On sharp scattering threshold for the focusing critical NLS & NLKG equations

Presented by: **Slim Ibrahim**

University of Victoria, BC CANADA

Short Pulse workshop, Fields Institute, ON CANADA, May2-6, 2011

joint works with :
N. Masmoudi & K. Nakanishi
T. Akahori

Talk's plan

1. The NLKG (NLS) equations and our goals
2. What's known and our results
3. Very brief idea of the proof
4. The critical case : variational characterization

Consider the NLKG

$$\begin{cases} \ddot{u} - \Delta u + u - f'(u) = 0, & u : \mathbb{R}^{1+N} \to \mathbb{R}, \\ u(0,\cdot) = u_0(\cdot) \in H^1(\mathbb{R}^N), & \partial_t u(0,\cdot) = u_1(\cdot) \in L^2(\mathbb{R}^N), \end{cases} \tag{1}$$

or NLS

$$\begin{cases} 2i\dot{u} + \Delta u + f'(u) = 0, & u : \mathbb{R}^{1+N} \to \mathbb{C}, \\ u(0,\cdot) = u_0(\cdot) \in \dot{H}^1(\mathbb{R}^N) \end{cases} \tag{2}$$

where $N \geq 3$ and f typically has the form

$$f(u) = a_1|u|^{p_1+1} + \dots . a_k|u|^{p_k+1},$$

where $a_k \geq 0$ and $1 + \frac{4}{N} < p_1 < \cdot\cdot < p_k \leq \frac{N+2}{N-2} = 2^* - 1$, and for $N \geq 3$, the index $2^* := \frac{2N}{N-2}$ is related to the Sobolev embedding

$$\|f\|_{L^{2^*}} \leq C_* \|\nabla f\|_{L^2}, \quad \text{for all } f \in \dot{H}^1(\mathbb{R}^N). \tag{3}$$

Conservation laws :

1. The NLKG • Energy

$$E(u, \dot{u}) = \|\nabla u\|_{L^2}^2 + \|\dot{u}\|_{L^2}^2 + \|u\|_{L^2}^2 - 2F(u) = E(u_0, u_1).$$

   • Momentum $\int \nabla u \partial_t u \ dx = \int u_1 \nabla u_0 \ dx$

2. The NLS • Energy

$$E(u) = \|\nabla u\|_{L^2}^2 - 2F(u) = E(u_0)$$

   • Mass

$$\|u\|_{L^2}^2 = \|u_0\|_{L^2}^2$$

   • Momentum

$$\mathscr{I} \int \overline{u} \nabla u \ dx$$

Our goals are :

- Does the Cauchy problem (1) have a global/finite time solution
- When global, what is the asymptoctic of the solution as $t \to \infty$?

Theorem (Small data scattering, Strauss .., Ginibre-Velo)

There exists $\delta > 0$ such that for any $\|(u_0, u_1)\|_{H^1 \times L^2(\mathbb{R}^N)} \leq \delta$, the solution u to (1) with initial data (u_0, u_1) at $t = 0$ exists globally in time and scatters.
If on some interval I

$$\|u\|_{ST(I)} = +\infty,$$

then, u blows up in I.

- It is known that finite energy solutions to (1) satisfy the conservation of the energy. Recall that Cazenave 79' has shown that if $E(u, \dot{u})(0) < 0$, then the corresponding solution blows up in finite time.
- W. Strauss showed that if the initial data is sufficiently small in the energy space (hence with positive energy), then the corresponding solution is global in time.

Main goal : to quantify how large should/can be the initial data to have an optimal bound for the global wellposedness and scattering. A first attempt in this direction was done by Shatah.

- Shatah : Global wellposedness versus instability
- Ohta-Todorova : Strong instablity via blow up
- For the wave equation (Klein-Gordon without mass) : Kenig-Merles, $'06$: Sharp global existence and scattering threshold for the energy critical NLW.
- For the nonlinear Schrödinger equation : Kenig-Merle, $'05$: Similar result as for the wave, but for radially symmetric solutions.

