Constant-Frequency Waves

John Hunter, UC Davis

Fields Institute, May 2, 2011

Outline

- 1. Lower order dispersion
- 2. Constant-frequency waves
- 3. Burgers-Hilbert equation
- 4. Asymptotic equation
- 5. Surface waves on vorticity discontinuities

1. Lower-order dispersion

Model equation

Inviscid Burgers equation with linear dispersion

$$u_t + \left(\frac{1}{2}u^2\right)_x = Lu_x$$

L self-adjoint, linear, translation-invariant spatial operator with symbol $\hat{L}$. Advective nonlinearity typical for fluid problems. Linearized dispersion relation:

$$\omega = W(k) \qquad W(k) = -k\hat{L}(k)$$

Whitham (1974) observed can get arbitrary linearized dispersion relation by choosing L appropriately, but have to be careful how it combines with nonlinearity.

Long and short waves

Weakly dispersive long waves: $\omega = k^3$, $L = \partial_x^2$, KdV eq.

$$u_t + \left(\frac{1}{2}u^2\right)_x = u_{xxx}$$

Weakly dispersive short waves: $\omega = 1/k$, $L = \partial_x^{-2}$, Ostrovsky-Hunter eq.

$$\partial_x \left[u_t + \left(\frac{1}{2}u^2\right)_x\right] = u$$

Weak long-short wave dispersion: Ostrovsky eq.

$$\partial_x \left[u_t + \left(\frac{1}{2}u^2\right)_x + \sigma u_{xxx}\right] = u \qquad \sigma = \pm 1$$

Lower-order dispersion

If linear dispersive operator $L\partial_x$ in

$$u_t + \left(\frac{1}{2}u^2\right)_x = Lu_x$$

is lower order as for short waves, then can't stop wave-breaking in sufficiently steep solutions (unlike KdV or Benjamin-Ono). Wave-breaking effects and long time dynamics in such equations can be subtle. Fornberg and Whitham (1978), Shefter and Rosales (1999), Stefanov, Pelinovsky, Sakovich, ...

Resonant media

Example: Sound waves in bubbly fluid. Rybak and Skrynnikov (1989), Tan (1991), H. (1995). Ignoring dissipation, get

$$\left(\partial_x^2 + 1\right)\left[u_t + \left(\frac{1}{2}u^2\right)_x\right] = u_x$$

Operator $(\partial_x^2 + 1)$ not invertible, unlike $(-\partial_x^2 + 1)$
Long waves: $(\partial_x^2 + 1)^{-1} \sim 1 - \partial_x^2$, get KdV

$$u_t - u_x + \left(\frac{1}{2}u^2\right)_x + u_{xxx} = 0$$

Short waves $\partial_x^2 + 1 \sim \partial_x^2$, get Ostrovsky-Hunter (after integration)

$$\partial_x\left[u_t + \left(\frac{1}{2}u^2\right)_x\right] = u$$

Pulse solution

Write resonant-media equation as evolution equation

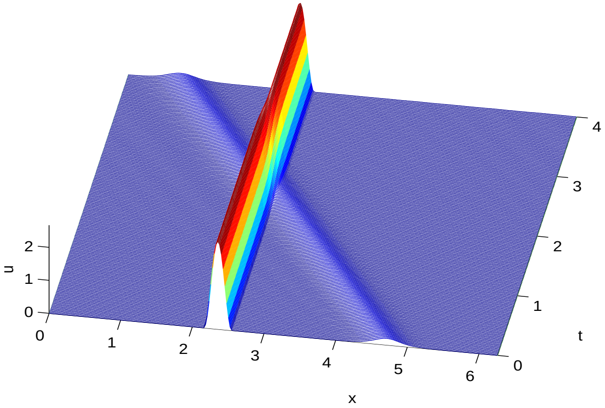
$$u_t + \mathbf{P}\left(\frac{1}{2}u^2\right)_x = (\partial_x^2 + 1)^{-1} u_x \qquad \mathbf{P}u = 0$$

$\mathbf{P}$ = orthogonal projection onto $\langle \cos x, \sin x \rangle^{\perp}$
Has exact stationary pulse solution with compact support

$$u(x,t) = \begin{cases} (8/3)\cos^2(x/4) & |x| \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Also have long-wave KdV-type solitons.

