Birkhoff Normal forms for the problem of water waves

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Workshop on Wave Breaking and Global Solutions
in the Short-Pulse Dispersive Equations

Fields Institute

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Outline

- Two ODEs
- Free surface water waves
- Birkhoff normal forms
- Implications of the normal form
- Possibilities in the KdV scaling limit

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Contrast two ODEs

- Quadratic case

$$\dot{z} = z^2 \ , \qquad z(0) = \varepsilon$$
$$z(t) = \frac{\varepsilon}{1 - \varepsilon t} \ , \qquad T = \frac{1}{\varepsilon}$$

- Cubic case

$$\dot{w} = w^3 \ , \qquad w(0) = \varepsilon$$
$$w(t) = \sqrt{\frac{\varepsilon^2}{1 - 2\varepsilon^2 t}} \ , \qquad T = \frac{1}{2\varepsilon^2}$$

- The general time of existence does not change when these ODE are replaced by

$$\dot{z} = i\omega z + z^2 \ , \qquad \dot{w} = i\omega w + w^3$$

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Free surface water waves

- Incompressible and irrotational flow

$$\nabla \cdot u = 0 \; , \qquad \nabla \wedge u = 0$$

which is a potential flow

$$u = \nabla \varphi \; , \quad \Delta \varphi = 0 \; , \qquad \partial_N \varphi = 0 \qquad \text{bottom BC}$$

in the fluid domain $-h < y < \eta(x,t)$, $\quad x \in \mathbb{R}^{d-1}$

- Free surface conditions

$$\begin{aligned} \partial_t \eta &= \partial_y \varphi - \partial_x \eta \cdot \partial_x \varphi && \text{kinetic BC} \\ \partial_t \varphi &= -g\eta - \tfrac{1}{2}|\nabla \varphi|^2 && \text{Bernoulli condition} \end{aligned}$$

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Zakharov's Hamiltonian

- The energy functional

$$\begin{aligned} H &= K + P \\ &= \int \int_{-h}^{\eta(x)} \tfrac{1}{2} |\nabla \varphi|^2 \, dy dx + \int_x \tfrac{g}{2} \eta^2 \, dx \end{aligned}$$

- Zakharov's choice of variables

$$z := (\eta(x), \xi(x) = \varphi(x, \eta(x)))$$

That is $\varphi = \varphi[\eta, \xi](x, y)$

- Expressed in terms of the Dirichlet – Neumann operator $G(\eta)$

$$H(\eta, \xi) = \int \tfrac{1}{2} \xi G(\eta) \xi + \tfrac{g}{2} \eta^2 \, dx$$

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Dirichlet – Neumann operator

- Laplace's equation on the fluid domain $-h < y < \eta(x)$

$$\xi(x) \mapsto \varphi(x,y) \mapsto N \cdot \nabla\varphi \, (1 + |\nabla_x \eta|^2)^{1/2} := G(\eta)\xi(x)$$

- In these coordinates

$$\begin{aligned} \partial_t \eta &= G(\eta)\xi = \text{grad}_\xi H \\ \partial_t \xi &= -g\eta - \text{grad}_\eta K = -\text{grad}_\eta H \end{aligned}$$

A calculation related to the variational formula of Hadamard [Collège de France lectures (1911)(1916) on the Green's function]

- PDE in Hamiltonian form

$$\partial_t z = J \text{grad}_z H \,, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

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Hamiltonian PDEs

Other Hamiltonian partial differential equations

- Korteweg de Vries

$$\partial_t r = -\partial_x\left(\frac{1}{6}\partial_x^2 r + \frac{3}{2}r^2\right)$$

$$J := -\partial_x \,, \qquad H_{KdV} = \int_x -\frac{1}{12}(\partial_x r)^2 + \frac{1}{2}r^3 \, dx$$

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  Boussinesq system
  nonlinear Schrödinger equation (NLS), …

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Lemma (properties of the Dirichlet - Neumann operator)

1. $G(\eta) \geq 0$ *and* $G(\eta)1 = 0$
2. $G(\eta)^* = G(\eta)$ *Hermetian symmetric*
3. $G(\eta) : H^1_\xi \to L^2_\xi$ *is analytic in* η *for* $\eta \in C^1$

$$G(\eta)\xi = G^{(0)}\xi + G^{(1)}(\eta)\xi + G^{(2)}(\eta)\xi + \dots$$

[using a theorem of Christ & Journé (1987)]

