $\eta(t,x) = x + t\big(u_0(x) - u_0(0)\big) + \eta(t,0)$ for all sufficiently small t.

- integrating the μB equation twice in x gives

$$u_t + uu_x = \mu(u_t), \tag{15}$$

- integrating $\ddot\eta(t,x) = \int_0^1 \ddot\eta \circ \eta^{-1}(t,x)\,dx = \ddot\eta(t,0)$ twice in t gives the explicit formula for the flow.

Corollary

Suppose that $u(t,x)$ is a smooth solution of the μB equation and let $u(0,x) = u_0(x)$.

1. *The following integrals are conserved by the flow of u*

$$\int_0^1 \big(u - \mu(u)\big)^p dx = \int_0^1 \big(u_0 - \mu(u_0)\big)^p dx, \qquad p = 1, 2, 3 \ldots.$$

2. *There exists $T_c > 0$ such that $\|u_x(t)\|_\infty \nearrow \infty$ as $t \nearrow T_c$.*

Peakons

$$\mu(u_t) - u_{txx} + b\mu(u)u_x = bu_x u_{xx} + uu_{xxx}, \qquad b \in \mathbb{Z} \tag{16}$$

$$b = 2 \Longrightarrow \mu\text{CH and } b = 3 \Longrightarrow \mu\text{DP}.$$

Theorem

For any $c \in \mathbb{R}$ and $b \neq 0, 1$, equation (16) admits the peaked period-one traveling-wave solution $u(t,x) = \varphi(x - ct)$ where

$$\varphi(x) = \frac{c}{26}(12x^2 + 23) \tag{17}$$

for $x \in [-\frac{1}{2}, \frac{1}{2}]$ and φ is extended periodically to the real line.

- one-peakon solutions of (16) are the same for any b,
- they travel with a speed equal to their height.

Multi-peakons

$$m(t,x) = \sum_{i=1}^{N} p_i(t)\delta(x - q^i(t)) \tag{18}$$

Theorem

The multi-peakon (18) satisfies the μ-equation (16) in the nonlocal form in distributional sense if and only if $\{q^i, p_i\}_1^N$ evolve according to

$$\dot{q}^i = u(q^i) \quad , \quad \dot{p}_i = -(b-1)p_i\{u_x(q^i)\} \tag{19}$$

where $\{u_x(q^i)\}$ denotes the regularized value of u_x at q^i defined by $\{u_x(q^i)\} := \sum_{j=1}^N p_j g'(q^i - q^j)$ and the Green function is given by $g(x) = \frac{1}{2}x(x-1) + \frac{13}{12} \quad \text{for} \quad x \in [0,1) \simeq S^1$.

Shock-peakons for μDP

$$u = \sum_{i=1}^{N} \Big(p_i g(x - q^i) + s_i g'(x - q^i) \Big) \tag{20}$$

Theorem

The shock-peakon (20) satisfies μDP in distributional sense if and only if $\{q^i, p_i, s_i\}_1^N$ evolve according to

$$\begin{aligned} \dot{q}^i &= u(q^i), \quad \dot{p}_i = 2\big(s_i\{u_{xx}(q^i)\} - p_i\{u_x(q^i)\}\big), \\ \dot{s}_i &= -s_i\{u_x(q^i)\}, \end{aligned} \tag{21}$$

where
$\{u_x(q^i)\} = \sum_{j=1}^N p_j g'(q^i - q^j) + \sum_{j=1}^N s_j, \qquad \{u_{xx}(q^i)\} = \sum_{j=1}^N p_j.$

Euler-Poincaré equations

G Lie group and $\mathfrak{g}$ its Lie algebra

$L : TG \to \mathbb{R}$ right invariant Lagrangian determined by $l : \mathfrak{g} \to \mathbb{R}$

The Euler-Lagrange equation for L

$$\frac{d}{dt}\frac{\delta l}{\delta u} = -\operatorname{ad}_u^* \frac{\delta l}{\delta u} \qquad \text{(EP)}$$

is called the *right Euler-Poincaré equation*

- $u = \dot{\gamma}\gamma^{-1}$ is the logarithmic derivative of the curve γ in G
- $(\operatorname{ad}_\xi^* \alpha, \eta) = (\alpha, \operatorname{ad}_\xi \eta)$ for $\alpha \in \mathfrak{g}^*$ and $\xi, \eta \in \mathfrak{g}$.