Compact pulse–KdV soliton interaction (rescaled space variable)



Attempted bi-Hamiltonian form

To investigate integrability of

$$u_t + \left(\frac{1}{2}u^2\right)_x = Lu_x$$

might try writing equation as bi-Hamiltonian system

$$(1) \qquad u_t = \partial_x \left[\frac{\delta \mathcal{H}_1}{\delta u}\right] \qquad \mathcal{H}_1 = \int \left(\frac{1}{2}uLu - \frac{1}{6}u^3\right)\, dx$$

$$(2) \qquad u_t = \mathbf{J}(u)\left[\frac{\delta \mathcal{H}_0}{\delta u}\right] \qquad \mathcal{H}_0 = \int \frac{1}{2}u^2\, dx$$

$$\mathbf{J}(u) = L\partial_x - \frac{1}{3}\left(u\partial_x + \partial_x u\right)$$

Jacobi identity

Is linearly perturbed Lie-Poisson operator $\mathbf{J}(u)$ Hamiltonian?

$$u_t = \mathbf{J}(u)\left[\frac{\delta \mathcal{H}_0}{\delta u}\right] \qquad \mathcal{H}_0 = \int \frac{1}{2}u^2\,dx$$

$$\mathbf{J}(u) = L\partial_x - \frac{1}{3}\left(u\partial_x + \partial_x u\right)$$

Theorem

The only self-adjoint translation invariant operator L, whose symbol satisfies a mild regularity condition, such that $\mathbf{J}(u)$ satisfies the Jacobi identity is $L = a + b\partial_x^2$, corresponding to the KdV equation.

Nevertheless, Ostrovsky-Hunter equation $L = \partial_x^{-2}$ is completely integrable (Vakhnenko and Parkes, 2002)

CH and DP equations

Lower-order nonlinear dispersive terms. Includes completely integrable DP and CH equations

Degasperis-Procesi (DP): perturbation of Burgers

$$\partial_x^2 \left[u_t + \left(\frac{1}{2} u^2 \right)_x \right] = u_t + \left(2u^2 \right)_x$$

Camassa-Holm (CH): perturbation of Hunter-Saxton (HS)

$$\partial_x^2 \left[u_t + \left(\frac{1}{2} u^2 \right)_x - \frac{1}{2} u_x^2 \right] = u_t + \left(\frac{3}{2} u^2 \right)_x$$

2. Constant-Frequency Waves

Constant-frequency waves

Waves whose linearized frequency ω is independent of wavenumber k e.g.

$$\omega^2 = 1$$

Linearized wave motion consists of decoupled simple harmonic oscillators at different spatial locations

$$u_t = v$$
$$v_t = -u$$

For systems that aren't invariant under spatial reflection get single mode (ω odd function of k) e.g.

$$\omega = \operatorname{sgn} k = \begin{cases} 1 & k > 0 \\ 0 & k = 0 \\ -1 & k < 0 \end{cases}$$

Constant-frequency waves are nondispersive

Single-mode constant-frequency wave with

$$\omega = \omega_0 \operatorname{sgn} k$$

is nondispersive with zero group velocity ($\omega'' = 0$ for $k \neq 0$)
Dominant weak nonlinearity is cubic, and get resonant four-wave interactions among many spatial harmonics (unlike dispersive waves)

$$\omega_0 + \omega_0 = \omega_0 + \omega_0, \qquad k_1 + k_2 = k_3 + k_4 \qquad k_j > 0$$

Need two spatial harmonics to start spectral cascade

Nondispersive hyperbolic waves

Constant-frequency waves nondispersive but qualitatively different from nondispersive hyperbolic waves with dispersion relation

$$\omega = c_0 k$$

For these waves, dominant weak nonlinearity is quadratic and get resonant three-wave interactions among many spatial harmonics

$$\omega_1 + \omega_2 = \omega_3, \qquad k_1 + k_2 = k_3$$

Will consider some model equations for constant-frequency waves

Model equation I

Pressureless 1-d rotating shallow water equations

$$u_t + uu_x = v$$
$$v_t + uv_x = -u$$

Same characteristic in each equation. Simple harmonic oscillations in characteristic coordinates with $dx/dt = u$

$$\frac{du}{dt} = v, \qquad \frac{dv}{dt} = -u$$

Global smooth time-periodic solutions if transformation from characteristic to spatial coordinates remains smooth over period (Liu and Tadmor, 2004).

