On Hirota-Satsuma's equation

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$u = u(x,t) \in \mathbb{R}$, $x, \; t \in \mathbb{R}$, and ϵ is a small parameter.
Note that $u_t = -u_x + \mathcal{O}(\epsilon) \Rightarrow$ these models are formally equivalent.

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We fix $\epsilon = 1$. $\Rightarrow$

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and

$$F(\phi) = \int_{\mathbb{R}} \left(\frac{1}{2}\phi'(x)^2 + \frac{1}{2}\phi''(x)^2 - \frac{1}{3}\phi(x)^3 + \frac{1}{6}\phi(x)^4 - \frac{3}{2}\phi(x)\phi'(x)^2 \right) dx$$

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2– Existence and stability of solitary wave solutions.

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Lemma

Ω_1 is an open subspace of $H^1(\mathbb{R})$.

Local well-posedness

For all $\phi \in \Omega_1$, let us denote

$$r_j(\phi) := \|\partial_x^j(-\partial_x^2 - \phi + 1)^{-1}\|_{\mathcal{B}(L^2)}, \quad j = 0, 1, 2,$$

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Theorem (Iório, —)

Let $u_0 \in \Omega_1$. Then $\exists\ T = T(u_0) > 0$, $\exists!\ u \in C([0,T]; H^1(\mathbb{R}))$ solution of (HS) such that $u(\cdot, 0) = u_0$. Moreover the flow map solution $u_0 \mapsto u$ is smooth.

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Observation: $T = T(\|u_0\|_{L^2}, r_j(u_0))$, $j = 0, 1, 2$.

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Then F is a contraction in the space

$$X_T^1(u_0) := \big\{ u \in C([0,T]; H^1(\mathbb{R})) \ : \ \sup_{t \in [0,T]} \|u(t) - u_0\|_{H^1} \le \alpha \big\},$$

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where $\alpha = \frac{1}{\sqrt{2} r_0(u_0)}$ and $T = T(r_j(u_0), \|u_0\|_{L^2})$ is small enough.

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\|F(u)(t) - u_0\|_{H^1} &= \|F(u)(t) - u_0\|_{L^2} + \|\partial_x(F(u)(t) - u_0)\|_{L^2} \\
&\leq \|\int_0^t \partial_x R_{u(t')} u(t')dt'\|_{L^2} + \|\int_0^t \partial_x^2 R_{u(t')} u(t')dt'\|_{L^2} \\
&\leq \int_0^t \frac{r_1(u_0) + r_2(u_0)}{1 - \frac{1}{\sqrt{2}} r_0(u_0)\|u(t') - u_0\|_{H^1}} \|u(t')\|_{L^2} dt' \\
&\leq 2T(r_1(u_0) + r_2(u_0))(\alpha + \|u_0\|_{L^2}).
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$\Rightarrow$ If T is small enough, $T = T(r_0(u_0), r_1(u_0), r_2(u_0), \|u_0\|_{L^2})$

$$F(X^1_T(u_0)) \subset X^1_T(u_0).$$

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Let $0 < \delta_0 < 1$*. If* $\phi \in B(0, \|\varphi^\star\|_{H^1})$ *and* $E(\phi) \le (1-\delta_0)E(\varphi^\star)$*. Then* $\exists\ \delta = \delta(\delta_0) > 0$ *such that*

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$$\Rightarrow \quad f(\bar{y}) \leq E(\phi) \leq (1-\delta_0)E(\varphi^\star).$$

Moreover,

$$f'(y) = 0 \Leftrightarrow y = 4\|\varphi^\star\|_{H^1}^2 \quad \text{and} \quad f(\|\varphi^\star\|_{H^1}^2) = E(\varphi^\star).$$

$\square$

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Theorem (Iório,—)

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$\rightarrow$ *a priori* bounds on $r_j(u(t)) = \|\partial_x^j(-\partial_x^2 - u(t) + 1)^{-1}\|_{\mathcal{B}(L^2)}$, $j = 0, 1, 2$.

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- $\partial_x^2 R_{u(t)}(-1) = -1 - u(t)R_{u(t)}(-1) + R_{u(t)}(-1)$.
$$\Rightarrow \|\partial_x^2 R_{u(t)}(-1)\|_{\mathcal{B}(L^2)} = \mathcal{O}(\delta, \|\varphi^\star\|_{H^1}).$$

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- $\partial_x^2 R_{u(t)}(-1) = -1 - u(t)R_{u(t)}(-1) + R_{u(t)}(-1)$.
$$\Rightarrow \|\partial_x^2 R_{u(t)}(-1)\|_{\mathcal{B}(L^2)} = \mathcal{O}(\delta, \|\varphi^\star\|_{H^1}).$$

- $\|\partial_x R_\phi(-1)\|_{\mathcal{B}(L^2)} \le \left(\|R_\phi(-1)\|_{\mathcal{B}(L^2)}\|\partial_x^2 R_\phi(-1)\|_{\mathcal{B}(L^2)}\right)^{\frac{1}{2}}$. $\square$

Solitary waves

- For $c > 0$, (HS) admits special solutions of the form

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- $\|\phi_\mu\|_{H^1} < \|\varphi^\star\|_{H^1}$ and $E(\phi_\mu) < E(\varphi^\star)$.
- $\Rightarrow$ *Question:* Stability of these solitary waves?