To state our results, we start first with the subcritical case.

$$\text{Let}\quad K(u) = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 - \int_{R^N} u(x) f'(u(x))\, dx$$

and denote by Q the *ground state* i.e. the positive radially symmetric solution to

$$-\Delta Q + Q - f'(Q) = 0 \tag{4}$$

minimizing the energy J

$$J(u) := \frac{1}{2}\big(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 - 2F(u)\big), \tag{5}$$

under the constraint $K = 0$. Then consider

$$\mathscr{K}^{\pm} := \{(u_0, u_1) :\ E(u_0, u_1) < E(Q, 0) \text{and} \pm K(u_0) > 0\}.$$

Theorem (IMN, '10, NLKG : the subcritical case)

Let $N \geq 3$ and f be energy subcritical. Then

- *Any $(u_0, u_1) \in \mathscr{K}^-$ leads to a finite time blow up solution :*

$$\limsup_{t \to T^*} \|u(t,\cdot)\|_{H^1(\mathbb{R}^N)} = +\infty, \quad 0 < T^* < +\infty.$$

- *Any $(u_0, u_1) \in \mathscr{K}^+$ leads to a global solution which scatters at both $\pm\infty$: there exists a unique $(u_0^\pm, u_1^\pm)$ such that*

$$\lim_{t \to \pm\infty} \|(u, \dot{u})(t) - U_0(t)(u_0^\pm, u_1^\pm)\|_{(H^1 \times L^2)(\mathbb{R}^N)} = 0,$$

where $U_0(t)(u_0, u_1)$ is the KG propagator.

Remark

- *Hypothesis $p_1 > 1 + \frac{4}{N}$ is only required to prove the scattering part in Theorem 2.2.*
- *For NLS : Akahori-Kikushi-Nawa, '11 similar result but with a different K*
- *In the critical case, the elliptic equation does not have a solution in H^1 contrary to the massless equation.*
- *In the critical case, take*

$$f'(u) = |u|^{p^*-2}u. \tag{6}$$

Our results : the critical case

Let W be the positive, radially symmetric solution of

$$-\Delta W = W^{2^*-1}, \tag{7}$$

minimizing the "wave energy"

$$J_w := \frac{1}{2}\big(\|\nabla u\|_{L^2}^2 - \frac{2}{p^*}\|u\|_{L^{p^*}}^{p^*}\big).$$

Namely W is Talenti function. Similarly, define

$$K^w(u) = \|\nabla u\|_{L^2}^2 - \int_{R^N} |u(x)|^{p^*}\, dx = K - \|u\|_{L^2}^2$$

and

$$E^w(u,\dot{u})(t) := \frac{1}{2}\Big(\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\dot{u}(\cdot,t)\|_{L^2}^2 - \frac{2}{p^*}\|u\|_{L^{p^*}}^{p^*}\Big) = E - \|u\|_{L^2}^2.$$

Theorem (IMN, '10, NLKG : the critical case)

Define the sets

$$\mathscr{K}^{\pm} := \{(u_0, u_1) : \ E_N(u_0, u_1) \leq E_N^w(W, 0) \text{ and } \ \pm K(u_0) > 0\}.$$

Then

- *Data in* $(u_0, u_1) \in \mathscr{K}^-$ *leads to a finite time blow up solution.*
- *Data in* $(u_0, u_1) \in \mathscr{K}^+$ *leads to a global solution which scatter at both* $\pm\infty$.

Variational characterization, splitting & invariances

$$K_{\alpha,\beta}(u) = \partial_\lambda J(u^\lambda_{\alpha,\beta}), \quad \text{with} \quad u^\lambda_{\alpha,\beta}(x) = e^{\alpha\lambda} u(e^{\beta\lambda} x).$$

Then we have

Lemma

If $\alpha \geq 0, \quad 2\alpha - (N-2)\beta > 0, \quad 2\alpha - N\beta \geq 0$ *then*

$$\begin{aligned} m_{\alpha,\beta} :&= \inf\{J(u) \mid K_{\alpha,\beta}(u) = 0,\ u \neq 0\} \\ &\overset{J = c_{\alpha,\beta}K_{\alpha,\beta} + H_{\alpha,\beta}}{=} \inf\{H_{\alpha,\beta}(u) \mid K_{\alpha,\beta}(u) = 0,\ u \neq 0\} \\ &\overset{u(\frac{\cdot}{\lambda})}{=} \inf\{H_{\alpha,\beta}(u) \mid K_{\alpha,\beta}(u) \leq 0,\ u \neq 0\} \\ &\overset{(3)}{=} J(Q). \end{aligned}$$

Splitting & Invariance

Define the subsets

$$\mathscr{K}^{\pm}_{\alpha,\beta} := \{(u_0, u_1) : \ E_N(u_0, u_1) < E_N(Q, 0) \text{ and } \ \pm K_{\alpha,\beta}(u_0) > 0\}$$

Then,

Lemma

$$\mathscr{K}^{\pm}_{\alpha,\beta} = \mathscr{K}^{\pm}_{1,0}.$$

Moreover, $\mathscr{K}^{\pm}_{1,0}$ are invariant sets of $H^1 \times L^2$ under the NLKG flow. That is : if data are in $\mathscr{K}^{\pm}$ then solution stays there (as long as it exists).