Model equation II

Coupled inviscid Burgers equations e.g. in gas dynamics describes resonant reflection of sound waves off an entropy wave (Majda, Rosales, Schonbeck, 1983).

$$
\begin{aligned}
u_t + \left(\frac{1}{2}u^2\right)_x &= v \\
v_t + \left(\frac{1}{2}v^2\right)_x &= -u
\end{aligned}
$$

Different characteristics in each equation.

Asymptotic equation

Asymptotic solution for weakly nonlinear constant-frequency wave (arbitrary spatial profile)

$$u(x,t;\varepsilon) \sim \varepsilon\psi(x,\varepsilon^2 t)e^{-it} + \text{c.c.}$$
$$v(x,t;\varepsilon) \sim -i\varepsilon\psi(x,\varepsilon^2 t)e^{-it} + \text{c.c.}$$

For model equation II, complex-valued amplitude $\psi(x,\tau)$ satisfies degenerate quasilinear Schrödinger equation

$$2i\psi_\tau + \left(|\psi|^2\,\psi_x\right)_x = 0$$

Free quantum-mechanical particle with mass inversely proportional to probability density. Dispersive analog of porous medium equation

$$u_t = (u^2 u_x)_x$$

Model equation III

Burgers-Hilbert equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = \mathbf{H}[u]$$

where $\mathbf{H}$ is spatial Hilbert transform (singular integral operator)

$$\mathbf{H}[e^{ikx}] = -i(\operatorname{sgn} k)e^{ikx} \qquad \mathbf{H}[u] = \left(\text{p.v.}\frac{1}{\pi x}\right) * u$$

Consists of inviscid Burgers equation with lower-order, conservative linear integral term. Linearized dispersion relation is $\omega = \operatorname{sgn} k$. Crucial difference from previous integro-differential equations with lower-order linear terms is that lower-order Hilbert transform term is nondispersive

3. Burgers-Hilbert equation

Burgers-Hilbert equation

Inviscid Burgers-Hilbert equation for $u(x, t)$

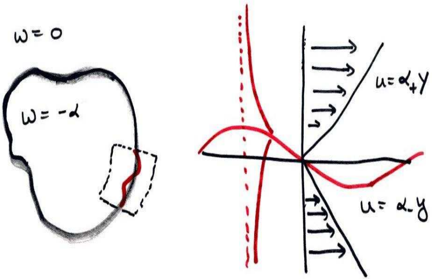
$$u_t + \left(\frac{1}{2}u^2\right)_x = \mathbf{H}[u]$$

Conservation law + singular integral operator (seems to makes global theory of weak solutions tricky)

Dimensional analysis shows this is model equation for constant-frequency, Hamiltonian surface waves: Effective equation for surface waves on vorticity discontinuity in 2-d incompressible Euler equations. Biello and H. (2010). Marsden and Weinstein (1983) wrote down equation (didn't analyze it or consider resonant cubic nonlinearities)

Vorticity Discontinuities

Planar discontinuity in vorticity in incompressible, inviscid fluid flow. Example: local behavior on boundary of vortex patch.



Surface waves propagate along discontinuity, decay exponentially into the interior. Rayleigh (1895). Location $y = \eta(x, t)$ of discontinuity described by Burger-Hilbert eq.

