1-D Schrödinger's Equation with Quadratic Nonlinearity

Normal form approach

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Problem of local well-posedness in low regularity

In this presentation, we will consider the local well-posedness (l.w.p) of the equation

$$\left| \begin{array}{l} u_t + iu_{xx} = u^2 \\ u(0,x) \in H^{-\alpha}(\mathbf{R}). \end{array} \right. \tag{1}$$

By *local well-posedness*, we mean that there exists $T := T(\|u(0)\|_{H^{-\alpha}}) > 0$ and a continuous solution map of (1) from $H^{-\alpha}(\mathbf{R})$ to $C_t H_x^{-\alpha}([0,T] \times \mathbf{R})$.

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Previous works on (1)

- Kenig, Ponce and Vega '96 established local well-posedness of (1) up to $H^{-\frac{3}{4}+}$.
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Preliminaries

We define the spatial Fourier transform of $f(x) \in \mathcal{S}(\mathbf{R})$

$$\widehat{f}(\xi) = \int_{\mathbf{R}} f(x)e^{-ix\xi}\,dx.$$

and space-time Fourier transform of $f(t,x) \in \mathcal{S}(\mathbf{R} \times \mathbf{R})$

$$\widetilde{f}(\tau,\xi) = \iint_{\mathbf{R}\times\mathbf{R}} f(t,x)e^{-i(t\tau+x\xi)}\,dx\,dt.$$

For a reasonable function p, We define the symbol $p(\nabla)$ via the Fourier transform: $\widehat{p(\nabla)f}(\xi) = p(\xi)\widehat{f}(\xi)$.

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Free solution and $X^{s,b}$ spaces

The solution of linear Schrödinger's equation $u_t + iu_{xx} = 0$ with $u(0) = f \in L^2$ is $e^{-it\partial_x^2} f$, defined via Fourier transform as before.

Note $(\tau - \xi^2)\widetilde{u}(\tau, \xi) = 0$. So $\widetilde{e^{-it\partial_x^2} f}$ is supported on $\tau = \xi^2$.

$$\|u\|_{X^{s,b}} := \|\langle \xi \rangle^s \langle \tau - \xi^2 \rangle^b \widetilde{u}\|_{L^2_{\tau,\xi}}.$$

For all Strichartz pairs (p, q): $\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$, $2 \leq p, q \leq \infty$

$$\|u\|_{L^q_t L^p_x} \lesssim \|u\|_{X^{0,\frac{1}{2}+}}.$$

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Change of Variable

We begin by changing the variable in (1) by setting $v = \langle\nabla\rangle^{-\alpha}u$.

$$\begin{cases} v_t + iv_{xx} = \langle\nabla\rangle^{-\alpha}\left(\langle\nabla\rangle^{\alpha}v\langle\nabla\rangle^{\alpha}v\right) \\ v(0,x) = f(x) \in L^2(\mathbf{R}). \end{cases} \tag{2}$$

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Another change of variable

Another change of variable: $v = e^{-it\partial_x^2} f_{>0} + z_{>0}$.

$$\begin{cases} z_t + iz_{xx} = \langle\nabla\rangle^{-\alpha}\left(\langle\nabla\rangle^{\alpha}(e^{-it\partial_x^2} f_{>0} + z_{>0})\langle\nabla\rangle^{\alpha}(e^{-it\partial_x} f_{>0} + z_{>0})\right) \\ z(0,x) = 0. \end{cases} \tag{3}$$

We want to construct an explicit solution h (normal form) for

$$h_t + ih_{xx} = \langle\nabla\rangle^{-\alpha}\left(\langle\nabla\rangle^{\alpha} e^{-it\partial_x^2} f_{>0}\langle\nabla\rangle^{\alpha} e^{-it\partial_x} f_{>0}\right). \tag{4}$$

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Construction of Normal Form

We define a bilinear multiplier T_σ with symbol $\sigma = \sigma(\xi, \eta)$

$$T_\sigma(u, v)(x) := \frac{1}{4\pi^2} \iint_{\mathbf{R}\times\mathbf{R}} \sigma(\xi, \eta)\widehat{u_{>0}}(\xi)\widehat{v_{>0}}(\eta)e^{i(\xi+\eta)x}\, d\xi\, d\eta.$$

Formally, applying $\partial_t + i\partial_x^2$ to the integrand above would give the following symbol multiplied to σ:

$$i(\tau + \omega) - i(\xi + \eta)^2 = i(\tau - \xi^2) + i(\omega - \eta^2) - 2i\xi\eta.$$

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\begin{aligned}
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& + T_\sigma\left(u, (\partial_t + i\partial_x^2) v\right) - 2i T_{\xi\eta\cdot\sigma(\xi,\eta)}(u, v).
\end{aligned}
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If $u = v = e^{-it\partial_x} f$, the first two terms disappear.

Thus

$$\begin{aligned}(\partial_t + i\partial_x^2)T_\sigma(u,v) =& T\left((\partial_t + i\partial_x^2)u, v\right)\\ &+ T_\sigma\left(u, (\partial_t + i\partial_x^2)v\right) - 2iT_{\xi\eta\cdot\sigma(\xi,\eta)}(u,v).\end{aligned}$$

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Normal form (continued)

In order to satisfy (4), we want

$$-2iT_{\xi\eta\cdot\sigma(\xi,\eta)}(e^{-it\partial_x^2}f, e^{-it\partial_x^2}f) = \langle\nabla\rangle^{-\alpha}\left(\langle\nabla\rangle^{\alpha}e^{-it\partial_x^2}f\langle\nabla\rangle^{\alpha}e^{-it\partial_x^2}f\right)$$

So we let $\sigma(\xi,\eta) = -\frac{1}{2i}\frac{1}{\xi\eta}\frac{\langle\xi\rangle^{\alpha}\langle\eta\rangle^{\alpha}}{\langle\xi+\eta\rangle^{\alpha}}$.

