# Some problems concerning a Quasilinear Schrödinger Equation

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# The Physical Model

The protagonist of the seminar will be the following Quasilinear Schrödinger Equation:

$$i\partial_t \phi(t,x) + \Delta\phi(t,x) + \lambda\phi(t,x)\Delta|\phi(t,x)|^2 + |\phi(t,x)|^{p-1}\phi(t,x) = 0$$

where $i$ is the imaginary unit, $N \geq 1$, $p > 1$, $\lambda > 0$, $(t,x) \in (0,\infty) \times \mathbf{R}^N$ and $\phi : \mathbf{R} \times \mathbf{R}^N \to \mathbf{C}$.

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This equation is more accurate for a lot of physical phenomena than the classical semilinear one ($\lambda = 0$). It appears in different models, such as the superfluid film equation in plasma physics. The Quasilinear term has a stabilizing effect.

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# The Initial Value Problem

There are no satisfactory results for the short time dynamics. The main difficulty which arises in the study of the Cauchy problem is the presence of the quasilinear term, which causes the phenomenon called *loss of derivatives*. In particular there are no local wellposedness results in the natural energy space

$$X_{\mathbf{C}} = \Big\{ u \in H^1(\mathbf{R}^N, \mathbf{C}) : \int_{\mathbf{R}^N} |u|^2 |\nabla |u||^2 dx < \infty \Big\}.$$

and so a Gagliardo-Nirenberg type inequality which is present [CJS]

$$\int_{\mathbb{R}^N} |u|^{p+1} dx \le K \Big( \int_{\mathbf{R}^N} |u|^2 dx \Big)^{1-\theta} \Big( \int_{\mathbf{R}^N} |u|^2 |\nabla u|^2 dx \Big)^{\frac{\theta N}{N-2}}. \qquad (1)$$

with

$$\theta = \frac{(p-1)(N-2)}{2(N+2)}$$

and some $K > 0$ depending only on $N$ cannot guarantee global wellposedness. CJS proved blow-up for $p > 3 + \frac{4}{N}$.

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The best result available at the moment is the following:

## Theorem

*[Colin-Jeanjean-Squassina] Let* $N \geq 1$, $s = 2E(\frac{N}{2}) + 2$ *and assume that* $a_0 \in H^{s+2}(\mathbf{R}^N)$. *Then there exists a positive* $T$ *and a unique solution to the Cauchy problem (1) satisfying*

$$\phi(0, x) = a_0(x),$$

$$\phi \in L^\infty(0, T; H^{s+2}(\mathbb{R}^N)) \cap C([0, T]; H^s(\mathbb{R}^N)),$$

*and the conservation laws*

$$\|\phi(t)\|_2 = \|a_0\|_2, \tag{2}$$

$$\mathcal{E}(\phi(t)) = \mathcal{E}(a_0), \tag{3}$$

*for all* $t \in [0, T]$.

The proof of Theorem is based on energy methods and to overcome the loss of derivatives induced by the quasilinear term, gauge transforms are used. I've cheated in the statement...

References

For the main results on the CP we refer to Colin, Colin-Jeanjean-Squassina, Kenig-Ponce-Vega, Lange, Poppenberg...
I apologize for the not complete set of references!

## Stationary solutions

Despite we still have not a satisfactory theory of local wellposedness, people have been able to prove the existence of *standing waves* $\phi(t,x) = u(x)e^{i\omega t}$. Here $u : \mathbf{R}^N \to \mathbf{C}$ solves the quasilinear elliptic equation:

$$-\Delta u - \lambda u \Delta |u|^2 + \omega u - |u|^{p-1}u = 0, \tag{4}$$

while $\omega > 0$ is the time-frequency and $\lambda > 0$. Equation (4) is variational and is the Euler-Lagrange equation of the associated energy functional $\mathcal{E}_\omega^\lambda$ which is

$$\mathcal{E}_\omega^\lambda(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 dx + \frac{\lambda}{4}\int_{\mathbf{R}^N} |\nabla |u|^2|^2 dx + \tag{5}$$

$$+\frac{\omega}{2}\int_{\mathbf{R}^N} |u|^2 dx - \frac{1}{p+1}\int_{\mathbf{R}^N} |u|^{p+1} dx. \tag{6}$$