Burgers-Hilbert equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = \mathbf{H}[u]$$

If $v = \mathbf{H}[u]$, then (u, v) satisfy ($\mathbf{H}^2 = -\mathbf{I}$)

$$u_t + \partial_x \left(\frac{1}{2}u^2\right) = v$$

$$v_t + |\partial_x| \left(\frac{1}{2}u^2\right) = -u$$

where $|\partial_x| = \mathbf{H}\partial_x$, $|\partial_x|\left[e^{ikx}\right] = |k|e^{ikx}$. Simple harmonic oscillators with nonlocal spatial nonlinearity. Seems to be intrinsically nonlinear — can't be reduced to local equation by linear transformation (unlike *e.g.* Constantin-Lax-Majda eq.)

Linearized equation

Linearization of equation gives spatially distributed simple harmonic oscillators

$$u_t = v$$
$$v_t = -u$$

where velocity

$$v = \mathbf{H}[u]$$

is spatial Hilbert transform of displacement u. Otherwise oscillators are spatially decoupled

Linearized IVP

Initial-value problem for linearized equation

$$u_t = \mathbf{H}[u]$$
$$u(x,0) = f(x)$$

Solution is ($\mathbf{H}^2 = -\mathbf{I}$)

$$u(x,t) = f(x)\cos t + g(x)\sin t \qquad g = \mathbf{H}[f]$$

Solution oscillates in time between two spatial profiles $f(x)$ and $g(x)$ where g is Hilbert transform of f

Steep slope in one phase gives sharp finger, or filament, in other phase

Strongly and weakly nonlinear regimes

Suppose $u_x(x,0) = O(\varepsilon)$ in Burgers-Hilbert equation

- Strongly nonlinear regime $\varepsilon \gg 1$:

$$u_t + \left(\frac{1}{2}u^2\right)_x = \mathbf{H}[u]$$

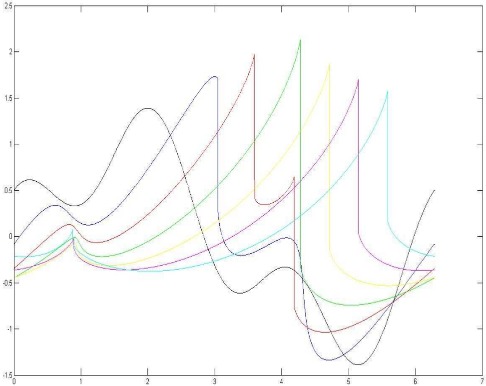
  Perturbation of inviscid Burgers equation. Singularity timescale $T_s = O\left(\varepsilon^{-1}\right) \ll 1$.

- Weakly nonlinear regime $\varepsilon \ll 1$:

$$u_t + \left(\frac{1}{2}u^2\right)_x = \mathbf{H}[u]$$

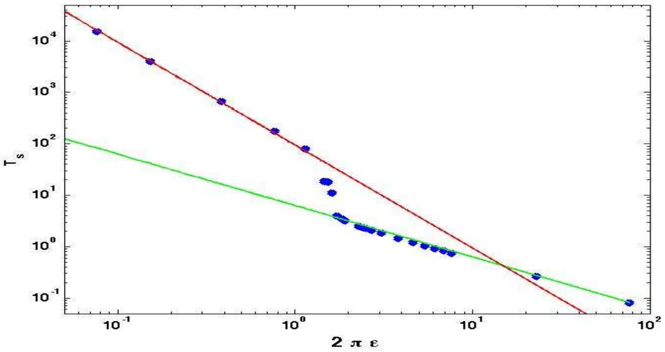
  Solutions oscillate in time and nonlinearity is effectively cubic (compression in one phase canceled by expansion in the other). Singularity timescale $T_s = O\left(\varepsilon^{-2}\right) \gg 1$.

Numerical solution for large amplitudes



Infinite derivative at shocks due to Hilbert transform, but otherwise qualitatively similar to *e.g.* Ostrovsky-Hunter eq.