Orbit invariants

The group coadjoint operator Ad_g^* is given by the formula

$$(\mathrm{Ad}_g^* \alpha, \xi) = (\alpha, \mathrm{Ad}_g \xi), \quad \forall \xi \in \mathfrak{g}.$$

Along solutions u of the EP equation the quantity

$$\mathrm{Ad}_\gamma^* m = \mathrm{Ad}_\gamma^* \frac{\delta l}{\delta u} = \text{const.} \tag{22}$$

is conserved for any curve γ in G satisfying $u = \dot\gamma\gamma^{-1}$.

Examples of EP equations

1. Burgers' equation
$$\partial_t u = -3uu' \tag{23}$$
   - $m = \frac{\delta l}{\delta u} = u$
   - orbit invariant $\mathrm{Ad}^*_\gamma u = (u \circ \gamma)(\gamma')^2$ is conserved.
2. Camassa-Holm equation
$$\partial_t u - \partial_t u'' + 3uu' - uu''' - 2u'u'' = 0 \tag{24}$$
   is the geodesic equation on $\mathrm{Diff}(S^1)$ for the right invariant H^1 metric.
   - 'momentum' is $m = \frac{\delta l}{\delta u} = u - u''$
   - orbit invariant $\mathrm{Ad}^*_\gamma m = (m \circ \gamma)(\gamma')^2$ is conserved.

3. μHS equation (introduced by Khesin, Lenells and Misiołek)

$$\partial_t u'' = 2\mu(u)u' - 2u'u'' - uu''', \tag{25}$$

where $\mu(u) = \int_{S^1} u dx$ is the geodesic equation on $\mathrm{Diff}(S^1)$ for the right invariant metric defined by

$$\langle u_1, u_2 \rangle_\mu = \int_{S^1} (\mu(u_1)\mu(u_2) + u_1' u_2') dx = \langle u_1, (\mu - \partial_x^2) u_2 \rangle_{L^2}.$$

- $m = \mu(u) - u''$
- orbit invariant $\mathrm{Ad}_\gamma^* m = (m \circ \gamma)(\gamma')^2$ is conserved.

Generalized Euler-Poincaré equations

A generalized Euler-Poincaré equation associated to a right invariant Lagrangian function $L : TG \to \mathbb{R}$ with value $l : \mathfrak{g} \to \mathbb{R}$ at the identity and to a G–action Θ on $\mathfrak{g}$ is

$$\frac{d}{dt}\frac{\delta l}{\delta u} = -\theta_u^* \frac{\delta l}{\delta u}, \qquad \text{(gEP)}$$

where

- u is the right logarithmic derivative of a curve γ in G,
- θ is the infinitesimal action associated to the (left) group action Θ,
- θ_ξ^* is the adjoint of θ_ξ for $\xi \in \mathfrak{g}$.

For $m = \frac{\delta l}{\delta u}$ the gEP equation $\frac{d}{dt} m = -\theta_u^* m$.

Orbit invariants for gEP

Proposition

The quantity

$$\Theta_\gamma^*(m) = \Theta_\gamma^*\left(\frac{\delta l}{\delta u}\right) = const.$$

is conserved along solutions u of the generalized Euler-Poincaré equation for any curve γ in G satisfying $u = \dot{\gamma}\gamma^{-1}$.

An abstract Noether theorem for gEP

Theorem

Given a G–manifold $\mathcal{C}$ and a G–equivariant map $\kappa : \mathcal{C} \to \mathfrak{g}^{**}$, i.e. it satisfies $\kappa(\gamma \cdot c) = \Theta_\gamma^{**}\kappa(c)$ for all $c \in \mathcal{C}$, with Θ_γ^{**} the adjoint of Θ_γ^{*}, the Kelvin quantity $I(c, u) = \left(\kappa(c), \frac{\delta l}{\delta u}\right)$ defined by κ is conserved for u solution of the generalized right Euler-Poincaré equation, where $c = \gamma \cdot c_0$, $c_0 \in \mathcal{C}$, for γ a curve in G with $u = \gamma'\gamma^{-1}$.

Examples of gEP equation

1. Degasperis-Procesi equation

$$\partial_t u - \partial_t u'' + 4uu' - uu''' - 3u'u'' = 0 \qquad \text{(DP)}$$

- It has a geometric interpretation on the space of tensor densities on the circle: Let Θ be the left action of $\text{Diff}(S^1)$ on $\mathcal{F}_{-2}$ and Θ^* the right action of $\text{Diff}(S^1)$ on $\mathcal{F}_3$ its dual. The corresponding generalized EP equation on $\text{Diff}(S^1)$ for the right invariant H^1 Lagrangian is the DP equation:

$$\partial_t m = -um' - 3u'm, \quad m = u - u''.$$

- orbit invariant is conserved

$$\Theta^*_\gamma(m) = (m \circ \gamma)(\gamma')^3, \quad m = u - u''$$

for $\dot{\gamma}\gamma^{-1} = u$.