4. *Setting* $D_x := -i\partial_x$

$$\begin{aligned} G^{(0)}\xi &= |D_x| \tanh(h|D_x|)\xi \\ G^{(1)}\xi &= (D_x \cdot \eta D_x - G^{(0)}\eta G^{(0)})\xi \end{aligned}$$

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conservation laws

- Mass $M = \int \eta \, dx$

$$\begin{aligned} \{M, H\} &= \int \mathrm{grad}_\eta M \, \mathrm{grad}_\xi H - \mathrm{grad}_\xi M \, \mathrm{grad}_\eta H \, dx \\ &= \int 1 \, G(\eta)\xi \, dx \\ &= \int G(\eta) 1 \, \xi \, dx = 0 \end{aligned}$$

- Momentum $I(\eta, \xi) = \int \eta \partial_x \xi \, dx$

$$\{I, H\} = 0$$

- Energy $H(\eta, \xi)$

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Taylor expansion of the Hamiltonian

- From the analyticity of $G(\eta)$

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\begin{aligned}
H &= H^{(2)} + H^{(3)} + H^{(4)} + \dots \\
&= \frac{1}{2}\int \xi G^{(0)}\xi + g\eta^2\,dx \;+\; \sum_{m\geq 3}\frac{1}{2}\int \xi G^{(m-2)}(\eta)\xi\,dx
\end{aligned}
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- The linearized equations are

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\partial_t \begin{pmatrix}\eta \\ \xi\end{pmatrix} = J\,\mathrm{grad}_{(\eta,\xi)}H^{(2)}
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namely

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A harmonic oscillator with frequencies $\omega(k) = \sqrt{g|k|\tanh(h|k|)}$

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normal forms

Restrict our considerations to the $d = 2$ periodic case; $x \in \mathbb{R}^1/2\pi\mathbb{Z}^1 = \mathbb{T}^1$

- The frequencies are discrete $\omega(k) = \sqrt{gk\tanh(hk)}$, $k \in \mathbb{Z}^1$
- Normal form - transform the equations to retain only essential nonlinearities

$$\tau : z = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \mapsto w$$

in a neighborhood $B_R(0) \subseteq H^r$

- Conditions:

1. The transformation τ is canonical, so the new equations are

$$\partial_t w = J\,\mathrm{grad}\widetilde{H}(w) , \qquad \widetilde{H}(w) = H(\tau^{-1}(w))$$

2. The new Hamiltonian is

$$\widetilde{H}(w) = H^{(2)}(w) + \left(Z^{(3)} + \cdots + Z^{(M)}\right) + \widetilde{T}^{(M+1)}$$

where each $Z^{(m)}$ retains only resonant terms

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- In particular if $Z^{(3)} = 0$ then the new equations will have no quadratic nonlinear terms; the lowest order nonlinear terms will be cubic.

- This transformation procedure is called the reduction to

  Birkhoff normal form.

  It is part of the theory of averaging for dynamical systems.

Theorem (C. Sulem and WC (2009))

Let $d = 2$ (and $h = +\infty$) and fix $r > 3/2$. There exists $R_0 > 0$ such that for any $R < R_0$, on every neighborhood $B_R(0) \subseteq H_\eta^{r+1} \times H_\xi^r$ the Birkhoff normal forms transformation $\tau^{(3)}$ is defined.

$$\tau^{(3)} \; : \; B_R(0) \to B_{2R}(0)$$
$$(\tau^{(3)})^{-1} \; : \; B_{R/2}(0) \to B_R(0)$$

The result is that $w = \tau^{(3)}(z)$ transforms $H(z)$ to

$$\widetilde{H}(w) = H^{(2)}(w) + 0 + \widetilde{T}^{(4)}(w)$$

A similar statement holds for $r > 5/2$ in the case of positive surface tension, with however a possible nonzero $Z^{(3)}$. Furthermore $\tau^{(3)}$ is an analytic diffeomorphism in this case.

NB The transformation mixes the domain η and the potential variables ξ.

