Workshop on Wave Breaking and Global Solutions in the Short-Pulse Dispersive Equations

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The Cauchy problem of weakly dispersive equations

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Abstract

We shall discuss well-posedness of the initial value problem for a class of weakly dispersive nonlinear evolution equations, including the Camassa-Holm, the Degasperis-Procesi, and the Novikov equation. The focus will be continuity properties of the data-to-solution map in Sobolev spaces. This talk is based on work in collaboration with Carlos Kenig, Gerard Misiolek and Curtis Holliman.

Outline

- Main Result
- Camassa-Holm type equations
- Well-posedness for the **Novikov equation**
- Non-uniform dependence for the **Novikov equation**
- References

Main Result (work with Curtis Holliman)

The Camassa-Holm type equation (discovered recently [2009] by V. Novikov to be integrable)

$$(1-\partial_x^2)\partial_t u = -4u^2\partial_x u + 3u\partial_x u\partial_x^2 u + u^2\partial_x^3 u \qquad (1)$$

is well-posed in H^s, $s > 3/2$, on both the line and the circle with continuous dependence on initial data. Furthemore, **the data-to-solution map is not uniformly continuous.**

integrable equations

Integrable equations possess special properties, like:

- Infinite hierarchy of higher symmetries (Novikov's test),
- Have a Lax Pair,
- Infinitely many conserved quantities,
- A bi-Hamiltonian formulation,
- Can be solved by the Inverse Scattering Method.

Camassa-Holm type equations [Novikov: J. Phys. A, 2009]

- These are integrable equations of the form

$$(1-\partial_x^2)u_t = F(u, u_x, u_{xx}, u_{xxx}, \cdots) \qquad (2)$$

where F is a polynomial of u and its x-derivatives.

- Definition of integrability: Existence of an infinite hierarchy of (quasi-) local higher symmetries.

CH equations with quadratic nonlinearities

Theorem A. [Novikov, 2009] If the equation

$$\begin{aligned}(1-\varepsilon\partial_x^2)u_t = c_1 uu_x &+ \varepsilon[c_2 uu_{xx} + c_3 u_x^2] \\ &+ \varepsilon^2[c_4 uu_{xxx} + c_5 u_x u_{xx}] \\ &+ \varepsilon^3[c_6 uu_{xxxx} + c_7 u_x u_{xxx} + c_8 u_{xx}^2] \\ &+ \varepsilon^4[c_9 uu_{xxxxx} + c_{10} u_x u_{xxxx} + c_{11} u_{xx} u_{xxx}]\end{aligned}$$

is integrable then up to rescaling is one of the following 10:

1• Camassa-Holm (CH): $(1-\partial_x^2)u_t = -3uu_x + 2u_x u_{xx} + uu_{xxx}$
2• Degasperis-Procesi (DP): $(1-\partial_x^2)u_t = -4uu_x + 3u_x u_{xx} + uu_{xxx}$
3• ...

CH equations with cubic nonlinearities

Theorem C. [Novikov, 2009] If the equation

$$(1-\varepsilon\partial_x^2)u_t = c_1u^2u_x + \varepsilon[c_2u^2u_{xx} + c_3uu_x^2]$$
$$+\varepsilon^2[c_4u^2u_{xxx} + c_5uu_xu_{xx} + c_6u_x^3]$$
$$+\varepsilon^3[c_7u^2u_{xxxx} + c_8uu_xu_{xxx} + c_9uu_{xx}^2 + c_{10}u_x^2u_{xx}]$$
$$+\varepsilon^4[c_{11}u^2u_{xxxxx} + c_{12}uu_xu_{xxxx} + c_{13}uu_{xx}u_{xxx} + c_{14}u_x^2u_{xxx} + c_{15}u_xu_{xx}^2]$$

is integrable then up to rescaling is one of the following 10:

1• Novikov equation: $(1-\partial_x^2)u_t = -4u^2u_x + 3uu_xu_{xx} + u^2u_{xxx}$
2• Fokas-Qiao eqn: $(1-\partial_x^2)u_t = \partial_x(-u^3 + uu_x^2 + u^2u_{xx} - u_x^2u_{xx})$
3• ...

