Global well-posedness and small data scattering for the Ostrovsky equation

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Ostrovsky/Ostrovsky-Hunter/Vakhnenko

Consider

$$\left| \begin{array}{l} u_{tx} = u + (u^p)_{xx}, \quad (t,x) \in \mathbf{R}^1_+ \times \mathbf{R}^1 \\ u(0) = f \in H^s(\mathbf{R}^1), \end{array} \right. \tag{1}$$

The case $p = 2$

- Also called reduced Ostrovsky, Ostrovsky-Hunter, Vakhnenko equation, ...
- modeling of small-amplitude long waves in rotating fluids of finite depth
- Ostrovsky'78, Hunter'90, Boyd '05, Morrisson-Parkes-Vakhnenko'99, Vakhnenko-Parkes'02.

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The Short pulse equation

The case $p = 3$

- Short Pulse Equation
- completely integrable, equivalent to sine-Gordon equation, infinite hierarchy of conserved quantities, Sakovich-Sakovich'05,
- explicit analytical solutions of loop and of breather form, Sakovich-Sakovich'06
- wave breaking, Liu-Pelinovsky-Sakovich, 09 (both in periodic and whole line context)

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Local Solutions to (gO)

Schäfer-Wayne'04 showed the following

Theorem

The (gO) equation is locally well-posed in $H^2(\mathbf{R}^1)$.

Question: What is a solution?

Definition

$u \in H^{1/2+}(\mathbf{R}^1)$ is a weak solution, if

$$\int_0^\infty \int_{-\infty}^{\infty} u(t,x)\psi_{tx}(t,x)dxdt + \int_{-\infty}^{\infty} f\psi_x dx = \\ = \int_0^\infty \int_{-\infty}^{\infty} [u\psi + u^p\psi_{xx}]dxdt,$$

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Local solutions - cont.

Definition

We say that the equation (1) is locally well-posed in H^{s_0}, $s_0 \geq 0$, if

1. For $s_1 >> 1$, $f \in H^{s_1}$, there exists $T_0 = T(\|f\|_{H^{s_0}})$ and a classical solution $u \in C[(0, T_0), H^{s_1}) \cap C^1[(0, T_0), H^{s_1-1})$.
2. There exists $s_2 : 0 \leq s_2 \leq s_0$, so that for any $f, g \in H^{s_1}$, there exists $C = C(\|f\|_{H^{s_0}}, \|g\|_{H^{s_0}})$, so that

$$\sup_{0 \leq t \leq T_0} \|u(t) - v(t)\|_{H^{s_2}} \leq C(\|f\|_{H^{s_0}}, \|g\|_{H^{s_0}})\|f - g\|_{H^{s_2}}$$

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Local well-posedness for the gO equation

Theorem

Let $p \geq 2$ be an integer, $s > 3/2$.

Then (gO) is locally well-posed in $H^s(\mathbf{R}^1)$.

In particular, for any $f \in H^s(\mathbf{R}^1)$, there exists a time $0 < T_0 = T_0(\|f\|_{H^s}) \leq \infty$, so that the problem (1) has a unique strong solution in the space $C([0, T_0), H^s(\mathbf{R}^1))$.

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Remarks

- This is a *quasilinear wave equation*. Thus, one does not expect to produce a solution via a fixed point argument.
- Such equations will in general not have Lipschitz dependence on the initial data (which would be one of the consequences of a fixed point iteration procedure).
- In fact, for this problem, we only get

$$\sup_{0\leq t\leq T_0} \|u(t)-v(t)\|_{H^{\frac{1}{2}+}} \leq C(\|f\|_{H^{\frac{3}{2}+}}, \|g\|_{H^{\frac{3}{2}+}})\|f-g\|_{H^{\frac{1}{2}+}}$$

Is that last result sharp?

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Global solutions for the case $p = 3$

Pelinovsky and Sakovich, [2010] have shown that for $p = 3$, the problem has global solutions for small data.