Singularity Time T_s vs. Amplitude ε



Green line = quadratic Burgers equation asymptotics; Red line = cubic asymptotics; Blue dots = numerical solution

Complex form of linearized solution

Consider weakly nonlinear asymptotics for Burgers-Hilbert. Linearized equation

$$u_t = \mathbf{H}[u]$$

has solution in complex form

$$u(x,t) = \psi(x)e^{-it} + \psi^*(x)e^{it}, \qquad \psi(x) = \int_0^\infty \hat{\psi}(k)e^{ikx}\,dk$$

where ψ has only positive wavenumber components

$$\mathbf{P}\psi = \psi \qquad \mathbf{P} = \frac{1}{2}\left(\mathbf{I} + i\mathbf{H}\right)$$

$\mathbf{P}$ = projection onto positive wavenumber components. Solution oscillates between $\Re\psi$ and $\Im\psi$.

Asymptotic equation

Weakly nonlinear solutions of Burgers-Hilbert equation

$$u(x,t;\varepsilon) = \varepsilon\psi(x,\varepsilon^2 t)e^{-it} + \text{c.c.} + O(\varepsilon^2)$$

where ψ has only positive wavenumber components

$$\mathbf{P}\psi = \psi, \qquad \mathbf{P} = \frac{1}{2}\left(\mathbf{I} + i\mathbf{H}\right)$$

Equation for $\psi(x,\tau)$ is nonlocal, cubically quasilinear, singular integro-differential equation

$$\psi_\tau = \mathbf{P}\partial_x\left(\psi\,|\partial_x|\,n - n\,|\partial_x|\,\psi\right) \qquad n = |\psi|^2$$

where $|\partial_x| = \mathbf{H}\partial_x$ has symbol $|k|$

Hamiltonian structure

Hamiltonian form of Burgers-Hilbert equation

$$u_t = -\partial_x \left(\frac{\delta \mathcal{H}}{\delta u} \right)$$

$$\mathcal{H}(u) = \int \left\{ \frac{1}{2} u \, |\partial_x|^{-1} u + \frac{1}{6} u^3 \right\} dx$$

Asymptotic equation has Hamiltonian form

$$\psi_t = -\partial_x \left[\frac{\delta \mathcal{H}}{\delta \psi^*} \right] \qquad \mathbf{P}\psi = \psi \qquad n = \psi^* \psi$$

$$\mathcal{H}(\psi, \psi^*) = \int \left\{ \frac{i}{4} n \left(\psi \psi_x^* - \psi^* \psi_x \right) - \frac{1}{2} n \, |\partial_x| \, n \right\} dx$$

Near-identity transformation

Consider Burgers-Hilbert equation with scaled nonlinearity

$$u_t + \varepsilon u u_x = \mathbf{H}[u]$$

Asymptotic equation also follows from near-identity transformation that eliminates nonresonant $O(\varepsilon)$ cubic terms from Burgers-Hilbert Hamiltonian and retains resonant $O(\varepsilon^2)$ quartic terms

Rigorous analysis seems difficult because remainder terms are singular perturbations that involve higher-order derivatives

Lower-order linear term $\mathbf{H}[u]$ give poor control on higher-order nonlinearities uu_x

Near-identity transformation

Near-identity transformation

$$v = u + \frac{1}{2}\varepsilon \left|\partial_x\right| \left(h^2\right) + \frac{1}{2}\varepsilon^2 \left[\partial_x^2 \left(h^2 u\right) - \partial_x \left(h \left|\partial_x\right| \left(u^2\right)\right)\right]$$

where $h = \mathbf{H}[u]$ gives

$$v_t + \varepsilon^2 \partial_x \left[\frac{1}{6}\left|\partial_x\right|(v^3) - \frac{1}{2} v \left|\partial_x\right| (v^2) + \frac{1}{2} v^2 \left|\partial_x\right| v\right] = H[v] + \varepsilon^3 R(u;\varepsilon)$$

Real form of asymptotic equation for ψ if $O(\varepsilon^3)$ terms neglected
Pick up extra derivative for every power of u that is eliminated
e.g. $R(u;\varepsilon) \approx \partial_x^3 u^4$

Compare with KdV

$$u_t + \varepsilon u u_x = u_{xxx}$$

Near-identity transformation

$$v = u - \frac{1}{6}\varepsilon \left(\partial_x^{-1} u\right)^2$$

gives

$$v_t - \frac{1}{6}\varepsilon^2 v^2 (\partial_x^{-1} v) = v_{xxx} + \varepsilon^3 R(u;\varepsilon)$$

where $R(u;\varepsilon) \approx \partial_x^{-3} u^4$. Error terms are smoother *c.f.* Craig, Schneider, Wayne, Germain, Masmoudi, Shatah, Wu, ... for KdV, water wave asymptotics etc.