Common phenomenon of CH, DP and NE

- They all have **peakon** solutions:

$$u(x,t) = ce^{-|x-ct|}$$

- The discovery of the CH equation (Camassa-Holm [1993]) was partly driven by the desire to find a water wave equation which has traveling wave solutions that break. The Korteweg-de Vries equation (KdV),

$$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0, \tag{3}$$

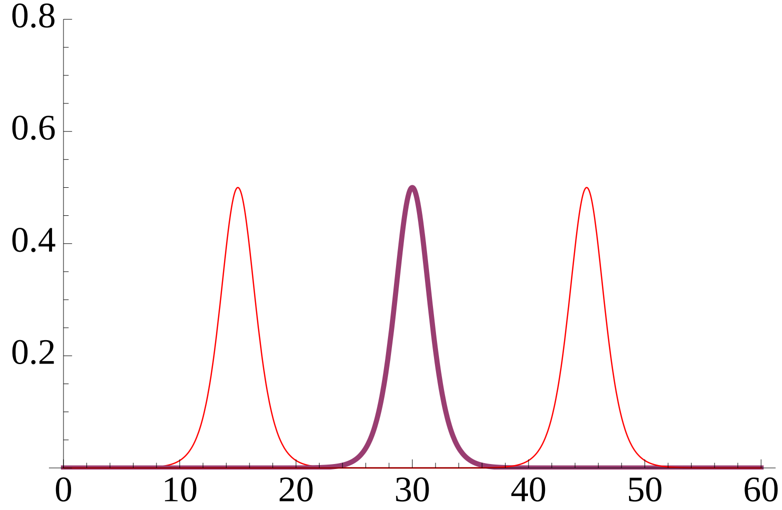
which was derived in 1895 as a model of long water waves propagating in a channel has only **smooth solitons**.

- Also, CH, DP and NE have **multipeakon** solutions:

$$u(x,t) = \sum_{j=1}^{n} p_j(t)e^{-|x-q_j(t)|}$$

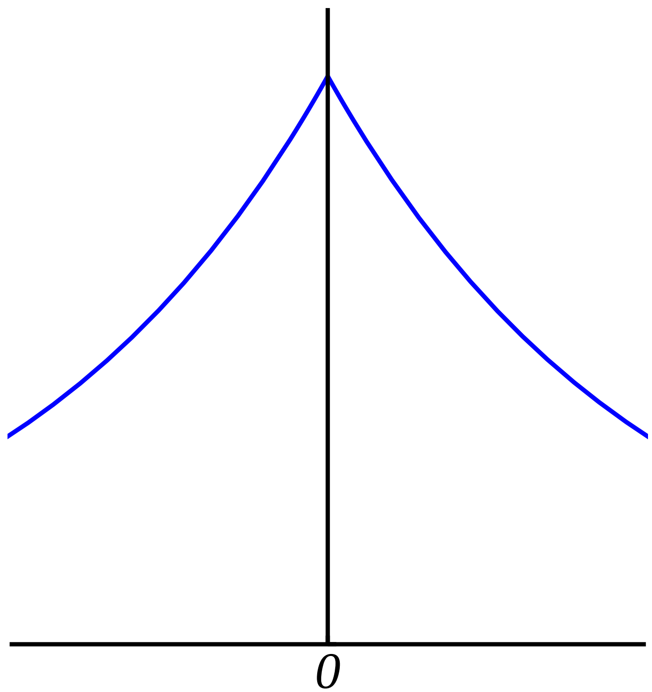
KdV Soliton: $u(x,t) = f(x - ct)$

$$f(x) = \frac{c}{2}\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}\,x\right)$$



CH Peakon: $u(x,t) = f(x - ct)$

$$f(x) = ce^{-|x|}$$



Conserved Quantities

CH, DP and NE have ∞-many conserved quantities, which among other things are used for proving **global** solutions.

- **Main conserved quantity by CH and NE is the H^1-norm:**

$$\|u\| \doteq \int [u^2 + u_x^2] dx$$

- **While, the main conserved quantity by DP is a twisted L^2-norm:**

$$\|u\|_{\tilde{L}^2}^2 \doteq \int (1 - \partial_x^2) u \cdot (4 - \partial_x^2)^{-1} u \, dx$$

The Cauchy problem for NE

• In its nonlocal form the Cauchy problem for NE is

$$\partial_t u + \frac{1}{3}\partial_x(u^3) + F(u) = 0 \tag{4}$$

$$u(x,0) = u_0(x),\ u \in H^s, \tag{5}$$

where

$$F(u) \doteq D^{-2}\partial_x\Big[u^3 + \frac{3}{2}\Big(u(\partial_x u)^2\Big)\Big] + D^{-2}\Big[\frac{1}{2}(\partial_x u)^3\Big] \tag{6}$$

and $D^{-2} = (1-\partial_x^2)^{-1}$.

• Observe that $F(u) \in H^s$ for any $u \in H^s$. Thus, applying a Galerkin-type approximation we prove the following local well-posedness result.

