Laser Plasma interactions: Zakharov's System and solitary waves

M-T Colins, M. Ohta

INRIA FUTUR, Projet MC2
Université de Bordeaux, FRANCE

Wave Breaking and Global Solutions in the Short-Pulse Dispersive Equations

OUTLINE

1. Motivations
2. Three waves interaction model.
3. The Cauchy problem.
4. The solitary waves.

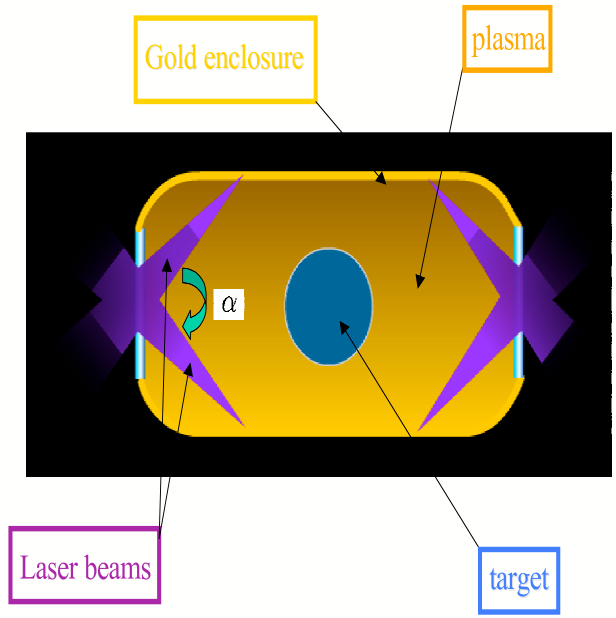
1. Motivations

Simulate fusion by inertial confinement in laboratory

1. Motivations

Simulate fusion by inertial confinement in laboratory

I. Setting of the problem

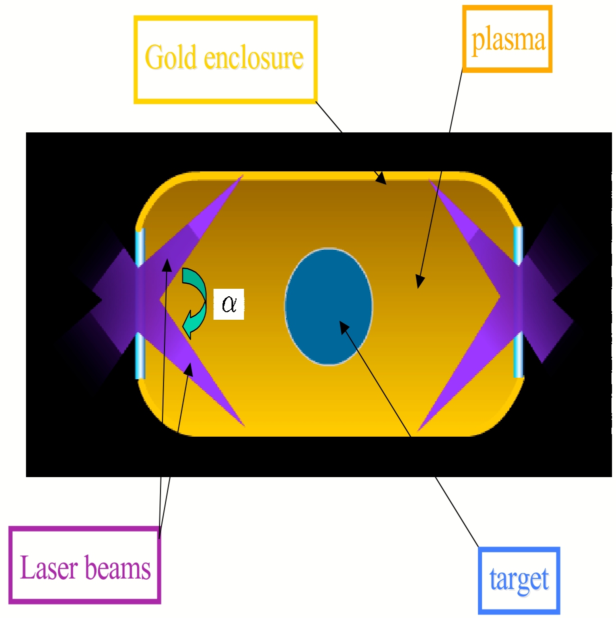


α : Angle between two crossing beams.