Examples of gEP equation

2. μDP equation

$$\mu(\partial_t u) - \partial_t u'' + 3\mu(u)u' - 3u'u'' - uu''' = 0 \qquad (26)$$

- coadjoint action ad_u^* replaced by $\theta_u^* = u\partial_x + 3u'$
- orbit invariant is conserved:

$$\Theta_\gamma^*(m) = (m \circ \gamma)(\gamma')^3, \quad m = \mu(u) - u''$$

  for $\dot{\gamma}\gamma^{-1} = u$.

3. generalized EPDiff equation

$$\partial_t \mathbf{m} + u \cdot \nabla \mathbf{m} + (\nabla u)^\top \cdot \mathbf{m} + (b-1)\mathbf{m}(\operatorname{div} u) = 0. \qquad (27)$$

- In the special case $b = 3$ and $\mathbf{m} = u - \Delta u$ it extends the Degasperis-Procesi equation to higher dimensions.
- the Kelvin quantity $\int_c \frac{1}{f}\mathbf{m}^\flat$ is conserved along generalized EPDiff.

Given a smooth curve $\bar{\gamma} : I = [0,1] \to G/H$, we compare the left logarithmic derivatives of two smooth lifts $\gamma, \gamma_1 : I \to G$ of $\bar{\gamma}$, i.e. $\bar{\gamma} = \pi \circ \gamma = \pi \circ \gamma_1$. There exists a smooth curve $h : I \to H$ such that $\gamma_1 = \gamma h$, hence

$$u_1 = \delta^l \gamma_1 = \delta^l(\gamma h) = h^{-1}\gamma^{-1}(\gamma' h + \gamma h') = \mathrm{Ad}(h^{-1})u + \delta^l h$$

for $u = \delta^l \gamma : I \to \mathfrak{g}$.

We notice that u_1 is obtained from u via a right action of the group element $h \in C^\infty(I, H)$:

$$u \cdot h = \mathrm{Ad}(h^{-1})u + \delta^l h. \tag{28}$$

It is a right action because of the identity $\delta^l(h_1 h_2) = \mathrm{Ad}(h_2^{-1})\delta^l h_1 + \delta^l h_2$.

This means one can define the *left logarithmic derivative* $\bar{\delta}^l$ of a curve $\bar{\gamma}$ in G/H as an orbit under the right action (28) of $C^\infty(I, H)$ on $C^\infty(I, \mathfrak{g})$, namely the orbit $u \cdot C^\infty(I, H)$ of the left logarithmic derivative u of an arbitrary lift $\gamma : I \to G$ of $\bar{\gamma}$, so

$$\bar{\delta}^l : C^\infty(I, G/H) \to C^\infty(I, \mathfrak{g})/C^\infty(I, H), \quad \bar{\delta}^l \bar{\gamma} = \delta^l \gamma \cdot C^\infty(I, H).$$

When the subgroup H is trivial, we recover the ordinary logarithmic derivative δ^l for curves in G.

Proposition

The following are equivalent data:

1. *left G–invariant function $\bar{L}$ on $T(G/H)$;*
2. *right TH–invariant and left G–invariant function L on TG;*
3. $\mathfrak{h}$*–invariant and* $\mathrm{Ad}(H)$*–invariant function l on* $\mathfrak{g}$*;*
4. $\mathrm{Ad}(H)$*–invariant function $\bar{l}$ on* $\mathfrak{g}/\mathfrak{h}$*.*

Theorem

A solution of the Euler-Lagrange equation for a left G–invariant Lagrangian $\bar{L}: T(G/H) \to \mathbb{R}$ is a curve $\bar{\gamma}$ in G/H such that the left logarithmic derivative $u = \gamma^{-1}\dot{\gamma}$ of a lift γ of $\bar{\gamma}$ satisfies the left Euler-Poincaré equation

$$\frac{d}{dt}\frac{\delta l}{\delta u} = \mathrm{ad}_u^* \frac{\delta l}{\delta u}, \tag{29}$$

for l the ($\mathfrak{h}$–invariant and $\mathrm{Ad}(H)$–invariant) restriction of $L = \bar{L} \circ T\pi$ to $\mathfrak{g}$.