4. Asymptotic equation

Short-time existence of smooth solutions

Asymptotic equation

$$\psi_t = \mathbf{P}\partial_x \left[\psi\,|\partial_x|\,n - n\,|\partial_x|\,\psi\right] \qquad n = |\psi|^2 \qquad \mathbf{P} = \frac{1}{2}(\mathbf{I} + \mathbf{H})$$

Theorem

(Ifrim + H.) The asymptotic equation has unique, smooth spatially-periodic solution for time $T = T(\|\psi_0\|_{H^2})$

$$\psi \in C\left([-T, T], H^2(\mathbb{T})\right) \cap C^1\left([-T, T], H^1(\mathbb{T})\right)$$

Commutator form

Asymptotic equation

$$\psi_\tau = \mathbf{P}\partial_x \left[\psi \left|\partial_x\right| n - n \left|\partial_x\right| \psi\right] \qquad n = |\psi|^2 \qquad \mathbf{P} = \frac{1}{2}\left(\mathbf{I} + \mathbf{H}\right)$$

Commutator form

$$\psi_t = \partial_x \left[\psi, \left[\psi, \left|\partial_x\right|\right]\right] \psi^*$$

As written, equation ‘looks’ second-order in spatial derivatives, but it’s ‘really’ lower-order due to cancelation, so get good energy estimates

Minimal spectral growth

Spectral form of equation (spatially periodic)

$$\hat{\psi}_t(k_1, t) = ik_1 \sum_{k_1+k_2=k_3+k_4} \Lambda(k_1, k_2, k_3, k_4)\hat{\psi}^*(k_2, t)\hat{\psi}(k_3, t)\hat{\psi}(k_4, t)$$

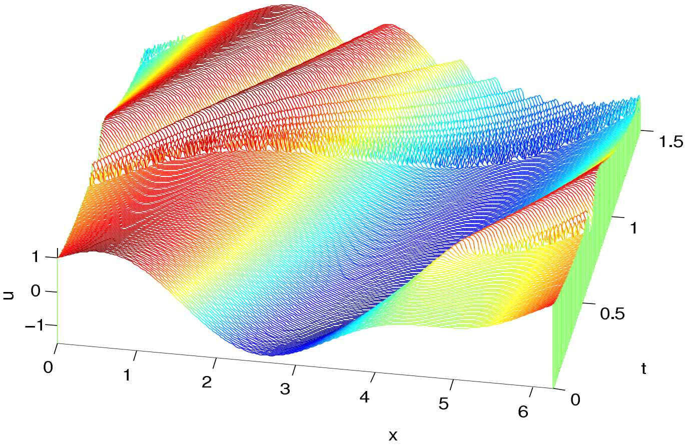
where interaction coefficient Λ for $k_3 + k_4 - k_2 \to k_1$

$$\Lambda(k_1, k_2, k_3, k_4) = 2 \min(k_1, k_2, k_3, k_4)$$

Hamiltonian property: $\Lambda(k_1, k_2, k_3, k_4)$ symmetric

Key point: Value of Λ bounded by *lowest* wavenumber: prevents loss of derivatives by nonlinear amplification of high wavenumbers

Singularity formation

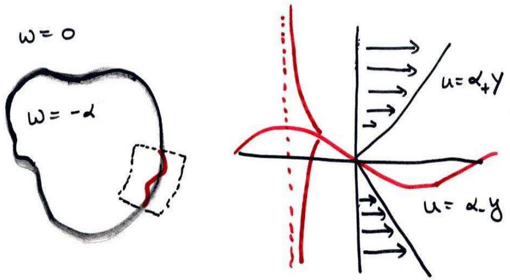


Solution for real part $u = \Re\psi$ of $\psi_t = \mathbf{P}\partial_x \left[\psi\,|\partial_x|\,n - n\,|\partial_x|\,\psi\right]$

5. Vorticity discontinuities

Vorticity Discontinuities

Planar discontinuity in vorticity in incompressible, inviscid fluid flow. Example: local behavior on boundary of vortex patch.