Thus $h(t,x) = T_\sigma(e^{-it\partial_x^2}f, e^{-it\partial_x^2}f)$ satisfies equation (4) with the initial condition $h(0,x) = T_\sigma(f,f)$.

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Normal form

$$T(u,v)(x) = \frac{i}{8\pi^2}\iint_{\mathbf{R}\times\mathbf{R}} \frac{\langle\xi\rangle^\alpha\langle\eta\rangle^\alpha}{\langle\xi+\eta\rangle^\alpha}\frac{1}{\xi\eta}\widehat{u_{>0}}(\xi)\widehat{v_{>0}}(\eta)e^{i(\xi+\eta)x}\,d\xi\,d\eta.$$

We claim $T : L^2 \times L^2 \to H^{\frac{1}{2}}$
and also $T : X^{0,\frac{1}{2}+\delta} \times X^{0,\frac{1}{2}+\delta} \to L^2_t H^1_x$.

Now $v = e^{-it\partial_x^2} f + h + w$ where w solves

$$\begin{cases} w_t + iw_{xx} = \mathcal{N}(e^{-it\partial_x^2} f, w+h) + \mathcal{N}(h+w, h+w) \\ w(0,x) = -T(f,f)(x) \in H^{\frac{1}{2}}(\mathbf{R}). \end{cases} \tag{5}$$

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$X^{s,b}$ estimate

We have the Strichartz estimate for $b > 1/2$

$$\| \int_0^t e^{-i(t-s)\partial_x^2} \mathcal{N}(s)\, ds \|_{X_T^{s,b}} \leq \|\mathcal{N}\|_{X^{s,b-1}}.$$

So if we want to construct a fixed point argument for $w_t + iw_{xx} = \mathcal{N}$, the we need to estimate $\|\mathcal{N}\|_{X_T^{s,b-1}}$.

We construct this argument for $w \in X^{\alpha-\frac{1}{2},\frac{1}{2}+\delta}$, so we need to estimate

$$\|\mathcal{N}(e^{-it\partial_x^2} f_{>0}, h_{>0}, w_{>0})\|_{X_T^{\alpha-\frac{1}{2},-\frac{1}{2}+\delta}}.$$

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Non-linearities of (5)

We need to consider

$$\mathcal{N}(e^{-it\partial_x^2}f, w),\ \mathcal{N}(w, w),\ \mathcal{N}(h, h),\ \mathcal{N}(w, h),\ \mathcal{N}(e^{-it\partial_x^2}f, h),$$

After some work, we arrive at the estimates:

$$\|\mathcal{N}(u, v)\|_{X_T^{\alpha-\frac{1}{2}, -\frac{1}{2}+\delta}} \leq C_{T,\delta}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{\alpha-\frac{1}{2},\frac{1}{2}+\delta}}$$

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Problematic case

But the estimate

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is false!

However, $h = T(e^{-it\partial_x^2}f, e^{-it\partial_x^2}f)$.

So the problematic term is in a trilinear form $\mathcal{N}(T(f,g),v)$ with $f,g,v \in X^{0,\frac{1}{2}+\delta}$. Using the trilinear form, we show

$$\|\mathcal{N}(T(f,g),v)\|_{X^{\alpha-\frac{1}{2},-\frac{1}{2}+\delta}} \leq C\|f\|_{X^{0,\frac{1}{2}+\delta}}\|g\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}}.$$

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is false!

However, $h = T(e^{-it\partial_x^2}f, e^{-it\partial_x^2}f)$.

So the problematic term is in a trilinear form $\mathcal{N}(T(f,g),v)$ with $f,g,v \in X^{0,\frac{1}{2}+\delta}$. Using the trilinear form, we show

$$\|\mathcal{N}(T(f,g),v)\|_{X^{\alpha-\frac{1}{2},-\frac{1}{2}+\delta}} \leq C\|f\|_{X^{0,\frac{1}{2}+\delta}}\|g\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}}.$$

Derivative quadratic non-linearity

$$\left| \begin{array}{l} u_t + iu_{xx} = \langle \nabla \rangle^{\beta} [u^2] \\ u(0,x) \in H^{-\alpha}(\mathbf{R}). \end{array} \right. \tag{6}$$

- M. Christ: (6) is ill-posed when $\beta = 1$ for large data.
- A. Stefanov, '07: (6) is l.w.p. for small data in H^1 when $\beta = 1$.
- Bejenaru-Tataru, '09 and '10: (6) is l.w.p. in a weighted Sobolev space $H^{s,\gamma}$.
- Currently, there is no result for $\beta < 1$.

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Main result

Theorem

(6) *is l.w.p. in* $H^{-\alpha}$ *when* $\alpha + \beta < 1$, $\beta < 1/2$.

Also, we can write $u = e^{-it\partial_x^2}u(0) + h + w$ *where* $h \in L_t^\infty H_x^{\frac{1}{2}-\alpha} \cap L_t^2 H_x^{1-\alpha}$ *and* $w \in X_T^{-\frac{1}{2},\frac{1}{2}+\delta}$.

We can also write a Lipschitz property for a smoother space

$$\|u - v\|_{L_T^\infty H_x^{-\frac{1}{2}}} \leq C\|u_0 - v_0\|_{H^{-\frac{1}{2}}}$$

where $C := C(\|u_0\|_{H^{-\alpha}}, \|v_0\|_{H^{-\alpha}})$.