The blow-up side.

Define $y(t) := \|u(t,\cdot)\|_{L^2}^2$. Then

$$\ddot{y} = 2\|\dot{u}\|_{L^2}^2 - 2K_{1,0}(u) > 0, \tag{8}$$

and using the energy conservation, the superquadratic growth of f and Cauchy-Schwarz we have

$$\begin{aligned} \ddot{y} &\geq (4+\varepsilon)\|\dot{u}\|_{L^2}^2 - 2(2+\varepsilon)E + \varepsilon(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2) \quad (9)\\ &\geq \frac{4+\varepsilon}{4}\frac{\dot{y}^2}{y}, \quad \text{for large } t. \quad (10) \end{aligned}$$

Since $\frac{4+\varepsilon}{4} > 1$, then necessarily y blows up in finite time, a contradiction.

Idea of the proof

The main task is to show the finiteness of a global space-time norm (Strichartz norm).

- Let $S_\delta := \{(u_0, u_1) \in \mathscr{K}^+$ such that $E(u_0, u_1) \leq \delta\}$, and define

  $$E^* := \sup\{\delta : \text{ any solution } u \text{ with data } (u_0, u_1) \in S_\delta \text{ scatters}\}$$

- If $\|(u_0, u_1)\|_{H^1 \times L^2} \ll 1$, then (u_0, u_1) is in $\mathscr{K}^+$, and from small data scattering, we have $\delta > 0$.
- Main task : $E^* \overset{?}{=} E(Q, 0)$
- By contradiction : If not, there's a sequence of solutions u_n with initial data $(\varphi_n, \psi_n) \in \mathscr{K}^+$ such that $E(\varphi_n, \psi_n) \to E^* < E(Q, 0)$, and $\|u_n\|_{ST(\mathbb{R})} = +\infty$.

Lemma

construction of a critical element

1. *There exists an initial data* $(u_{0,c}, u_{1,c}) \in \mathscr{K}^+$ *with* $E_N(u_{0,c}, u_{1,c}) = E^*$ *such that the corresponding solution* u_c *to* (1) *satisfies* $\|u_c\|_{ST_s(\mathbb{R})} = +\infty$.
2. *One can find a path* $x(t)$ *such that the set*
$$K := \{(u, \dot{u})(x - x(t), t), \quad t \in (0, \infty)\}$$
is precompact in $H^1 \times L^2$.
3. *For any* $\varepsilon > 0$, *there exists* $R_0(\varepsilon) > 0$ *and* $c(t) : \mathbb{R} \to \mathbb{R}^N$ *such that at any* $t \in \mathbb{R}$ *we have*
$$E_{R_0, c(t)}(t) \leq \varepsilon.$$

Lemma (Properties of the critical element)

Let u be a critical element. Then

$$M(u) := \int u_t \nabla u dx = 0$$

Moreover, let $R_0(\varepsilon) > 0$, $c(t) \in \mathbb{R}^N$ such that the exterior energy is small, and $\delta_\infty > 0$ be a lower bound of $\frac{K_{N/2,1}(u)}{\|u\|_{H^1}^2}$ If $0 < \varepsilon \ll \delta_\infty$ and $R \gg R_0(\varepsilon)$ then we have

$$|c(t) - c(0)| \leq R - R_0(\varepsilon),$$

for $0 < t < t_0$ till some $t_0 \gtrsim \delta_\infty R/\varepsilon$.