Thanks to the variational structure, solutions of equation (4) can be found by means of critical point theory. But... Is this quasilinear functional Frechet differentiable in the energy space? In principle I should care about this, but I do not, because... We perform a MAGIC change of unknown by setting $v = r^{-1}(u)$, where the function $r : \mathbf{R} \to \mathbf{R}$ is the unique solution to the Cauchy problem

$$r'(s) = \frac{1}{\sqrt{1 + 2\lambda r^2(s)}}, \qquad r(0) = 0. \tag{7}$$

Here $u \in X_{\mathbf{C}}$ is assumed to be real valued. Then, in [CJ] it is proved that, if $v \in H^1(\mathbb{R}^N) \cap C^2(\mathbf{R}^N)$ is a real solution to

$$-\Delta v = \frac{1}{\sqrt{1 + 2\lambda r^2(v)}}\Big(|r(v)|^{p-1} r(v) - \omega r(v)\Big), \tag{8}$$

then $u = r(v) \in X_{\mathbf{C}} \cap C^2(\mathbb{R}^N)$ and it is a real solution of (4). Now thanks to Berestycki-Lions we have the following theorem:

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## Theorem (CJS)

*For all $\omega > 0$, $\lambda = 1$ and $1 < p < \frac{3N-2}{N-2}$ for $N \geq 3$ there exists a ground state $u(x)$ of the form*

$$u(x) = e^{i\theta}|u(x)|, \quad x \in \mathbf{R}^N,$$

*for some $\theta \in \mathbb{S}^1$. In particular, the ground states are, up to a constant complex phase, real-valued and non-negative. Furthermore any real non-negative ground state $u$ satisfies the following properties*

i) $u > 0$ *in* $\mathbb{R}^N$,

ii) $u$ *is a radially symmetric decreasing function,*

iii) $u \in C^2(\mathbb{R}^N)$,

iv) *for all $\alpha \in \mathbf{N}^N$ with $|\alpha| \leq 2$, there exists $(c_\alpha, \delta_\alpha) \in (\mathbf{R}_+^*)^2$ such that*

$$|D^\alpha u(x)| \leq C_\alpha e^{-\delta_\alpha |x|}, \quad \text{for all } x \in \mathbf{R}^N.$$

*Moreover, in the case $N = 1$, there exists a unique positive ground state to (4) up to translations.*

## Main open problems for the elliptic equation

- Uniqueness of the ground state for $N \geq 2$
- Nondegeneracy of the ground state (later for the precise definition)
- Orbital stability of the ground state (not just elliptic indeed)

One can easily have some partial results on the first two questions (unfortunately the trasformation does not work well with also the time derivative), because...

We notice that for small $\lambda$ the transformed equation (8) is a perturbation of the semilinear one ($\lambda = 0$). Can we deduce uniqueness and nondegeneracy from the properties of the ground state of the semilinear equation?

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## Theorem (Uniqueness)

*(S., 2010) Let* $1 < p < 2^*$*. There exists* $\bar{\lambda}$ *such that for* $0 < \lambda < \bar{\lambda}$ *there exists only one real, positive, radial and exponentially decaying ground state* $u_\lambda(x)$ *of equation (4). Moreover suppose* $v \in \mathcal{G}_\omega^\lambda$*, then there exists* $\xi \in \mathbf{R}^N$ *and* $\theta \in [0, 2\pi]$ *such that* $v_\lambda(x) = u_\lambda(x - \xi)e^{i\theta}$*, where* $u_\lambda(x)$ *is the only real, positive, radial and exponentially decaying ground state of (4).*

## Theorem (Nondegeneracy)

*(S., 2010) Let* $1 < p < 2^*$*. There exists* $\bar{\lambda}$ *such that for* $0 < \lambda < \bar{\lambda}$ *the ground state* $u_\lambda$ *of equation (4) is nondegenerate.*

Actually in the same paper we proved also that for any $\lambda > 0$, the ground state is also $C^\infty$: [CJS] forgot to do it, but it is almost obvious thanks to standard bootstrap. Slightly slower bootstrap due the quasilinear term.