Local well-posedness for NE

Theorem 1. [H.–Holliman] *If $s > 3/2$ and $u_0 \in H^s$ then there exists $T > 0$ and a unique solution $u \in C\Big([0,T]; H^s\Big)$ of the initial value problem (4)–(5), which depends continuously on the initial data u_0. Furthermore, we have the estimate*

$$\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}, \quad \text{for} \quad 0 \leq t \leq T \leq \frac{1}{4c_s\|u_0\|^2_{H^s}}, \qquad (7)$$

where $c_s > 0$ is a constant depending on s.

• In the periodic case and when $s > 5/2$ NE's well-posedness has been proved by Tiglay [T]. Her proof is based on Arnold's geometric framework as it was further developed in Ebin-Marsden [EM] for proving well-posedness of the Euler equations in Sobolev spaces.

Non-uniform dependence for NE

Theorem 2. [H.–Holliman] If $s > 3/2$ then the solution map $u_0 \to u(t)$ for the NE equation is **not** uniformly continuous from any bounded set of $H^s(\mathbb{R})$ into $C([0, T]; H^s(\mathbb{R}))$.

Remark. The analogous to Theorem 2 result has been prooved:

- For CH on the line by [H.–Kenig, Diff. Int. Eqns 2009]

- For CH on the circle by [H.–Kenig-Misiolek, CPDE 2010]

- For DP on the line and the circle by [H.–Holliman, DCDS 2011]

Idea of Proof

We shall prove that there exist two sequences of NE solutions $u_n(t)$ and $v_n(t)$ in $C([0,T]; H^s(\mathbb{R}))$ such that:

1• $$\sup_n \|u_n(t)\|_{H^s} + \sup_n \|v_n(t)\|_{H^s} \lesssim 1,$$

2• $$\lim_{n\to\infty} \|u_n(0) - v_n(0)\|_{H^s} = 0,$$

3• $$\liminf_n \|u_n(t) - v_n(t)\|_{H^s} \gtrsim f(t), \quad 0 \le t < T \le 1,$$

where $f(t) > 0$.

Remark. For DP and CH $f(t) = \sin t$.

Approximate solutions motivation (from Euler eqns)

- For any $\omega \in \mathbb{R}$ and $n \in \mathbb{Z}^+$ the divergence free vector field

$$u^{\omega,n}(t,x) = \Big(\omega n^{-1} + n^{-s}\cos(nx_2 - \omega t),\, \omega n^{-1} + n^{-s}\cos(nx_1 - \omega t)\Big)$$

is a solution! to the Euler equations on $\mathbb{T}^2$.

- The corresponding to $\omega \pm 1$ sequences

$$u^{+1,n}(t,x) \qquad \text{and} \qquad u^{-1,n}(t,x)$$

Satisfy conditions (1)-(3) for non-uniform dependence of the periodic Euler equations in 2-D.

Remark. Non-uniform dependence for the Euler equations in n-D was proved in [H.–Misiołek, CMP 2010].

Approximate solutions for CH and DP

- The approximate solutions

$$u^{\omega,n}(x,t) = \omega n^{-1} + n^{-s}\cos(nx - \omega t), \quad \text{for} \quad \omega = -1, 1, \qquad (8)$$

where $n \in \mathbb{Z}^+$, satisfy conditions (1)-(3) for non-uniform dependence of the periodic CH and DP equations **but they are not solutions.**

- However, the **error**

$$E = CH(u^{\omega,n}) \quad \text{or} \quad E = DP(u^{\omega,n})$$

is **small**.

NE approximate solutions on the circle

- They are

$$u^{\omega,n} = \omega n^{-1/2} + n^{-s}\cos(nx - \omega t), \quad \text{for} \quad \omega = 0, 1. \qquad (9)$$

Note. One of them ($\omega = 0$) has **no low frequency**. Unlike in the case of CH and DP they are **asymmetric**.

Approximate Solutions on the Line

Again we will construct the $u^{\omega,n}$ as a high-low frequency combination. We will take the cutoff function

$$\varphi(x) = \begin{cases} 1, & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| \geq 2. \end{cases} \tag{10}$$

• **High Frequency Part:** This will be very similar to the high frequency of the periodic approximate solution; however, as we have the cutoff φ we can introduce a parameter δ to help control decay in later estimates.

$$u^h = u^{h,\omega,n} = n^{-\delta/2-s}\varphi(\frac{x}{n^\delta})\cos(nx - \omega t). \tag{11}$$

Approximate Solutions on the Line

• **Low Frequency Part:** Recall that in the periodic case the low frequency part was simply $\omega n^{-1/2}$. Here we will introduce the cutoff in the following way. We take $u_\ell = u_{\ell,\omega,n}$ to solve Novikov's initial value problem

$$(u_\ell)_t + (u_\ell)^2(u_\ell)_x + F(u_\ell) = 0, \tag{12}$$

$$u_\ell(x,0) = \omega n^{-1/2}\tilde{\varphi}(\frac{x}{n^\delta}), \tag{13}$$

where we have $\tilde{\varphi} \in C_0^\infty(\mathbb{R})$ and

$$\tilde{\varphi}(x) = 1, \quad \text{if} \quad x \in \text{supp}\,\varphi. \tag{14}$$

Now that we have approximate solutions, we can define actual solutions and follow the same program as in the periodic case.