Orbit invariants on homogeneous spaces

Proposition

The quantity $\mathrm{Ad}^*_{\gamma^{-1}} \frac{\delta l}{\delta u} \in \mathfrak{g}^*$ *is conserved along the left Euler Poincaré equation* (29) *on* G/H *with* $\mathfrak{h}$*–invariant and* $\mathrm{Ad}(H)$*–invariant Lagrangian function* l *on* $\mathfrak{g}$*, where* γ *is any lift of* $\bar{\gamma}$ *and* $u = \delta^l \gamma$*.*

The independence on the choice of the lift γ is immediate: for $\gamma_1 = \gamma h$,

$$\mathrm{Ad}^*_{\gamma_1^{-1}} \frac{\delta l}{\delta u_1} = \mathrm{Ad}^*_{(\gamma h)^{-1}} \mathrm{Ad}^*_h \frac{\delta l}{\delta u} = \mathrm{Ad}^*_{\gamma^{-1}} \frac{\delta l}{\delta u}.$$

An abstract Noether theorem for homogeneous spaces

Theorem

Considering a G–manifold $\mathcal{C}$ and a map $\kappa : \mathcal{C} \to \mathfrak{g}^{**}$ which is G–equivariant, the Kelvin quantity

$$I : \mathcal{C} \times \mathfrak{g} \to \mathbb{R}, \quad I(c, u) = \left(\kappa(c), \frac{\delta l}{\delta u} \right) \tag{30}$$

is conserved along solutions $\bar{\gamma}$ of the Euler-Lagrange equation on G/H with left invariant Lagrangian $\bar{L}$, namely for γ a curve in G lifting $\bar{\gamma}$, u its left logarithmic derivative and $c = \gamma^{-1} \cdot c_0$, $c_0 \in \mathcal{C}$.

- The H–equivariance of A ensures the $\mathrm{Ad}(H)$–invariance of l and l descends to an $\mathrm{Ad}(H)$–invariant Lagrangian $\overline{l} : \mathfrak{g}/\mathfrak{h} \to \mathbb{R}$.
- $m = \frac{\delta l}{\delta u} = Au$, so the left Euler-Poincaré equation on G/H writes $\frac{d}{dt}Au = \mathrm{ad}_u^*(Au)$. It is the image under the inertia operator A of the Euler equation, the left invariant version of the Euler equation:

$$\frac{d}{dt}u = \mathrm{ad}(u)^\top u.$$

  This can be interpreted as the geodesic equation on G/H for the left invariant Riemannian metric coming from the degenerate inner product $\langle \xi, \eta \rangle = (A\xi, \eta)$ on $\mathfrak{g}$.

Examples of EP equations on homogeneous spaces

1. The Hunter-Saxton equation

$$\partial_t u'' = -2u'u'' - uu''' \tag{31}$$

is a geodesic equation on the homogeneous space $S^1 \setminus \mathrm{Diff}(S^1)$ of right cosets with the right invariant metric defined by $\langle u_1, u_2 \rangle = \int_{S^1} u_1' u_2' dx$ on $\mathfrak{X}(S^1)$.

It fits into our framework above when $A(u) = -u''$. The two conditions are easily verified: the kernel of A is $\mathbb{R}$, the Lie algebra of the subgroup of rigid rotations, and A is S^1–equivariant.

In this case $l(u) = \frac{1}{2}\langle u, u\rangle = \frac{1}{2}\int_{S^1}(u')^2 dx$, so $m = \frac{\delta l}{\delta u} = -u''$ satisfies

$$\partial_t m = -um' - 2u'm,$$

which gives Hunter-Saxton equation.

A conserved quantity for the Hunter-Saxton equation is

$$\mathrm{Ad}^*_\gamma m = -(u'' \circ \gamma)(\gamma')^2,$$

where $\gamma : I \to \mathrm{Diff}(S^1)$ is any lift of the curve $\bar{\gamma}$.

Examples of EP equations on homogeneous spaces

2. Let K be a Lie group with Lie algebra $\mathfrak{k}$ possessing a K–invariant inner product $\langle\ ,\ \rangle_{\mathfrak{k}}$. The Lie algebra of the loop group $LK := C^\infty(S^1, K)$ is the loop algebra $L\mathfrak{k} = C^\infty(S^1, \mathfrak{k})$. The subgroup of constant loops, identified with K, defines the homogeneous space of right cosets $K \setminus LK$.
Each $\mathfrak{k}$–invariant and $\mathrm{Ad}(K)$–invariant Lagrangian l on $L\mathfrak{k}$ determines a right Euler-Poincaré equation on $K \setminus LK$:

$$\partial_t m = [u, m], \quad m = \frac{\delta l}{\delta u}. \tag{32}$$

Here m is a curve in $L\mathfrak{k}$, since the inner product $\langle\ ,\ \rangle_{\mathfrak{k}}$ permits the identification of the regular dual of $L\mathfrak{k}$ with $L\mathfrak{k}$.