# joint work with Jeanjean

We want to remove the innatural request of $H^1$-subcriticality and generalize the problem to $\lambda = 1$, namely not small. We have the following results:

**Theorem**

*(almost complete) Let $3 \leq p < \frac{3N-2}{N-2}$ and $\lambda = 1$. Then there exists only one real, positive, radial and exponentially decaying ground state $u(x) := u_1(x)$ of equation (4). Moreover suppose $v$ is another ground state, then there exists $\xi \in \mathbf{R}^N$ and $\theta \in [0, 2\pi]$ such that $v(x) = u(x-\xi)e^{i\theta}$, where $u(x)$ is the only real, positive, radial and exponentially decaying ground state of (4).*

**Theorem**

*(complete) Let $3 \leq p < \frac{3N-2}{N-2}$ and $\lambda = 1$. Then the ground state $u(x)$ of equation (4) is nondegenerate, namely the following properties are true (Notation: $I(u) := \mathcal{E}_1^1(u)$):*

- *(ND)* $\ker[I''(u)] = span\left\{ iu(x), \frac{\partial u(x)}{\partial x_j} \quad j = 1, \cdots, N \right\}$;
- *(Fr) $I''(u)$ is an index 0 Fredholm map.*

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## Remark

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*Why nothing in the range* $1 < p < 3$*? For* $3 \leq p < \frac{3N-2}{N-2}$ *the ground state is a mountain pass, in the other range not so clear geometry... I'm wondering... Is there a relationship between the MP geometry and the Uniqueness of the ground state?*

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# Scheme of the proof of uniqueness

The proof of this result relies heavily on the analysis of Kwong and McLeod in the semilinear case. The argument we have used in the proof is exposed in the book of Tao who found the highway: a monotonicity formula!

## The shooting method

Without loss of generality we consider the case $\omega = 1$ and so

$$-\Delta u - u\Delta|u|^2 + u - |u|^{p-1}u = 0. \tag{9}$$

Since by [CJS] the ground state is radial, real up to phase shift and positive, we reduce to prove uniqueness for the following boundary value problem:

$$u''(r) + \frac{N-1}{r}u'(r) + u\left((u^2)''(r) + \frac{N-1}{r}(u^2)'(r)\right) - u + u^p = 0 \tag{10}$$

with $u > 0$ and the boundary conditions $u'(0) = 0$ and $\lim_{r\to+\infty} u(r) = 0$.

We classify the set of intial data in the folllowing way:

- *subcritical* if $\inf_{t>0} u_y(t) > 0$,
- *critical* if $\inf_{t>0} u_y(t) = 0$,
- *supercritical* if $\inf_{t>0} u_y(t) < 0$.

We have rephrased Theorem (UNIQUENESS): we need to show that there exists only one critical position.
The strategy is the following:

- prove that the sets of supercritcal and subcritical positions are open. This implies that there exists a minimal critical position $y^* > 0$
- prove that all $y > y^* > 0$ are supercritical.

INSIDE: a big mess of ODE techniques which I do not completely understand, but everything works!

# Proof of Nondegeneracy

The strategy is to study the linearized operator by decomposing it into spherical harmonics. It turns out that the harmonics can be divided into three different groups according to the method with which one analyzes them:

- *ODE harmonics*: This are usually the lowest harmonics, the most difficult to treat but they are not always present. One rules out the possibility of having solutions through comparison principle and monotonicty formulas.
- *Solution Harmonics*: These are interemediate harmonics and one identifies here all the possible solutions.
- *Variational harmonics*: Through variational methods, mainly Perron-Frobenius theory one exclude solutions of higher harmonics