1. Motivations

Simulate fusion by inertial confinement in laboratory

I. Setting of the problem

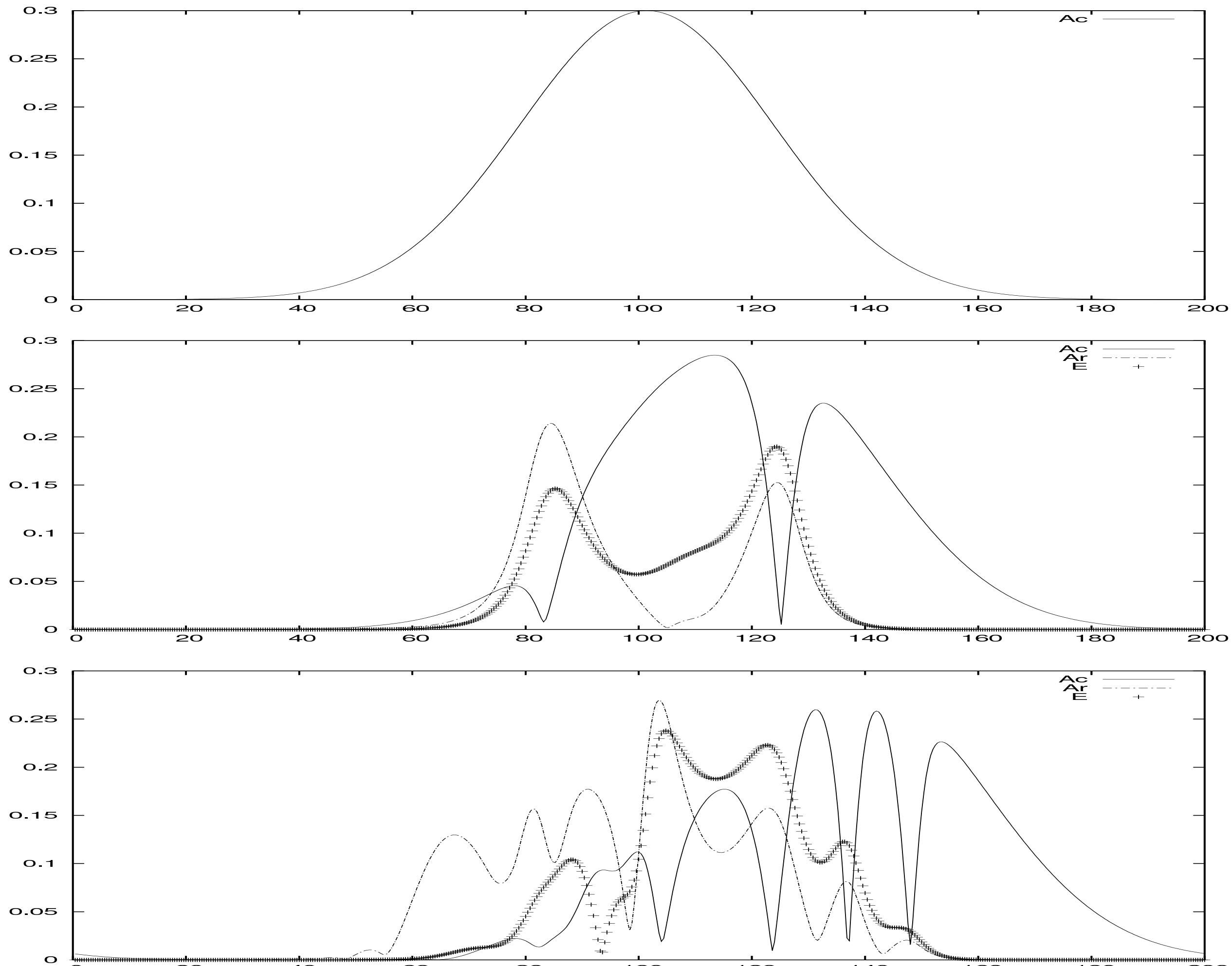


α : Angle between two crossing beams.

Collaborators: CEA: G. Gallice, R. Belaouar.

University of Bordeaux :T. Colin, C. Galusinski, G. Métivier, V. Tikhonchuk

1. Motivations.



2. Three waves interaction model.

a. The equations.

$$(n_0 + n_e)\left(\partial_t v_e + v_e \cdot \nabla v_e\right) = -\frac{\gamma_e T_e}{m_e}\nabla n_e - \frac{e(n_0 + n_e)}{m_e}\left(E + \frac{1}{c} v_e \times B\right)$$

$$(n_0 + n_i)\left(\partial_t v_i + v_i \cdot \nabla v_i\right) = -\frac{\gamma_i T_i}{m_i}\nabla n_i + \frac{e(n_0 + n_i)}{m_i}\left(E + \frac{1}{c} v_i \times B\right)$$

$$\partial_t n_e + \nabla \cdot ((n_0 + n_e) v_e) = 0$$

$$\partial_t n_i + \nabla \cdot ((n_0 + n_i) v_i) = 0.$$

2. Three waves interaction model.

For the electronic-plasma waves :

$$\partial_t B + c\nabla \times E = 0,$$

$$\partial_t E - c\nabla \times B = 4\pi e\left((n_0 + n_e)v_e\right),$$

For the electromagnetic waves :

$$\partial_t \psi = c\nabla A$$

$$\partial_t A + cE = c\nabla \psi$$

$$\partial_t E - c\nabla \times \nabla \times A = 4\pi e(n_0 + n_e)v_e$$

2. Three waves interaction model.

b. Polarization conditions.

Linearization + Decomposition

longitudinal part + transverse part

$$B = B_{\|} + B_{\perp}$$

with

$$\nabla \times B_{\|} = 0$$

and

$$\nabla \cdot B_{\perp} = 0$$

same for E et v_e

2. Three waves interaction model.

electronic-plasma waves:

$$\partial_t B_{\|} = 0,\ \partial_t E_{\|} = 4\pi e n_0 v_{e\|}$$

$$\partial_t v_{e\|} = -\frac{\gamma_e T_e}{m_e n_0}\nabla n_e - \frac{e}{m_e}E_{\|},\ \partial_t n_e + n_0 \nabla \cdot v_{e\|} = 0$$