Proof's ideas :

- If not, then the Lorentz transform reduces the energy while it preserves the L^p_{tx} (or Strichartz) norms.
- For any $R > 0$ let $P_R(t) = \int x\phi_R e(u)dx$. Then

$$\dot{P}_R(t) = -NM(u) + \int [N(1 - \phi_R) + (r\partial_r)\phi_R] u_t \nabla u,$$

  and if u is a critical element, the first term disappears by the above lemma.Since $M = 0$, then we have

$$|\dot{P}_R(t)| \lesssim E_R(t) \leq \varepsilon.$$

- Estimate P_R in the direction of $c(t)$.

step II : Extinction of the critical element

$$V_R(t) = \langle \phi_R u_t | r\partial_r u + \frac{N}{2} u \rangle.$$

Then we have for any solution u (local virial identity),

$$\begin{aligned} \dot{V}_R(t) &\leq -K_{N/2,1}(u(t)) + CE_R(t) \\ &\leq -\delta_2 + CE_R(t). \end{aligned}$$

Contradiction : Now we choose positive $\varepsilon \ll \delta_2\delta_\infty$ and $R \gg R_0(\varepsilon)$. Then by Lemma 5.2 there exists $t_0 \sim \delta_\infty R/\varepsilon$ such that $E_R(t) \leq \varepsilon$ for $0 < t < t_0$. Then from local viriel identity we have

$$-V_R(t_0) + V_R(0) \geq \delta_2 t_0 - C\varepsilon t_0 \gtrsim \delta_2 t_0 \sim \frac{\delta_2\delta_\infty}{\varepsilon} R, \tag{11}$$

while the left hand side is dominated by RE_∞, which is a contradiction when $\varepsilon/\delta_2\delta_\infty$ is sufficiently small.

Construction of the critical element

Recall that we are arguing by contradiction : assume there's a sequence of solutions u_n of

$$(\Box + 1)u = f(u), \quad u : \mathbb{R}^{1+d} \to \mathbb{R} \tag{12}$$

with initial data $(\varphi_n, \psi_n) \in \mathscr{K}^+$ such that $E_N(\varphi_n, \psi_n) \to E^* < E(Q)$, and $\|u_n\|_{ST(\mathbb{R})} = +\infty$. For u we associate $\vec{u} = \langle \nabla \rangle u - i\dot{u}$. Then

$$(\Box + 1)u = 0 \iff (i\partial_t + \langle \nabla \rangle)\vec{u} = 0, \quad E_0(u) = \|\vec{u}\|^2_{L^2_x}, \tag{13}$$

$$(\Box + 1)u = f(u) \iff (i\partial_t + \langle \nabla \rangle)\vec{u} = f(\langle \nabla \rangle^{-1} \operatorname{Re} \vec{u}). \tag{14}$$

Using the profile decomposition, we can write $\vec{u}_n(0) = \vec{v}_n(0)$ with

$$\begin{aligned} \vec{v}_n &= \sum_{k=1}^{K} \vec{v}_n^k + R_n^K, \quad \text{and} \\ \vec{v}_n^k &= e^{i(t-t_n^k)\langle\nabla\rangle} T_n^k \psi^k, \quad T_n^k f = (h_n^k)^{-d/2} f((x-c_n^k)/h_n^k), \end{aligned} \tag{15}$$

with orthogonality and smallness of R_n^K in $L^\infty \dot{B}^{-d/2}_{\infty,\infty}$.
To each sequence $\{\vec{v}_n^k\}_{n\in\mathbb{N}}$, we associate a nonlinear profile : a sequence $\{\vec{u}_n^k\}_{n\in\mathbb{N}}$ satisfying :

$$\vec{u}_n^k(t) = U(t-t_n^k) T_n^k \varphi_\infty^k, \quad \text{and} \quad \|(\vec{u}_n^k - \vec{v}_n^k)(0)\|_{L^2} = o(1), n \to \infty.$$

Hence

$$\vec{u}_n(0) = \sum_{k=1}^{K} U(-t_n^k) T_n^k \varphi_\infty^k + R_n^K + o(1)$$

$$\vec{u}_n(t) = U(t) \sum_{k=1}^{K} U(-t_n^k) T_n^k \varphi_\infty^k + R_n^K + o(1)$$

$$\vec{u}_n(t) = \sum_{k=1}^{K} U(t - t_n^k) T_n^k \varphi_\infty^k + e_n^K(t).$$

But $\sum_{k=1}^{K} U(t - t_n^k) T_n^k \varphi_\infty^k$ is an approximate solution which scatter, so by perturbation argument, does the actual solution $\vec{u}_n$. A contradiction.