- In the special case $K = SO(3)$ and $\dot{H}^{-1}$ Lagrangian

$$l(u) = \frac{1}{2}\int_{S^1} \langle \partial_x^{-1} u, \partial_x^{-1} u\rangle_{\mathfrak{k}} dx = -\frac{1}{2}\int_{S^1} \langle \partial_x^{-2} u, u\rangle_{\mathfrak{k}} dx$$

  we obtain the Landau-Lifschitz equation

$$\partial_t L = L \times L'',$$

  where one identifies the Lie algebras $(\mathfrak{so}(3), [\ ,\ ])$ and $(\mathbb{R}^3, \times)$.
- This equation is equivalent to the vortex filament equation $\partial_t c = c' \times c''$, for $L = c'$ the tangent vector to the filament, a closed arc-parametrized time-dependent curve c in $\mathbb{R}^3$.

Generalized Euler-Poincaré equations on homogeneous spaces

Proposition

Let H be a subgroup of G with Lie algebra $\mathfrak{h}$, and let θ^* be a Lie algebra action of $\mathfrak{g}$ on $\mathfrak{g}^*$. If the map $\theta^* : \mathfrak{g}\times\mathfrak{g}^* \to \mathfrak{g}^*$ is H–equivariant and if the action θ^* restricted to $\mathfrak{h}$ equals the coadjoint action ad^* restricted to $\mathfrak{h}$, then $C^\infty(I, H)$ is a symmetry group of the equation $\frac{d}{dt}\frac{\delta l}{\delta u} = -\theta^*_u \frac{\delta l}{\delta u}$, for the left action $h \cdot u = \mathrm{Ad}(h)u + \delta^r h$.

Example of generalized EP equation on homogeneous spaces

μBurgers equation

$$-\partial_t u'' - 3u'u'' - uu''' = 0 \qquad (33)$$

is a generalized EP equation on homogeneous space $S^1 \setminus \mathrm{Diff}(S^1)$.

- The Lagrangian is given here by the $\dot{H}^1$ inner product:

$$l(u) = \frac{1}{2}\int_{S^1} (u')^2 dx. \qquad (34)$$

- Orbit invariant

$$\Theta^*_\gamma(m) = (m \circ \gamma)(\gamma')^3, \quad m = -u'',$$

  is conserved for any lift $\gamma : I \to \mathrm{Diff}(S^1)$ of the solution curve $\bar{\gamma} : I \to S^1 \setminus \mathrm{Diff}(S^1)$ whose right logarithmic derivative is $\bar{u} = C^\infty(I, S^1) \cdot u$, i.e. $\dot{\gamma} \circ \gamma^{-1} = u$.

Orbit invariants in global existence results

The four integrable equations: CH, μCH, DP and μDP are special cases of the equation

$$\partial_t m = -um' - \lambda u' m, \quad m = \Phi u, \tag{35}$$

where the operator Φ on the space of smooth functions on the circle is either

- a linear differential operator of the form $\sum_{j=0}^{r}(-1)^j \partial_x^{2j}$, or
- the linear operator $\mu - \partial_x^2$, where $\mu(u)$ is the mean of the function u on S^1.

The equation (35) is a generalized EP equation on the group of diffeomorphisms of the circle for the reduced Lagrangian

$$l(u) = \frac{1}{2}\int_{S^1} u\Phi u dx.$$

In this case the $\mathrm{Diff}(S^1)$ action Θ^* is the action on λ–densities on the circle, with associated infinitesimal action $\theta^*_u f = uf' + bu'f$. The coadjoint action is obtained for $b = 2$ and in this special case (35) is the geodesic equation on $\mathrm{Diff}(S^1)$ with respect to the right invariant metric defined by the H^r inner product.

We consider the periodic Cauchy problem for (35):

$$\partial_t u + uu' = -\Phi^{-1}\left([u, \Phi]u' + \lambda u'\Phi u\right), \quad x \in S^1, t \in \mathbb{R}^+ \tag{36}$$

$$u(0, x) = u_0(x) \tag{37}$$

where $\Phi : H^s \to H^{s-r}$, $\Phi = \sum_{j=0}^{r}(-1)^j \partial_x^{2j}$ and λ is an arbitrary real number.

We prove the following global (in time) existence and uniqueness theorem using orbit invariants.

Theorem

Let $s > 2r + \frac{1}{2}$. Assume that the initial data $u_0 \in H^s(S^1)$ satisfies

$$\Phi u_0 \geq 0.$$

Then the Cauchy problem (36)-(37) has a unique global solution u in
$C(\mathbb{R}^+, H^s(S^1)) \cap C^1(\mathbb{R}^+, H^{s-1}(S^1))$.

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