$$[\partial_t^2 - v_{th}^2 \Delta + {\omega_{pe}}^2] v_{e\|} = 0$$

Plasma Pulsation $\omega_{pe} = \sqrt{\frac{4\pi e^2 n_0}{m_e}}$

and thermal velocity $v_{th} = \sqrt{\frac{\gamma_e T_e}{m_e}}$

2. Three waves interaction model.

electronic-plasma waves:

$$\partial_t B_{\|} = 0,\ \partial_t E_{\|} = 4\pi e n_0 v_{e\|}$$

$$\partial_t v_{e\|} = -\frac{\gamma_e T_e}{m_e n_0}\nabla n_e - \frac{e}{m_e}E_{\|},\ \partial_t n_e + n_0 \nabla \cdot v_{e\|} = 0$$

$$[\partial_t^2 - v_{th}^2 \Delta + {\omega_{pe}}^2] v_{e\|} = 0$$

Plasma Pulsation $\omega_{pe} = \sqrt{\frac{4\pi e^2 n_0}{m_e}}$

and thermal velocity $v_{th} = \sqrt{\frac{\gamma_e T_e}{m_e}}$

dispersion relation :

$$\omega^2 = \omega_{pe}^2 + k^2 v_{th}^2.$$

2. Three waves interaction model.

Electromagnetic waves

$$\partial_t B_\perp + c\nabla \times E_\perp = 0$$

$$\partial_t E_\perp - c\nabla \times B_\perp = 4\pi e n_0 v_{e\perp}$$

$$\partial_t v_{e\perp} = -\frac{e}{m_e} E_\perp$$

2. Three waves interaction model.

Electromagnetic waves

$$\partial_t B_\perp + c\nabla \times E_\perp = 0$$

$$\partial_t E_\perp - c\nabla \times B_\perp = 4\pi e n_0 v_{e\perp}$$

$$\partial_t v_{e\perp} = -\frac{e}{m_e} E_\perp$$

$$=> \partial_t^2 E_\perp - c^2 \Delta E_\perp + \omega_{pe}^2 E_\perp = 0$$

dispersion relation :

$$\omega^2 = \omega_{pe}^2 + k^2 c^2.$$

2. Three waves interaction model.

$A_0 : (K_0, \omega_0)$, $A_R : (K_R, \omega_R)$, $E : (K_1, \omega_1 + \omega_{pe})$

Condition for interactions :