Concluding remarks

Remark

Under the assumption $E^w(u_0, u_1) < E^w(W, 0)$, K-M result states as follows

$$\pm\|\nabla u_0\|_{L^2} < \pm\|\nabla W\|_{L^2} \Rightarrow \textit{Scattering/finite time bup}$$

Note that

- $$\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2} \Longleftrightarrow K^w_{1,0} > 0$$
- $$K^w_{\alpha,\beta} = c_{\alpha,\beta} K^w_{1,0}.$$

Remark

For the NLS, the only functional which is relevant in both blow up and scattering is $K_{N/2,1}$. *Indeed, if*

$$\mathscr{V} = \int |x|^2 |u|^2 \, dx,$$

then

$$\frac{d^2}{dt^2}\mathscr{V} = K_{N/2,1}$$

from which one derives

- $\mathscr{V}(t) \lesssim \mathscr{V}(0) + t - t^2$ *if* $\sup_{t \geq 0} K_{N/2,1} < 0$
- $\mathscr{V}(0) - t + t^2 \lesssim \mathscr{V}(t)$ *if* $\inf_{t \geq 0} K_{N/2,1} > 0$

A typical model

$$\Delta u + \omega u - \mu |u|^{p-1} u - |u|^{\frac{4}{N-2}} u = 0$$

with $\omega, \mu > 0$, and $1 + \frac{4}{N} < p < \frac{N+2}{N-2}$. Let J_ω, T_λ and T'_λ be the L^2 and H^1 scalings, respectively. Recall that

$$K(u) = K_{\frac{N}{2},1}(u) = 2\|\nabla u\|_{L^2}^2 - \mu \frac{N(p-1)}{p+1} - 2\|u\|_{L^{2^*}}^{2^*}.$$

$$\begin{aligned} I_\omega(u) &= J_\omega(u) - \frac{2}{N(p-1)} K_{\frac{N}{2},1}(u) \\ &= \omega\|u\|_{L^2}^2 + \frac{N(p-1)-4}{N(p-1)}\|\nabla u\|_{L^2}^2 + \frac{4-(N-2)(p-1)}{N(p-1)}\|u\|_{L^2}^{2^*} \end{aligned}$$

Recall also

$$m_\omega = \inf\{J_\omega(u) : u \in H^1 - 0, K(u) = 0\},$$

$$\tilde{m}_\omega = \inf\{I_\omega(u) : u \in H^1 - 0, K(u) = 0\},$$

$$\sigma = \inf\{\|\nabla u\|_{L^2}^2 : u \in H^1, \|u\|_{L^{2^*}} = 1\},$$

$$\sigma^{N/2} = \|T'_{\varepsilon^{-1}} \nabla W\|_{L^2}^2 = \|T'_{\varepsilon^{-1}} W\|_{L^{2^*}}^{2^*}$$

Theorem (Akahori-Ibrahim '11)

Assume that $N \geq 4$, $\omega > 0$ and $\mu > 0$. Then there exists a minimizer for m_ω. Moreover,

$$m_\omega = \tilde{m}_\omega > 0$$

Theorem (Akahori-Ibrahim '11)

Assume that $N = 3$, $\omega > 0$ and $0 < \mu << 1$, then there is no a minimizer for m_ω. Also, there is no ground state.

Remark

The equation

$$\Delta u + \omega u - \mu |u|^{p-1} u - |u|^{\frac{4}{N-2}} u = 0$$

does not have a solution. Indeed,

$$0 = K_{0,1} - \frac{1}{2} K(u) = \omega \| Q \|_{L^2}^2 + (1 - \frac{N(p-1)}{2(p+1)}) \omega \| Q \|_{L^{p+1}}^{p+1}$$

Sketch of the proofs : Existence of a ground state

1. $m_\omega = \tilde{m}_\omega$
2. For $n \geq 4$ we have $m_\omega < \frac{2}{d}\sigma^{\frac{d}{2}}$: Take $W_\varepsilon = T'_{\varepsilon^{-1}}$ and choose $0 < \lambda_\varepsilon < 1$ such that

$$K(T_{\lambda_\varepsilon} W_\varepsilon) = 0$$

...computation $\lambda_\varepsilon = 1 - C_0 \varepsilon^{N - \frac{(N-2)(p+1)}{2}} + O(\varepsilon^{N-2})$. Then note that $\tilde{m}_\omega \leq I_\omega(T_{\lambda_\varepsilon} W_\varepsilon)$