Approximate Solutions and Error

Defining the NE approximate solution by

$$u^{\omega,n} = u_\ell + u^h \quad \omega = 0 \text{ or } 1. \tag{15}$$

we show that the error

$$E \doteq \partial_t u^{\omega,n} + (u^{\omega,n})^2 \partial_x u^{\omega,n} + F(u^{\omega,n}). \tag{16}$$

satisfies the following estimate.

Lemma. *Let* $s > 3/2$ *and* $1/4 < \delta < 1$. *Then,*

$$\|E(t)\|_{H^\sigma} \lesssim n^{-r_s}, \qquad \text{for } n \gg 1, \tag{17}$$

with

$$r_s \doteq s + 1 - \sigma - 2\delta > 0. \tag{18}$$

NE Actual Solutions and their difference

Actual solutions solve the Novikov's equation with initial data

$$u_{\omega,n}(0) = u^{\omega,n}(0). \tag{19}$$

Then the difference

$$v \doteq u^{\omega,n} - u_{\omega,n}. \tag{20}$$

satisfies the initial value problem

$$v_t = E - \frac{1}{3}\partial_x(vw) - F(u^{\omega,n}) + F(u_{\omega,n}), \tag{21}$$

$$v(0) = 0, \tag{22}$$

where

$$w = (u^{\omega,n})^2 + u^{\omega,n}u_{\omega,n} + (u_{\omega,n})^2. \tag{23}$$

Size of difference is small

Proposition. *If* $s > 3/2$ *and* $1/4 < \delta < 1$, *then*

$$\|v(t)\|_{H^\sigma} \lesssim n^{-r_s}, \quad 0 \le t \le T, \tag{24}$$

where $r_s = s + 1 - \sigma - 2\delta > 0$.

Proof. The last i.v.p. gives the following identity for v

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^\sigma} = -\int_{\mathbb{R}} D^\sigma E \cdot D^\sigma v dx - \frac{1}{3}\int_{\mathbb{R}} D^\sigma \partial_x(wv) D^\sigma v dx \tag{25}$$
$$- \int_{\mathbb{R}} D^\sigma \Big[F(u^{\omega,n}) - F(u_{\omega,n}) \Big] D^\sigma v dx.$$

The first term on the right hand side of (25) is estimated by applying Cauchy-Schwarz

$$\int_{\mathbb{R}} D^\sigma E D^\sigma v dx \le \|E\|_{H^\sigma} \|v\|_{H^\sigma} \tag{26}$$

Second Term of (25)

We begin by rewriting this term by commuting w with $D^\sigma \partial_x$ to arrive at

$$\int_{\mathbb{R}} D^\sigma \partial_x (vw) D^\sigma v dx = \int_{\mathbb{R}} [D^\sigma \partial_x, w] v D^\sigma v dx + \int_{\mathbb{R}} w D^\sigma \partial_x v D^\sigma v dx \tag{27}$$

The first integral can be handled by the following Calderon-Coifman-Meyer type commutator estimate that can be found in H.-Kenig-Misiolek [CPDE 2010].

Lemma. *If* $\sigma + 1 \geq 0$ *then*

$$\| [D^\sigma \partial_x, w] v \|_{L^2} \leq C \|w\|_{H^\rho} \|v\|_{H^\sigma} \tag{28}$$

provided that $\rho > 3/2$ *and* $\sigma + 1 \leq \rho$.

Second Term of (25) (cont.)

Applying this lemma and the solution size estimate of the well-posedness theorem we have

$$\int_{\mathbb{R}} [D^\sigma \partial_x, w] v D^\sigma v dx \lesssim \|w\|_{H^s} \, \|v\|_{H^\sigma}^2 \lesssim \|v\|_{H^\sigma}^2. \tag{29}$$

Next integrating by parts and using the Sobolev lemma we have

$$\int_{\mathbb{R}} w D^\sigma \partial_x v D^\sigma v dx \lesssim \|\partial_x w\|_{L^\infty} \int_{\mathbb{R}} (D^\sigma v)^2 dx \lesssim \|v\|_{H^\sigma}^2. \tag{30}$$

Third Term of (25)

- We use the following multiplier estimate that can be found in H.-Kenig-Misiolek [CPDE 2010].