$$\omega_0 = \omega_{pe} + \omega_R + \omega_1, \; K_0 = K_R + K_1$$

with

$$\omega_0^2 = \omega_{pe}^2 + c^2 |K_0|^2,$$

$$\omega_R^2 = \omega_{pe}^2 + c^2 |K_R|^2,$$

$$(\omega_{pe} + \omega_1)^2 = \omega_{pe}^2 + v_{th}^2 |K_1|^2$$

2. Three waves interaction model.

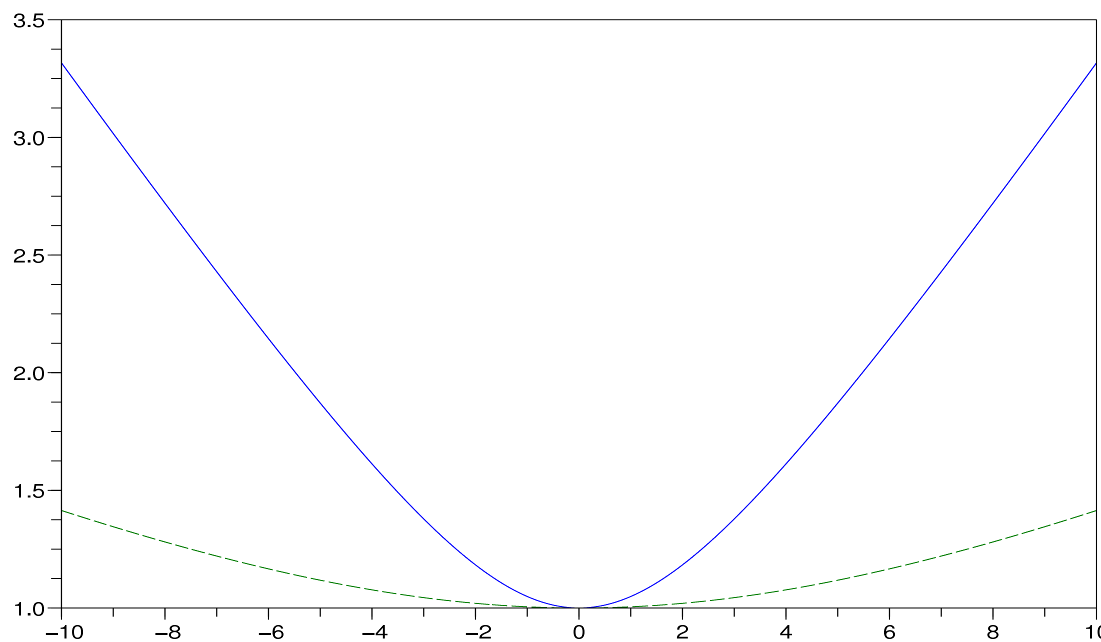
$$v_{th} << c$$

$=>$

$$[\partial_t^2 - v_{th}^2 \Delta + {\omega_{pe}}^2] E_{||} = 0$$

$$[\partial_t^2 - c^2 \Delta + \omega_{pe}^2] E_{\perp} = 0$$

have different status



2. Three waves interaction model.

- electromagnetic waves under the form:

$e^{i(kx-\omega t)} E_{\perp}(t,x)$ with $\partial_t E_{\perp} \ll \omega E_{\perp}$ et $\partial_x E_{\perp} \ll k E_{\perp}$.

- electronic-plasmas waves under the form $e^{-i\omega_{pe} t} E_{||}$ with $\partial_t E_{||} \ll \omega_{pe} E_{||}$.

2. Three waves interaction model.

• electromagnetic waves under the form:
$e^{i(kx-\omega t)}E_\perp(t,x)$ with $\partial_t E_\perp \ll \omega E_\perp$ et $\partial_x E_\perp \ll k E_\perp$.

• electronic-plasmas waves under the form $e^{-i\omega_{pe}t}E_{||}$
with $\partial_t E_{||} \ll \omega_{pe} E_{||}$.
Oscillations + nonlinearity create low frequency waves: ionic accoustic waves.

Scenario: send the laser,
$=>$ backscattered electromagnetic wave (Raman component).
$=>$ electronic plasma wave.
$=>$ ionic accoustic wave.
And retroaction!

2. Three waves interaction model.

Weakly nonlinear theory

$$F = F_0 e^{i(K_0 \cdot X - \omega_0 t)} + F_R e^{i(K_R \cdot X - \omega_R t)} + F_e e^{-i\omega_e t}$$

and collect !

$=>$ phase mismatch

$$e^{i\left(K_1 \cdot X - \omega_1 t\right)}$$

2. Three waves interaction model.

Electromagnetic waves:

$$\left(\partial_t^2 - c^2\Delta + \omega_{pe}^2\right)E_\perp = 0,$$

$$(A_\perp, E_\perp, v_{e\perp}) = e^{iK\cdot X-\omega t}(A_\perp, E_\perp, v_{e\perp}) + cc, \quad K = \begin{pmatrix} k \\ l \end{pmatrix}$$

$$E_\perp = \frac{i\omega}{c}A_\perp, \quad v_{e\perp} = \frac{e}{m_e c}A_\perp$$

$$-i\omega E_\perp + cK \times (K \times A) = 4\pi e n_0 v_{e\perp}$$

$$\Longrightarrow (E, A, v_e) \in K^\perp$$

2. Three waves interaction model.

Equation on A_0:

$$i\Big(\partial_t + \frac{c^2}{\omega_0} K_0 \cdot \nabla\Big) A_0 + \frac{1}{2\omega_0}\Big(c^2\Delta - \frac{c^4}{\omega_0^2}\big(K_0 \cdot \nabla\big)^2\Big) A_0$$

$$= \frac{2\pi e^2}{\omega_0 m_e} < n_e > A_0 - \frac{e}{2\omega_0 m_e}\big(\nabla \cdot E\big) A_R e^{-i\theta_1} \frac{K_0 \cdot K_R}{|K_0||K_R|}.$$

2. Three waves interaction model.

Equation on A_0:

$$i\Big(\partial_t + \frac{c^2}{\omega_0} K_0 \cdot \nabla\Big) A_0 + \frac{1}{2\omega_0}\Big(c^2 \Delta - \frac{c^4}{\omega_0^2}\big(K_0 \cdot \nabla\big)^2\Big) A_0$$

$$= \frac{2\pi e^2}{\omega_0 m_e} < n_e > A_0 - \frac{e}{2\omega_0 m_e}\big(\nabla \cdot E\big) A_R e^{-i\theta_1} \frac{K_0 \cdot K_R}{|K_0||K_R|}.$$

Equation on A_R:

$$i\Big(\partial_t + \frac{c^2}{\omega_R} K_R \cdot \nabla\Big) A_R + \frac{1}{2\omega_R}\Big(c^2 \Delta - \frac{c^4}{\omega_R^2}\big(K_R \cdot \nabla\big)^2\Big) A_R$$

$$= \frac{2\pi e^2}{\omega_R m_e} < n_e > A_R - \frac{e}{2\omega_R m_e}\big(\nabla \cdot E^*\big) A_0 e^{i\theta_1} \frac{K_0 \cdot K_R}{|K_0||K_R|}.$$