Their approach is based on exploiting the conservation laws and this particular equation relation's to the cubic $1 + 1$ Klein-Gordon model.

Question: Does (some norm of) the solution tend to zero as $t \to \infty$?

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Global well-posedness and scattering for the gO equation, $p \geq 4$

Theorem

Let $p \geq 4$ be an integer. If $\|f\|_{H^5} + \|f\|_{W^{3,1}} < \varepsilon$, the (gO) equation has an unique global solution in $C([0, \infty), H^5(\mathbf{R}^1))$. In addition,

$$\sup_{0<t<\infty} \|u(t)\|_{H^5} < 4\varepsilon$$

and it scatters

$$\|u\|_{L^q_t W^{3/2,r}} \leq C_p \varepsilon.$$

for all $q, r \in (2, \infty) : 1/q + 1/r < 1/2$, Heuristically,

$$\|u(t)\|_{L^r} \sim t^{-(r-2)/(2r)}.$$

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Remarks:

- One could push the p index down to the Strauss exponent

$$p > \frac{3+\sqrt{17}}{2} \sim 3.56..$$

Certainly not down to $p = 3$!

- In doing so, the smoothness requirement for initial data will increase!

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Energy estimate

Lemma

Let u solves

$$u_{tx} = u + \partial_x(F(t,x)u_x) + G(t,x), \quad t > 0 \tag{2}$$

Then, for every $s > 0$,

$$\begin{aligned} I_s'(t) &\leq C_s \|F_x(t,\cdot)\|_{L^\infty} I_s(t) + \\ &+ 2\sqrt{I_s(t)}(\|G(t,\cdot)\|_{H^{s-1}} + C\|u_x\|_{L^\infty_x} \|F(t,\cdot)\|_{\dot{H}^s}), \end{aligned}$$

where $I_s(t) = \|u(t,\cdot)\|^2_{\dot{H}^s}$.

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where $I_s(t) = \|u(t,\cdot)\|^2_{\dot{H}^s}$.

Energy estimate implies l.w.p.

- Proof of the energy estimate is via Littlewood-Paley theory.
- For existence, we construct the solutions as limits of

$$\partial_{tx} u_N = u_N + (p u_{N-1}^{p-1} \partial_x u_N)_x$$

- In order to close the iteration scheme (i.e. the energy estimates should have the same Sobolev norms on L.H.S. and R.H.S.) one needs to control (observe $G = 0$)

$$\|F(u)\|_{H^s} + \|\partial_x [F(u)]\|_{L^\infty} \lesssim \|u\|_{H^s}^{p-1} \quad \text{where } F \sim u^{p-1}$$

This AND Sobolev embedding necessitates $s > 3/2$. Uniqueness is similar (but G appears!)

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This AND Sobolev embedding necessitates $s > 3/2$.
Uniqueness is similar (but G appears!)

Energy estimate implies l.w.p.

- Proof of the energy estimate is via Littlewood-Paley theory.
- For existence, we construct the solutions as limits of

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Energy keeps it small until $\|u\|_{L_T^{p-1} W_x^{1,\infty}} << 1$

Applying the energy estimates to the local solution implies

$$\frac{d}{dt} J_s(t) \leq C_{s,p} J_s(t)(\|u_x(t)\|_{L_x^\infty}^{p-1} + \|u(t)\|_{L_x^\infty}^{p-1}).$$

for $J_s(u) = \|u(t)\|_{H^s}^2$.