Surface waves propagate along discontinuity, decay exponentially into the interior.

Surface waves

- Vorticity discontinuity is linearly stable (unlike vortex sheet, where velocity is discontinuous)
- Surface waves propagate along discontinuity
- Only parameters are shear rates α_+, α_-, equal to minus vorticity, which have dimensions of frequency
- Surface waves have constant frequency

$$\omega = \omega_0 \operatorname{sgn}(k) \qquad \omega_0 = \left(\frac{\alpha_+ - \alpha_-}{2}\right)$$

- Problem invariant under simultaneous time reversal and spatial reflection ($t \mapsto -t$, $x \mapsto -x$) but not under spatial reflection ($x \mapsto -x$) alone

Asymptotic equation

Location $y = \eta(x, t; \varepsilon)$ of discontinuity has asymptotic solution

$$\eta(x, t; \varepsilon) \sim \varepsilon\psi\left(x, \varepsilon^2 t\right) e^{-i\omega_0 t} + \varepsilon\psi^*(x, \varepsilon^2 t)e^{i\omega_0 t}$$

$\psi(x, \tau)$ satisfies exactly the *same* asymptotic equation as one from Burgers-Hilbert equation

$$\psi_\tau = \gamma_0 \mathbf{P}\partial_x \left[\psi\, |\partial_x|\, n - n\, |\partial_x|\, \psi\right] \qquad n = |\psi|^2$$

$$\gamma_0 = \frac{\alpha_+^2 + \alpha_-^2}{\alpha_+ - \alpha_-}$$

Effective equation

Burgers-Hilbert equation

$$\eta_t + \left(\frac{1}{2}\beta_0\eta^2\right)_x = \omega_0 \mathbf{H}[\eta]$$

$$\omega_0 = \frac{\alpha_+ - \alpha_-}{2}, \qquad \beta_0^2 = \frac{\alpha_+^2 + \alpha_-^2}{2}$$

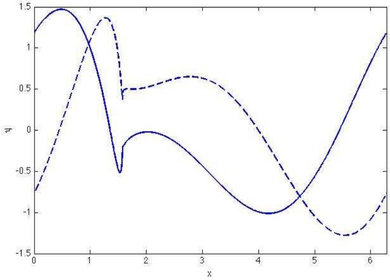
Provides effective equation for small-amplitude motion of vorticity discontinuity $y = \eta(x, t)$ between shears α_+, α_- on cubically nonlinear timescales

Note that can change sign of nonlinearity $\beta_0 \mapsto -\beta_0$ and still get same effective equation

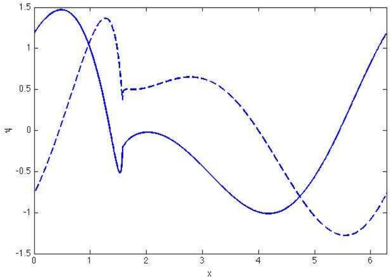
Filamentation and wavebreaking

- Wave breaking in Burgers-Hilbert equation corresponds to filamentation of vorticity discontinuity in weakly nonlinear regime. Happens on tiny spatial scale.
- Mechanism: discontinuity slowly folds over and overturned part stretched out into thin filament by underlying rapid time-periodic linearized oscillations. Get multiple filaments formed with repeated oscillations
- Contour dynamics numerics (Biello and H.)

Numerical solutions of filamentation

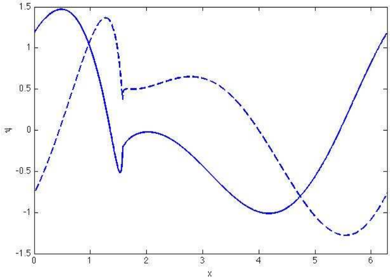


Filament formation



Strobed picture of filament formation: interface shown at same phase of oscillations

Detail of strobed picture of filament formation



Close-up of filament formation