2. Three waves interaction model.

Equation on E_0

$$i\partial_t E_0 + \frac{v_{th}^2}{2\omega_{pe}}\nabla\nabla\cdot E_0 - \frac{c^2}{2\omega_{pe}}\nabla\times\nabla\times E_0$$

$$= \frac{\omega_{pe}}{2} < n_e > E_0 + \frac{e\omega_{pe}}{2m_e c^2}\nabla\left(A_0\cdot A_R^* e^{i\theta_1}\frac{K_0\cdot K_R}{|K_0||K_R|}\right).$$

2. Three waves interaction model.

$$i\left(\partial_t + \frac{c^2}{\omega_0}K_0 \cdot \nabla\right)A_0 + \frac{\varepsilon}{2\omega_0}\left(c^2\Delta - \frac{c^4}{\omega_0^2}\left(K_0 \cdot \nabla\right)^2\right)A_0$$

$$= -\varepsilon\frac{e}{2\omega_0 m_e}\left(\nabla \cdot \mathbf{E}_{||}\right)A_R e^{-i\theta_1}\frac{K_0 \cdot K_R}{|K_0||K_R|},$$

$$i\left(\partial_t + \frac{c^2}{\omega_R}K_R \cdot \nabla\right)A_R + \frac{\varepsilon}{2\omega_R}\left(c^2\Delta - \frac{c^4}{\omega_R^2}\left(K_R \cdot \nabla\right)^2\right)A_R$$

$$= -\varepsilon\frac{e}{2\omega_R m_e}\left(\nabla \cdot \mathbf{E}_{||}^*\right)A_0 e^{i\theta_1}\frac{K_0 \cdot K_R}{|K_0||K_R|},$$

$$i\partial_t\mathbf{E}_{||} + \varepsilon\left(\frac{v_{th}^2}{2\omega_{pe}}\nabla\nabla \cdot \mathbf{E}_{||} - \frac{c^2}{2\omega_{pe}}\nabla \times \nabla \times \mathbf{E}_{||}\right)$$

$$= \varepsilon\frac{e\omega_{pe}}{2m_e c^2}\nabla\left(A_0 A_R^* e^{i\theta_1}\frac{K_0 \cdot K_R}{|K_0||K_R|}\right).$$

2. Three waves interaction model.

$E_{||} = \mathcal{E} e^{i(K_1 \cdot X - \omega_1 t)}$, with $\partial_t E_{||} << \omega_1 E_{||}, \;\; \nabla E_{||} << K_1 E_{||}$

$$(\omega_{pe} + \omega_1)^2 = \omega_{pe}^2 + v_{th}^2 |K_1|^2,$$

then

$$\omega_1 \approx \frac{v_{th}^2 |K_1|^2}{2\omega_{pe}}.$$

2. Three waves interaction model.

$E_{||} = \mathcal{E} e^{i(K_1 \cdot X - \omega_1 t)}$, with $\partial_t E_{||} << \omega_1 E_{||}, \quad \nabla E_{||} << K_1 E_{||}$

$$(\omega_{pe} + \omega_1)^2 = \omega_{pe}^2 + v_{th}^2 |K_1|^2,$$

then

$$\omega_1 \approx \frac{v_{th}^2 |K_1|^2}{2\omega_{pe}}.$$

$$\begin{aligned} i\Big(\partial_t + \frac{v_{th}^2}{\omega_{pe}} K_1 \cdot \nabla\Big)\mathcal{E} + \varepsilon \Delta \mathcal{E} \quad &= i \tfrac{e\omega_{pe}}{2m_e c^2} \Big(A_0 A_R^* \tfrac{K_0 \cdot K_R}{|K_0||K_R|} \Big) K_1 \\ &+ \varepsilon \tfrac{e\omega_{pe}}{2m_e c^2} \nabla \Big(A_0 A_R^* \tfrac{K_0 \cdot K_R}{|K_0||K_R|} \Big). \end{aligned}$$

2. Three waves interaction model.

$$f_0 = \frac{\omega_{pe}}{c} A_0, \;\; f_R = \frac{\omega_{pe}}{c} A_R, \;\; f = \frac{K_1 \cdot \mathcal{E}}{|K_1|},$$