By Gronwall's (for all $s \geq 0$!)

$$J_s(T) \leq J_s(0) \exp(c_{s,p} \|u\|_{L_t^{p-1}(0,T) W_x^{1,\infty}}^{p-1})$$

In particular,

$$\|u\|_{L_T^\infty H^s} \leq 3\|f\|_{H^s} \quad (3)$$

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Strichartz/decay estimates

Theorem

$$\left| \begin{array}{ll} u_{tx} = u & (t,x) \in \mathbf{R}^1_+ \times \mathbf{R}^1 \\ u(0,x) = f(x) & \end{array} \right.$$

We have $\hat{u}(t,\xi) = \hat{f}(\xi) \exp(-i\frac{t}{\xi})$

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Recall

Need to control $\|u\|_{L_T^{p-1} W_x^{1,\infty}}$

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We will show

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By the decay estimates

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Note by energy estimates $\|u\|_{L^\infty_t(0,T)H^{\alpha+2+3/2-3/r}} \leq C\|f\|_{H^{\alpha+2+3/2-3/r}}$.

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Recast the last estimate as

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Questions

1. Study the Lipschitzness of the solution map $S(t) : H^s \to H^s$ for any s.
   For example, for $s > 3/2$, what is the biggest $\alpha = \alpha(s)$, so that

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2. Is it true that solutions with small data to (SPE) ($p = 3$) and Ostrovsky-Hunter/Vakhnenko ($p = 2$) will be global and decaying?
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3. The argument here can be improved to powers $p > \frac{3+\sqrt{17}}{2}$
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Questions

1. Study the Lipschitzness of the solution map $S(t) : H^s \to H^s$ for any s.
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Some ideas

Normal forms

Idea: For (SPE), find "good" change variable $u = T(u, u, u) + v$, so that

$$v_{tx} = v + M(u, u, u, u, u),$$

Need spaces X, Y (based on Strichartz/decay estimates), so that

1. $w_{tx} = w + F$ implies $\|w\|_X \lesssim \|F\|_Y$,
2. $$\|T(u, v, w)\|_X \lesssim \|u\|_X \|v\|_X \|w\|_X$$
3. $$\|M(u_1, \ldots, u_5)\|_Y \lesssim \prod_{j=1}^{5} \|u_j\|_X$$

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It can be done for $p = 2$

For

$$u_{tx} = u + \partial_{xx}[u^2],$$

set $u = v + T(u, u)$, where

$$T(u_1, u_2) = \int \sigma(\xi_1, \xi_2)\hat{u_1}(\tau_1, \xi_1)\hat{u_2}(\tau_2, \xi_1)e^{i(\xi_1+\xi_2)x+(\tau_1+\tau_2)t} d\bar{\xi} d\bar{\tau}$$

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Note

$$(\tau_1 + \tau_2)(\xi_1 + \xi_2) + 1 = (\tau_1\xi_1 + 1) + (\tau_2\xi_2 + 1) + (\tau_1\xi_2 + \tau_2\xi_1 - 1$$

$$\tau_1\xi_2 + \tau_2\xi_1 - 1 = (\tau_1\xi_1 + 1)\frac{\xi_2}{\xi_1} + (\tau_2\xi_2 + 1)\frac{\xi_1}{\xi_2} - (\frac{\xi_1}{\xi_2} + \frac{\xi_2}{\xi_1} + 1)$$

So, select σ

$$\sigma(\xi_1, \xi_2) = c\frac{(\xi_1 + \xi_2)^2}{\frac{\xi_1}{\xi_2} + \frac{\xi_2}{\xi_1} + 1}$$

Note that σ is non-singular and Coifmann-Meyer symbol of order 2.

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It can be done for $p = 2$, III

we get

$$\begin{aligned}(\partial_{tx} - 1)T(u,u) &= 2T((\partial_{tx} - 1)u, u) + \partial_{xx}[u^2] + \\ &+ 2T((\partial_{tx} - 1)\partial_x^{-1}u, \partial_x u) = \\ &= 2T(\partial_{xx}[u^2], u) + 2T(\partial_x[u^2], \partial_x u) + \partial_{xx}[u^2].\end{aligned}$$

Thus, $u = v + T(u,u)$ satisfy

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Thank you for your attention.