# The linearized equation

The linearization of (4) around the ground state $u$ is the following

$$L(u)w := -\Delta w - w\Delta|u|^2 - 2u\Delta u Re(w) - 4\nabla u Re(\nabla w) + \\ -2uRe(\Delta w) + w - |u|^{p-1}w - (p-1)|u|^{p-1}Re(w). \quad (11)$$

We want to characterize the kernel of $L(u)$. In order to do this we split $w$ into its real and imaginary parts $w := w_1 + iw_2$, with $w_1, w_2$ real valued, and we decompose $L$ into two operators $L_+(u), L_-(u)$ acting on $w_1$ and $w_2$. They are defined as follows

$$\begin{aligned} L_+(u)w_1 \quad &:= \quad -\Delta w_1 - w_1\Delta|u|^2 - 2u\Delta u w_1 - 4\nabla u \nabla w_1 + \quad (12) \\ &- \quad 2u\Delta w_1 + \omega w_1 - p|u|^{p-1}w_1 \quad (13) \end{aligned}$$

and

$$L_-(u)w_2 := -\Delta w_2 - w_2\Delta|u|^2 + \omega w_2 - |u|^{p-1}w_2. \quad (14)$$

# Decomposition into spherical harmonics

We introduce some notation

$$r := |x| \in \mathbf{R}^+$$

is the radial variable, while

$$\Omega := \frac{x}{r} \in S^{N-1}$$

is the angular variable. The operators $\Delta_r$ and $\Delta_{S^{N-1}}$ are respectively the radial and the angular Laplacian and are defined as follows

$$\Delta_r := \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r}$$

and

$$\Delta_{S^{N-1}} := \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial y_j} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial y_i} \right).$$

Here $ds^2 := g_{ij} dy^i dy^j$ denotes the standard metric on $S^{N-1}$, $g := det(g_{ij})$ and $[g^{ij}] := [g_{ij}]^{-1}$.

Consider the spherical harmonics $Y_k(\Omega)$, which are the eigenvectors of the angular Laplacian $\Delta_{S^{N-1}}$:

$$-\Delta_{S^{N-1}} Y_k = \lambda_k Y_k, \tag{15}$$

where

$$\lambda_k := k(N+k-2), \quad k = 0, 1, 2, \dots$$

are the eigenvalues of $-\Delta_{S^{N-1}}$ with multiplicities $M_k - M_{k-2}$:

$$M_k := \frac{(N+k-1)!}{(N-1)!(k)!} \quad (k \geq 0), \quad M_k = 0 \quad \forall k < 0.$$

We split each $v \in X_{\mathbf{C}}$ into spherical harmonics

$$v(x) := \sum_{k \geq 0} v_k(r) Y_k(\Omega), \quad \text{where} \quad v_k(r) := \int_{S^{N-1}} v(r\Omega) Y_k(\Omega) d\Omega \in X_{\mathbf{C}}.$$

so that $L_+$ becomes

$$A_h w_h := -(1+2u^2)\Delta_r w_h + (1+2u^2)\frac{\lambda_h}{r^2} w_h + \omega w_h - 2u\Delta u w_h \tag{16}$$

$$-4uu' w_h' - p|u|^{p-1} w_h \tag{17}$$

for $h = 0, 1, 2, \cdots$, while $L_-$ becomes

$$B_k v_k := -\Delta_r v_k + \frac{\lambda_k}{r^2} v_k + v_k - v_k \Delta |u|^2 - |u|^{p-1} v_k = 0 \tag{18}$$

for $k = 0, 1, 2, \cdots$.