Take f_0 fixed (pump wave). Leading order in ε :

$$\left(\partial_t + \frac{c^2}{\omega_R} K_R \cdot \nabla\right) f_R = \frac{e|K_1|}{2m_e\omega_R} f^* f_0 \cos(\theta)$$

$$\left(\partial_t + \frac{v_{th}^2}{\omega_{pe}} K_1 \cdot \nabla\right) f = \frac{e|K_1|}{2m_e\omega_{pe}} f_0 f_R^* \cos(\theta).$$

$$\theta = \text{ angle between } K_0 \text{ and} K_R$$

2. Three waves interaction model.

$$f_0 = \frac{\omega_{pe}}{c} A_0, \;\; f_R = \frac{\omega_{pe}}{c} A_R, \;\; f = \frac{K_1 \cdot \mathcal{E}}{|K_1|},$$

Take f_0 fixed (pump wave). Leading order in ε :

$$\left(\partial_t + \frac{c^2}{\omega_R} K_R \cdot \nabla\right) f_R = \frac{e|K_1|}{2m_e\omega_R} f^* f_0 \cos(\theta)$$

$$\left(\partial_t + \frac{v_{th}^2}{\omega_{pe}} K_1 \cdot \nabla\right) f = \frac{e|K_1|}{2m_e\omega_{pe}} f_0 f_R^* \cos(\theta).$$

$$\theta = \text{ angle between } K_0 \text{ and} K_R$$

Amplification rate is proportional to

$$\beta^2 = \frac{\cos^2(\theta)|K_1|^2}{\omega_R\omega_{pe}} = \frac{|K_1|^2}{\omega_R} \frac{K_0 \cdot K_R}{|K_0||K_R|}$$

2. Three waves interaction model.

$$K_0 = \begin{pmatrix} k_0 \\ 0 \end{pmatrix}, \quad K_R = \begin{pmatrix} k \\ l \end{pmatrix}, \quad K_1 = \begin{pmatrix} k_1 \\ l_1 \end{pmatrix}.$$

$$K_0 = K_R + K_1, \quad \omega_0 = \omega_{pe} + \omega_R + \omega_1$$

$$l_1 = -l$$

2. Three waves interaction model.

$$K_0 = \begin{pmatrix} k_0 \\ 0 \end{pmatrix}, \quad K_R = \begin{pmatrix} k \\ l \end{pmatrix}, \quad K_1 = \begin{pmatrix} k_1 \\ l_1 \end{pmatrix}.$$

$$K_0 = K_R + K_1, \quad \omega_0 = \omega_{pe} + \omega_R + \omega_1$$

$$l_1 = -l$$

$$\omega_0^2 = 1 + {k_0}^2, \quad \omega_R^2 = 1 + (k^2 + l^2)$$

$$(1 + \omega_1^2) = 1 + \alpha^2({k_1}^2 + {l_1}^2), \qquad \alpha = \frac{v_{th}}{c} << 1$$

2. Three waves interaction model.

System

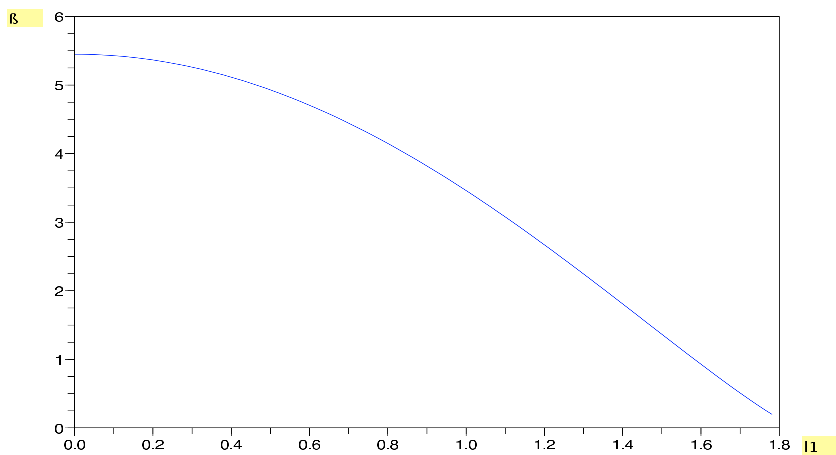
$$\text{Maximize } \beta = \frac{\sqrt{{k_1}^2 + {l_1}^2}}{\sqrt{1 + (k_0 - k_1)^2 + {l_1}^2}} \frac{|k_0 - k_1|}{\sqrt{(k_0 - k_1)^2 + {l_1}^2}}$$

$$\text{with } \sqrt{1 + {k_0}^2} = \sqrt{1 + (k_0 - k_1)^2 + {l_1}^2} + \sqrt{1 + \alpha({k_1}^2 + {l_1}^2)}$$