3. The above bound prevents concentration when getting a minimizer.

Sketch of the proofs : $N=3$, non existence a ground state

Suppose u is a minimizer for m_ω. Then u^* is a minimizer for $\tilde{m}_\omega$ solving the static NLKG. Set $\Phi(r) = ru^*(r)$, and take $g(r)$ smooth and multiply by gu :

1. $\frac{1}{4}\int_0^\infty (g''' - 4g)\Phi^2 \, dr =$
   $\mu \int_0^\infty (\frac{p-1}{p+1} g - \frac{p+3}{2(p+1)} rg') r^{-p} |\Phi|^{p+1} + \frac{2}{3}\int_0^\infty (g - rg') r^{-5} |\Phi|^6$
2. Choose $g(r) = \frac{1-e^{-2r}}{2}$ and note that

$$|\int_0^\infty (\frac{p-1}{p+1} g - \frac{p+3}{2(p+1)} rg') r^{-p} |\Phi|^{p+1}| \lesssim \|u\|_{L^{p+1}}^{p+1} \lesssim \|u\|_{H^1}^{p+1} \lesssim 1$$

Thank you

Lemma

The following two subsets

$$\mathscr{K}^{\pm} := \{(u_0, u_1) : \ E_N(u_0, u_1) < E_N^w(Q_w, 0) \text{ and } \ \pm K_{1,0}(u_0) > 0\}$$

are invariant sets of $H^1 \times L^2$ under the NLKG flow. That is : if data are in $\mathscr{K}^{\pm}$ then solution stays there (as long as it exists).

Thank you

Let $\vec{v}_n$ be a sequence of free solutions in the form

$$(i\partial_t + \langle\nabla\rangle)\vec{v}_n = 0, \quad \vec{v}_n(p_n) = T_n\psi. \tag{16}$$

We define a sequence $\vec{u}_n(t,x)$, and then $\varphi_n(x)$, by the equations

$$(i\partial_t + \langle\nabla\rangle)\vec{u}_n = f(\langle\nabla\rangle^{-1}\operatorname{Re}\vec{u}_n), \quad \vec{u}_n(0) = \vec{v}_n(0), \quad \vec{u}_n(p_n) = T_n\varphi_n. \tag{17}$$

Next we define $\widehat{v}_n$ and $\widehat{u}_n$ by undoing the transforms

$$\vec{v}_n = T_n\widehat{v}_n((t-p_n)/h_n), \quad \vec{u}_n = T_n\widehat{u}_n((t-p_n)/h_n). \tag{18}$$

Then they satisfy

$$\begin{aligned}&(i\partial_t + \langle\nabla\rangle_n)\widehat{v}_n = 0, \quad \widehat{v}_n(0) = \psi,\\ &(i\partial_t + \langle\nabla\rangle_n)\widehat{u}_n = f(\langle\nabla\rangle_n^{-1}\operatorname{Re}\widehat{u}_n), \quad \widehat{u}_n(0) = \varphi_n, \quad \widehat{u}_n(-p_n/h_n) = \widehat{v}_n(-\end{aligned} \tag{19}$$

where we denote $\langle\nabla\rangle_n = \sqrt{-\Delta + h_n^2}$.

Let $\gamma_n = -p_n/h_n$. Then we have equivalent integrated equations

$$\widehat{v}_n = e^{it\langle\nabla\rangle_n}\varphi, \quad \widehat{u}_n = \widehat{v}_n - i\int_{\gamma_n}^{t} e^{i(t-s)\langle\nabla\rangle_n} f(\langle\nabla\rangle_n^{-1} \operatorname{Re}\widehat{u}_n)\,ds. \tag{20}$$

Extracting subsequence, we may assume that

$$h_n \to \exists h_\infty \in [0,1], \quad \gamma_n \to \exists \gamma_\infty \in [-\infty,\infty]. \tag{21}$$

Then the limit equations are naturally given by

$$\widehat{v}_\infty = e^{it\langle\nabla\rangle_\infty}\varphi, \quad \widehat{u}_\infty = \widehat{v}_\infty - i\int_{\gamma_\infty}^{t} e^{i(t-s)\langle\nabla\rangle_\infty} f(\langle\nabla\rangle_\infty^{-1} \operatorname{Re}\widehat{u}_\infty)\,ds, \tag{22}$$

and $\varphi_n \to \varphi_\infty = \widehat{u}_\infty(0)$.
We can call $\widehat{u}_\infty$ or φ_∞ the nonlinear profile.