Ill-posedness in $\varphi^\star$

Observe that

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which is a contradiction since for all $R>0$,

$$u_c \underset{c\to+\infty}{\longrightarrow} 0 \quad \text{in} \quad L^\infty((-R,R)\times(0,T^\star]).$$

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Theorem (Bona,—)

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then $\forall\, t > 0$, $\exists\, \gamma = \gamma(t) \in \mathbb{R}$ satisfying

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Moreover, γ can be chosen as a C^1 function satisfying $|\gamma'(t) + (1+c)| \leq c\epsilon$, $\forall\, t > 0$.

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Involves global analysis like the "concentration compactness method".

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How to prevent a minimizing sequence from dichotomizing?

Lopes theorem

Theorem (Lopes)

Let E, $F : H^2(\mathbb{R}) \to \mathbb{R}$ *be translation invariant,* C^2 *functionals satisfying*

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(C) *If a minimizing sequence for (V_λ), $\phi_n \rightharpoonup \phi \neq 0$ in H^2, where ϕ is solution to $\Lambda'(\phi) = F'(\phi) + \mu E'(\phi) = 0$, then $\exists\, h \in H^2(\mathbb{R})$ s.t. $\Lambda''(\phi)(h, h) < 0$.*

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Then *Let* (ϕ_n) *a minimizing sequence for* (V_λ).

$$(\phi_n \rightharpoonup \phi \neq 0 \text{ in } H^2) \quad \Rightarrow \quad (\phi_n \to \phi \text{ in } W^{1,p},\ 2 < p \leq \infty).$$

Moreover, ϕ *is a solution to the Euler-Lagrange equation* $F'(\phi) + \mu E'(\phi) = 0$, *for some* $\mu \in \mathbb{R}$.

Technical results

Lemma

Let (ϕ_n) a bounded sequence of H^2. Then

$$(\phi_n(\cdot+c_n) \rightharpoonup 0 \text{ in } H^2, \ \forall (c_n) \subset \mathbb{R}) \ \Rightarrow \ (\phi_n \to 0 \text{ in } W^{1,p}, \ 2 < p \leq \infty).$$

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Lemma

If $\{\phi_n\}$ is a minimizing sequence for (V_λ) such that $\phi_n \rightharpoonup \phi$ in $H^1(\mathbb{R})$, then $\|\phi\|_{H^1} < \|\varphi^\star\|_{H^1}$.

Technical results

Lemma (Monotonicity)

(i) $e : (0,1) \to (0, e^\star),\ \mu \mapsto E(\phi_\mu)$ *is a strictly increasing bijection.*

(ii) $f : (0,1) \to (f^\star, 0),\ \mu \mapsto F(\phi_\mu)$ *is a strictly decreasing bijection.*

where $e^\star := E(\varphi^\star)$ *and* $f^\star := F(\varphi^\star)$.

Existence of global minimizers

Proposition

Let $\lambda \in (0, e^\star)$, *and* (ϕ_n) *a minimizing sequence for* (V_λ). *Then,* $\exists\, (c_n) \subset \mathbb{R},\ \tau \in \mathbb{R}$ *s.t.*

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Proof of the proposition

- Passing to the limit,

$$F(\phi_\alpha) \le \liminf(F(\psi_n)) = F_\lambda \text{ and } E(\phi_\alpha) \le \liminf(E(\psi_n)) = \lambda.$$

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- Then
  $\alpha = \mu, \; F(\phi_\mu) = F_\lambda, \; E(\phi_\mu) = \lambda,$ and $\|\psi_n\|_{H^2} \underset{n \to +\infty}{\longrightarrow} \|\phi_\mu\|_{H^2}$.

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- $\exists\, f \neq 0 \in H^2(\mathbb{R})$ such that $f_n \underset{n \to +\infty}{\rightharpoonup} f$ in H^2 and $\exists\, h \in C_0^\infty(\mathbb{R})$ such that $E'(f)h \neq 0$.

Proof of the theorem

- Consider the polynomial

$$P_n(t) = E(f_n + th) = a_n + b_n t + c_n t^2 + dt^3.$$

Then $a_n \longrightarrow \lambda$, $b_n \longrightarrow E'(f)h \neq 0$ and $c_n \longrightarrow \frac{1}{2}E''(f)(h,h)$.

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Contradiction $\square$

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