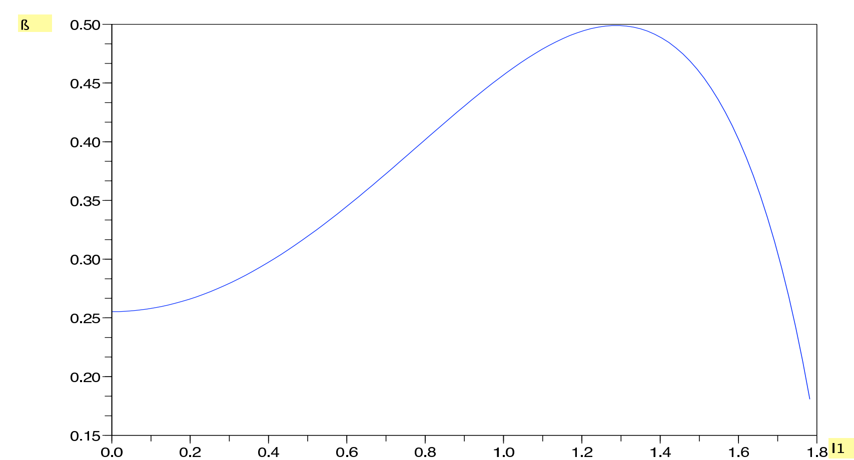
2. Three waves interaction model.

Results:

Backscaterred Raman :



Forward Raman:



2. Three waves interaction model.

$$K_0 = \begin{pmatrix} k_0 \\ 0 \end{pmatrix} = K_R^1 + K_1^1 = \begin{pmatrix} k_R^1 \\ l_R^1 \end{pmatrix} + \begin{pmatrix} k_1^1 \\ l_1^1 \end{pmatrix}$$

$$= K_R^2 + K_1^2 = \begin{pmatrix} k_R^2 \\ l_R^2 \end{pmatrix} + \begin{pmatrix} k_1^2 \\ l_1^2 \end{pmatrix}$$

$$= K_R^{2,sym} + K_1^{2,sym} = \begin{pmatrix} k_R^{2,sym} \\ l_R^{2,sym} \end{pmatrix} + \begin{pmatrix} k_1^{2,sym} \\ l_1^{2,sym} \end{pmatrix}$$

$$\begin{aligned} \omega_0 &= \omega_{pe} + \omega_R^i + \omega_1^i, \quad i = 1, 2 \\ &= \omega_{pe} + \omega_R^{2,sym} + \omega_1^{2,sym} \end{aligned}$$

2. Three waves interaction model.

$$E = E_0 e^{-i\omega_{pe} t} + \frac{i\omega_0}{c} A_0 e^{i(K_0 \cdot X - \omega_0 t)} + \frac{i\omega_R^1}{c} A_R^1 e^{i(K_R^1 \cdot X - \omega_R^1 t)}$$

$$+ \frac{i\omega_R^2}{c} A_R^2 e^{i(K_R^2 \cdot X - \omega_R^2 t)} + \frac{i\omega_R^{2,sym}}{c} A_R^{2,sym} e^{i(K_R^{2,sym} \cdot X - \omega_R^{2,sym} t)}$$

2. Three waves interaction model.

$$E = E_0 e^{-i\omega_{pe} t} + \frac{i\omega_0}{c} A_0 e^{i(K_0 \cdot X - \omega_0 t)} + \frac{i\omega_R^1}{c} A_R^1 e^{i(K_R^1 \cdot X - \omega_R^1 t)}$$

$$+ \frac{i\omega_R^2}{c} A_R^2 e^{i(K_R^2 \cdot X - \omega_R^2 t)} + \frac{i\omega_R^{2,sym}}{c} A_R^{2,sym} e^{i(K_R^{2,sym} \cdot X - \omega_R^{2,sym} t)}$$

$$i\Big(\partial_t + \frac{c^2}{\omega_0} K_0 \cdot \nabla\Big) A_0 + \frac{1}{2\omega_0}\Big(c^2 \Delta - \frac{c^4}{\omega_0^2}(K_0 \cdot \nabla)^2\Big) A_0$$

$$= \frac{2\pi e^2}{\omega_0 m_e} < n_e > A_0 - \frac{e}{2\omega_0 m_e}\Big((\nabla \cdot E) A_R^1 e^{-i\theta_1^1} \frac{K_0 \cdot K_R^1}{|K_0||K_R^1|}$$

$$+ (\nabla \cdot E) A_R^2 e^{-i\theta_1^2} \frac{K_0 \cdot K_R^2}{|K_0||K_R^2|} + (\nabla \cdot E) A_R^{2,sym} e^{-i\theta_1^{2,sym}} \frac{K_0 \cdot K_R^{2,sym}}{|K_0||K_R^{2,sym}|}\Big)$$