# The disconjugacy interval

## Definition

*(Disconjugacy Interval)[Kwong] Consider the following second order linear equation*

$$U''(x) + f(x)U'(x) + g(x)U(x) = 0, \quad x \in [0, +\infty) \tag{19}$$

*with $f$ and $g$ continuous real valued function. Suppose that (19) has solutions which do not vanish in a neighborhood of $+\infty$. The largest neighborhood $(c, +\infty)$ of $+\infty$ of which there exists a solution of (19) without zeros is called the disconjugacy interval of (19).*

## Remark

*No non-trivial solution of (19) can have more than one zero in $(c, +\infty)$. On the other hand, unless $c = 0$, any solution of (19) that has a zero before $c$ must have another zero in $(c, +\infty)$.*

## Lemma

*(Unboundedness)[Kwong] Let* $(c, +\infty)$ *be the disconjugacy interval of (19) with*

$$f(x) = \frac{N-1}{r} + \frac{4u'}{1+2u^2}$$

*and*

$$g(x) = -\frac{1 - \Delta|u|^2 - u\Delta u - p|u|^{p-1}}{1+2u^2}.$$

*Every solution of (19) with a zero in* $(c, +\infty)$ *is unbounded. Conversely, if the last zero of the unbounded solution of (19) is* $\rho$*, than* $\rho$ *is an interior point of the disconjugacy interval (*$\rho > c$*).*

We will discuss now just the $A_h$'s because $B_k$'s are very similar, actually easier.

## Lemma

*[ODE harmonic] There are no non-trivial solution of the equation $A_0 w = 0$.*

## Proof.

Consider the equation

$$\begin{aligned} A_0 w_0 \quad & := -(1+2u^2)\Delta_r w_0 + (1+2u^2)\frac{\lambda_0}{r^2} w_0 + \omega w_0 = \\ & -2u\Delta u w_0 - 4uu' w_0' - p|u|^{p-1} w_0 \end{aligned} \tag{20}$$

which becomes

$$A_0 w_0 = -(1+2u^2)\Delta_r w_0 + \omega w_0 - 2u\Delta u w_0 - 4uu' w_0' - p|u|^{p-1} w_0 \tag{21}$$

since $\lambda_0 = 0$. By Tao's highway $w_0$ has to change sign once in $[0, +\infty)$. □

Proof.

Then by the definition of disconjugacy interval, $w_0$ has to change sign once also in its disconjugacy interval and so by Lemma (UNBOUNDEDNESS) $w_0$ is unbounded. This means that there are no non-trivial solution $w_0 \in X_{\mathbf{C}}$ of (21) apart from $w_0 = 0$. This completes the proof. □

Now we prove that the only solutions of $A_1 w_1 = 0$ are $w_1(r) = u'(r)$.

## Lemma

*[Solutions harmonics] The only solution of $A_1 w_1 = 0$ and of $B_0 v_0 = 0$ are respectively $w_1(r) = u'(r)$ and $v_0(r) = u(r)$.*

## Proof.

We consider

$$A_1 w_1 : \quad = \quad -(1+2u^2)\Delta_r w_1 + (1+2u^2)\frac{N-1}{r^2} w_1 + \omega w_1 + \quad (22)$$

$$- \quad 2u\Delta u w_1 - 4uu' w_1' - p|u|^{p-1} w_1 = 0. \quad (23)$$

By differentiating (4) with respect to $r$, it is easy to see that $w_1(r) = u'(r)$ is a solution of (22). We try now to find solutions of the form $\tilde{w}_1(r) := d(r) w_1(r)$. Apart from $\tilde{w}_1(r) := d_0 w_1(r)$ with $d_0 \in \mathbf{R}$, $d$ must satisfy the following ODE

$$\frac{d''}{d'} = \frac{N-1}{r} + \frac{4u'}{1+2u} + 2\frac{u''}{u'}.$$

□

### Proof.

The solution of these ODE is

$$d'(r) = \frac{1}{r^{N-1}|u'(r)|^2(1+2u(r))^2}.$$

This means that $d'(r) \simeq e^{2\delta r}$ as $r \to +\infty$ and so

$$\tilde{w}_1(r) = u(r)d(r) \simeq e^{-\delta r}e^{2\delta r} \simeq e^{\delta r}.$$

Hence $\tilde{w}_1$ cannot belong to $X_{\mathbf{C}}$ if $d'(r)$ is not identically zero and this completes the proof of the lemma. □

By variational techniques we prove that there are no other solutions.