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Outline of the proof

- The transformation $\tau^{(3)}$ is constructed as the time $s = 1$ flow of an auxiliary Hamiltonian system.
  Define complex symplectic coordinates

$$z(x) = \sqrt[4]{\frac{g}{4G^{(0)}(D_x)}}\,\eta(x) + i\sqrt[4]{\frac{G^{(0)}(D_x)}{4g}}\,\xi(x) = \sum_{k\in\mathbb{Z}^1} \hat{z}_k e^{ikx}$$

- In these coordinates (dropping 'hat' notation)

$$H = \sum_k \omega(k)|z_k|^2 + \sum_{m\geq 3}\Big[\sum_{|p|+|q|=m} c(p,q) z^p \overline{z}^q\Big]$$

where $|p| + |q| := \sum_\ell p_\ell + q_\ell = m$ and

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Proposition

- *One can choose initial data* $\eta_0(x) = \eta(x,0)$ *such that* $M = 2\pi\hat{\eta}(0) = 0$
- *Unless* $\langle k, p - q \rangle = 0$ *the coefficients satisfy*

$$c(p,q) = 0$$

  (conservation of momentum)
- *There are no nonzero* $m = 3$ *resonances. Indeed*

$$\omega(k_1) \pm \omega(k_2) \pm \omega(k_3) = 0 \qquad \text{and} \qquad k_1 \pm k_2 \pm k_3 = 0$$

  implies $k_\ell = 0$ *for some* $\ell = 1, 2, 3$

The auxiliary Hamiltonian is determined by a cohomological equation

$$\{K^{(3)}, H^{(2)}\} + H^{(3)} = 0$$

to be solved for $K^{(3)}$. This is a linear equation

- To solve $\{H^{(2)}, K^{(3)}\} = H^{(3)}$ for $K^{(3)}$, one determines the eigenvalues of the operator $ad_{H^{(2)}}(\cdot) := \{H^{(2)}, \cdot\}$. These are precisely

$$\pm i\big(\omega(k_1) \pm \omega(k_2) \pm \omega(k_3)\big)$$

  By the above proposition, $ad_{H^{(2)}}(\cdot)$ has a formal inverse.

- Question: Does the flow $\varphi_s(z)$ of the vector field exist, and on which Banach spaces?

$$\partial_s z = X_{K^{(3)}}(z) = J\,\mathrm{grad}_z K^{(3)}(z) \qquad (1)$$

Theorem (Main new technical result)

The system of equations (1)
(i) Satisfies good energy estimates on $H^{r+1}_\eta \times H^r_\xi$ *when* $h = +\infty$ *for* $r > 3/2$
(ii) With nonzero surface tension it defines a locally Lipschitz vector field on $H^r_\eta \times H^{r+1}_\xi$*, for* $r > 5/2$

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Long time existence theorem

This next section describes our hopes for the Birkhoff normal form

Theorem (work in progress)

Let $d = 2$ and consider the case of periodic data $x \in \mathbb{T}^1$. There exist $s_0 > 0$ and $\varepsilon_0 > 0$ such that for initial data $(\eta_0, \xi_0) = z_0 \in H^s_\eta \times H^{s+1/2}_\xi := H^s_*$, $s > s_0$ and $\varepsilon < \varepsilon_0$, satisfying

$$\|z_0\|_{H^s_*} < \varepsilon$$

then there exists a solution $z(x,t) = \varphi_t(z_0)$ to the equations for surface water waves for all $t \in [-C/\varepsilon^2, C/\varepsilon^2]$, satisfying

$$\|z(\cdot,t)\|_{H^s_*} < 4\varepsilon$$

This is to be compared with other recent work in the case $x \in \mathbb{R}^{d-1}$, for noncompact regions.

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then there exists a solution $z(x,t) = \varphi_t(z_0)$ *to the equations for surface water waves for all* $t \in [-C/\varepsilon^2, C/\varepsilon^2]$*, satisfying*

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This is to be compared with other recent work in the case $x \in \mathbb{R}^{d-1}$, for noncompact regions.

Long time existence theorem

This next section describes our hopes for the Birkhoff normal form

Theorem (work in progress)

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Recent work on $\mathbb{R}^{d-1}$

Theorem (S. Wu (2009))

For $d = 2$, there exists $\varepsilon_0 > 0$ such that given Sobolev data $\|(\eta_0, \xi_0)\|_{H^s_*} = \varepsilon < \varepsilon_0$ then solutions exist over exponentially long time intervals

$$|t| < C_1 e^{C_2/\varepsilon}$$

- The proof uses the property of L^∞ time decay for the linear equations, following work of J. Shatah on the nonlinear Klein - Gordon equation

Theorem (P. Germain, N. Masmoudi & J. Shatah (2009))

For $d = 3$, there exists $\varepsilon_0 > 0$ such that given Sobolev data $\|(\eta_0, \xi_0)\|_{H^s_*} < \varepsilon < \varepsilon_0$ then solutions exist globally in time.