Lemma. If $\sigma \in (1/2, 1)$ then

$$\|fg\|_{H^{\sigma-1}} \lesssim \|f\|_{H^{\sigma-1}} \|g\|_{H^{\sigma}}, \tag{31}$$

- Then, one can prove that for any u and w in H^σ, we have

$$\|F(u) - F(w)\|_{H^\sigma} \lesssim (\|u\|_{H^{\sigma+1}} + \|w\|_{H^{\sigma+1}})^2 \|u - w\|_{H^\sigma}.$$

- Therefore

$$\int_{\mathbb{R}} D^\sigma \Big[F(u^{\omega,n}) - F(u_{\omega,n}) \Big] D^\sigma v dx \lesssim \|v\|_{H^\sigma}^2. \tag{32}$$

End of proposition's proof

Putting these results together we arrive at the differential inequality

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^\sigma}^2 \lesssim \|E\|_{H^\sigma}\|v\|_{H^\sigma} + \|v\|_{H^\sigma}^2. \tag{33}$$

Solving this inequality gives

$$\|v\|_{H^\sigma} \lesssim \|E\|_{H^\sigma} \lesssim n^{-r}. \tag{34}$$

H^s norm of the difference is small

- By the well-posedness theorem obtain the following estimate for the H^k-norm of the difference of $u^{\omega,n}$ and $u_{\omega,n}$

$$\|v\|_{H^k} = \|u^{\omega,n}(t) - u_{\omega,n}(t)\|_{H^k} \lesssim n^{k-s}, \quad 0 \leq t \leq T. \tag{35}$$

- **Interpolating** between with $s_1 = \sigma$ and $s_2 = [s] + 2 = k$ gives

$$\begin{aligned}\|v(t)\|_{H^s} &\leq \|v(t)\|_{H^\sigma}^{(k-s)/(k-\sigma)} \|v(t)\|_{H^k}^{(s-\sigma)/(k-\sigma)} \\ &\lesssim n^{(-r_s)[(k-s)/(k-\sigma)]} n^{(k-s)[(s-\sigma)/(k-\sigma)]} \\ &\lesssim n^{-(1-2\delta)[(k-s)/(k-\sigma)]}.\end{aligned}$$

From the last inequality we obtain that

$$\|u^{\omega,n}(t) - u_{\omega,n}(t)\|_{H^s(\mathbb{R})} \lesssim n^{-\varepsilon_s}, \ \ 0 \leq t \leq T, \tag{36}$$

where ε_s is given by

$$\varepsilon_s = (1 - 2\delta)/(s + 2) > 0, \ \text{ if } \delta < \frac{1}{2}. \tag{37}$$

End of Proof of Theorem 2 on the Line

- **Behavior at time zero.** We have

$$\|u_{1,n}(0) - u_{0,n}(0)\|_{H^s} \leq n^{(-1+\delta)/2}\|\tilde{\varphi}\|_{H^s} \longrightarrow 0 \text{ as } n \to \infty.$$

- **Behavior at time $t > 0$.** We have

$$\begin{aligned} \|u_{1,n}(t) - u_{0,n}(t)\|_{H^s} \geq\ & \|u^{1,n}(t) - u^{0,n}(t)\|_{H^s} \\ & - \|u^{1,n}(t) - u_{1,n}(t)\|_{H^s} \\ & - \|u^{0,n}(t) - u_{0,n}(t)\|_{H^s}. \end{aligned} \tag{38}$$

Observe that for the second and third terms we have

$$\lim_{n\to\infty} \|u^{1,n}(t) - u_{1,n}(t)\|_{H^s} = \lim_{n\to\infty} \|u^{0,n}(t) - u_{0,n}(t)\|_{H^s} = 0 \tag{39}$$

For the remaining term observe

$$\liminf_{n\to\infty} \|u_{1,n}(t) - u_{0,n}(t)\|_{H^s} \geq \liminf_{n\to\infty} \|u^{1,n}(t) - u^{0,n}(t)\|_{H^s}. \quad (40)$$

Therefore, the inequality

$$\liminf_{n\to\infty} \|u^{1,n}(t) - u^{0,n}(t)\|_{H^s} \geq \frac{1}{\sqrt{2}}\|\varphi\|_{L^2(\mathbb{R})}\Big[\sin t + \cos t - 1\Big] \quad (41)$$

completes the proof of Theorem 2 on the line. $\square$

Thank you!

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