### Lemma

*[Variational harmonics] The only solutions of $B_k v_k = 0$ for $k \geq 1$ and of $A_h w_h = 0$ for $h \geq 2$ are $v_k = 0$ and $w_h = 0$.*

### Proof.

We have proved in the previous lemma that the only solution of $A_1 w_1 = 0$ is $w_1(r) = u'(r)$. Since $w_1$ have constant sign in its domain of definition which is $\mathbf{R}^+ = (0, +\infty)$, then $A_1$ is a non-negative operator. Why? If $\nu$ is the smallest eigenvalue of $A_1$, then any corresponding eigenfunction $v_\nu$ does not change sign in $\mathbf{R}^+$. If $\nu < 0$, then $v_\nu$ should be orthogonal to $w_1(r) = u'(r) > 0$, which is impossible. Hence $\nu \geq 0$ which implies that $A_1$ is non-negative.
We now prove that $A_h$ is a positive operator for $h \geq 2$. We can write $A_h$ as $A_h = A_1 + \frac{\gamma_h}{r^2}(1 + 2u^2)$ with $\gamma_h := \lambda_h - \lambda_1$ which is positive for $h \geq 2$. As before this implies that $A_h$ is a positive operator for $h \geq 2$ and so $A_h w_h = 0$ implies $w_h = 0$ for $h \geq 2$. □

We are now ready to prove Theorem (NONDEGENERACY)

## Proof.

(NONDEGENERACY)First of all we have that $z(x) := w(x) + iv(x) \in Kernel I''(u)$ (here $w$ and $v$ are real valued) if and only if $w \in Kernel(L_+(u))$ and $v \in Kernel(L_-(u))$. By ODE, SOLUTIONS and VARIATIONAL Lemmas we have that each solution of $L_+(u)w = 0$ is a multiple of $u'(r)Y_1(\Omega)$ and each solution of $L_-(u)v = 0$ is a multiple of $u(r)Y_0(\Omega)$. Here $Y_0(\Omega)$ belongs to the 1-dimensional kernel of $-\Delta_{S^{N-1}}$ $(M_0 = 0)$ while $Y_1(\Omega)$ belongs to the $N$-dimensional kernel of $-\Delta_{S^{N-1}} - \lambda_1$ $(M_1 = N)$. If we define $Y_{1,j}, \cdots, Y_{N,j}$ as the basis of the kernel of $-\Delta_{S^{N-1}} - \lambda_1$, we have that

$$w \in span\{u'(r)Y_{1,j} : j = 1, \cdots, N\} = span\{u'(r)Y_{1,j} : j = 1, \cdots, N\}$$

and

$$w \in span\{u(r)Y_0\} = span\{u(r)\}$$

which proves (ND). □

## Proof.

For what concerns (Fr) we can write $I''(u)$ in the following way

$$(I''(u)z, z)_2 = (z, z)_{X_c} - (|u|^{p-1}w + (p-1)|u|^{p-1}Re(w), w)_2$$

with $P(x)w := |u|^{p-1}w + (p-1)|u|^{p-1}Re(w)$. Since $u$ is exponentially decaying, then $P(x)$ is a compact operator and so $I''(u)$ is a compact perturbation of the identity. □

Open problems for the complete evolution

- Local wellposedness in the energy space $X_{\mathbf{C}}$
- Orbital stability of the ground state (also elliptic indeed)
- Illposedness for initial data with low regularity

**Remark**

*I could go on because more or less everything is unknown... Well, I can prove Morawetz and Intercation Morawetz estimates in the defocusing case because the quasilinear term helps the quantum flux of the semilinear equation to disperse mass: same proof!*

## Open problems for the complete evolution

- Local wellposedness in the energy space $X_{\mathbf{C}}$
- Orbital stability of the ground state (also elliptic indeed)
- Illposedness for initial data with low regularity

### Remark

*I could go on because more or less everything is unknown... Well, I can prove Morawetz and Intercation Morawetz estimates in the defocusing case because the quasilinear term helps the quantum flux of the semilinear equation to disperse mass: same proof!*

Thank you for your attention!