- The proof uses a different method, which however is again related to averaging theory of dynamical systems

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Analysis of the modulational regime

- We intend to use this newly won long-time existence theory to address the modulational scaling regime of the water wave problem.
  Question: [C. Sulem, P.-L. Sulem & C. (1992), plus present work] Solutions of the water waves problem in modulational form

$$\begin{aligned} z_0(x) &= (\eta_0, \xi_0)(x) \\ &= \varepsilon \cos(k_0 x)(N(\varepsilon x), \Xi(\varepsilon x)) \end{aligned}$$

  converge to solutions of the cubic nonlinear Schrödinger equation

$$i\partial_T Z(Y,T) = \partial_k^2 \omega(k_0) \partial_Y^2 Z - c(k_0)|Z|^2 Z$$

  where $Y = \varepsilon(x - \partial_k \omega(k_o)t)$ and $T = \varepsilon^2 t$. The time interval of convergence is

$$|T| \sim \mathcal{O}(1) \; , \qquad i.e. \quad |t| \sim \mathcal{O}(1/\varepsilon^2)$$

- References: Guido Schnieder & Eugene Wayne, Nathan Totz & Sijue Wu

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The KdV scaling regime

Return to the KdV equation

$$\partial_T r = -\partial_X \, \mathrm{grad} H_{KdV}(r) \; , \qquad H_{KdV} = \int -\frac{1}{12}(\partial_X r)^2 + \frac{1}{2} r^3 \, dX$$

Question: How this emerges as a model equation for water waves.

The scaling regime

$$\begin{aligned} X &:= \varepsilon x \; , & \varepsilon D_X = D_x \\ \xi &\mapsto \varepsilon \xi \; , & \eta \mapsto \varepsilon^2 \eta \end{aligned}$$

The Dirichlet – Neumann operator will behave as follows

$$\varepsilon D_X \tanh(\varepsilon h D_X) \sim \varepsilon^2 h D_X^2 + \frac{1}{6} \varepsilon^4 h^3 D_X^4 + \mathcal{O}(\varepsilon^6 D_X^6)$$

Transform the symplectic structure

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} \mapsto \begin{pmatrix} \eta \\ u := \partial_X \xi \end{pmatrix} \quad \text{giving rise to} \quad \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -\partial_X \\ -\partial_X & 0 \end{pmatrix}$$

- On the level of the Hamiltonian functional for water waves

$$H = \frac{1}{2}\int hu^2 + g\eta^2 - \varepsilon^2 \frac{h^3}{6}(\partial_X u)^2 \, dX + \frac{\varepsilon^2}{2}\int \eta u^2 \, dX + \mathcal{O}(\varepsilon^4)$$

- Transform to a moving reference frame using the momentum integral $\int \eta u \, dX$

$$H - \sqrt{gh} I = \frac{1}{2}\int \left(\sqrt{h}u - \sqrt{g}\eta\right)^2 dX + \frac{\varepsilon^2}{2}\int \left(-\frac{h^3}{6}(\partial_X u)^2 + \eta u^2\right) dX + \mathcal{O}(\varepsilon^4)$$

- The KdV description is valid in the region of phase space in which $(\sqrt{h}u - \sqrt{g}\eta) \sim o(\varepsilon)$. In this region $u = \sqrt{g/h}\,\eta + o(\varepsilon)$, thus

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A 'derivation' of complete integrability

- Perform the above sequence of transformations on the Birkhoff normal form $\tau_\varepsilon^{(3)}$ for water waves.
  The limit $\varepsilon \to 0$ before the normal forms transformation

$$\begin{aligned} H - \sqrt{gh}I &= H^{(2)} + H^{(3)} + T^{(4)} \\ &\mapsto \int -\frac{1}{12}(\partial_X r)^2 + \frac{1}{2}r^3 \, dX + 0 \end{aligned}$$

- After the normal forms transformation

$$\begin{aligned} \left(H - \sqrt{gh}I\right) \circ (\tau_\varepsilon^{(3)})^{-1} &= H^{(2)} + 0 + \widetilde{T}^{(4)} \\ &\mapsto \int -\frac{1}{12}(\partial_X R)^2 + 0 \end{aligned}$$

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$$\lim_{\varepsilon \to 0} \tau_\varepsilon^{(3)} := \tau_0^{(3)}$$

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Thank you