2. Three waves interaction model.

$$i\Big(\partial_t + \frac{c^2}{\omega_R^1} K_R^1 \cdot \nabla\Big) A_R^1 + \frac{1}{2\omega_R^1}\Big(c^2\Delta - \frac{c^4}{(\omega_R^1)^2}(K_R^1 \cdot \nabla)^2\Big) A_R^1$$

$$= \frac{2\pi e^2}{\omega_R^1 m_e} < n_e > A_R^1 - \frac{e}{2\omega_R^1 m_e}(\nabla \cdot E^*) A_0 e^{i\theta_1^1} \frac{K_0 \cdot K_R^1}{|K_0||K_R^1|}$$

$$i\Big(\partial_t + \frac{c^2}{\omega_R^2} K_R^2 \cdot \nabla\Big) A_R^2 + \frac{1}{2\omega_R^2}\Big(c^2\Delta - \frac{c^4}{(\omega_R^2)^2}(K_R^2 \cdot \nabla)^2\Big) A_R^2$$

$$= \frac{2\pi e^2}{\omega_R^2 m_e} < n_e > A_R^2 - \frac{e}{2\omega_R^2 m_e}(\nabla \cdot E^*) A_0 e^{i\theta_1^2} \frac{K_0 \cdot K_R^2}{|K_0||K_R^2|}$$

2. Three waves interaction model.

$$i\partial_t E + \frac{v_{th}^2}{2\omega_{pe}}\Delta E = \frac{\omega_{pe}}{2}P_{||}\left(< n_e > E\right) + \frac{\omega_{pe}e}{2m_e c^2}\nabla\Big(A_0 A_R^{1}{}^* \frac{K_0\cdot K_R^1}{|K_0||K_R^1|}e^{i\theta_1^1}$$

$$+A_0 A_R^{2}{}^* \frac{K_0\cdot K_R^2}{|K_0||K_R^2|}e^{i\theta_1^2} + A_0 A_R^{2,sym}{}^* \frac{K_0\cdot K_R^{2,sym}}{|K_0||K_R^{2,sym}|}e^{i\theta_1^{2,sym}}\Big)$$

$$\left(\partial_t^2 - c_s^2\Delta\right) < n_e > = \frac{1}{4\pi n_i}\Delta\Big(|E|^2 + \frac{\omega_{pe}^2}{c^2}\left(|A_0|^2 + |A_R^1|^2 + |A_R^2|^2 + |A_R^{2,sym}|^2\right)\Big)$$

Pertinent parameter : $\sqrt{\frac{\omega_R^1}{\omega_R^2}}$

2. Three waves interaction model.

x $\in [0, 150]$, y$\in$[0,80], N_x=150, N_y=80

T=50, $\mathrm{N}_t = 288$
Collision test case + (A_R^1,K_0) in the x-axis.

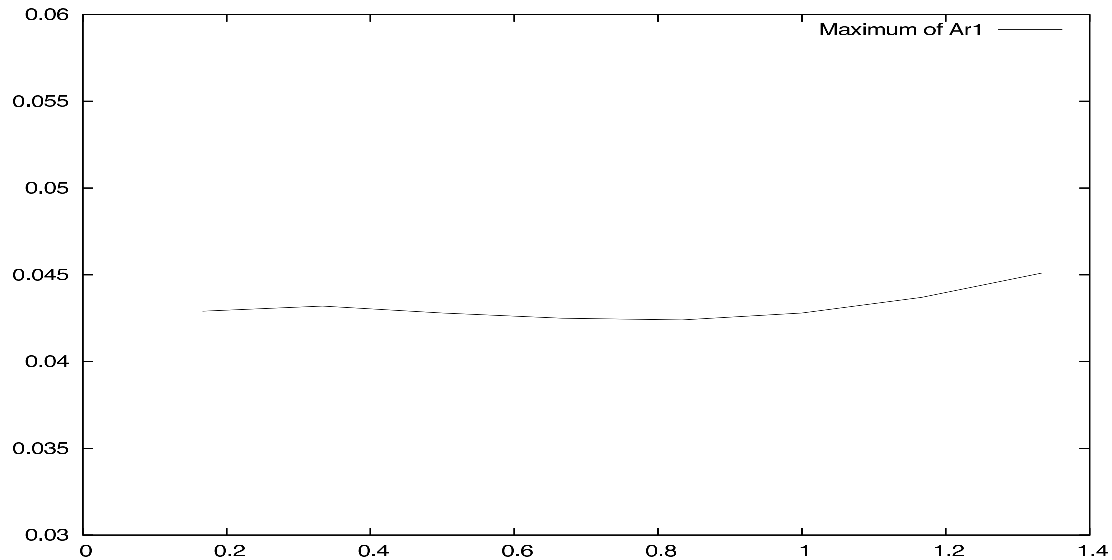
$$A_C(0) = 0.3e^{-0.03(x-75)^2}e^{-0.01(y-40)^2}.$$

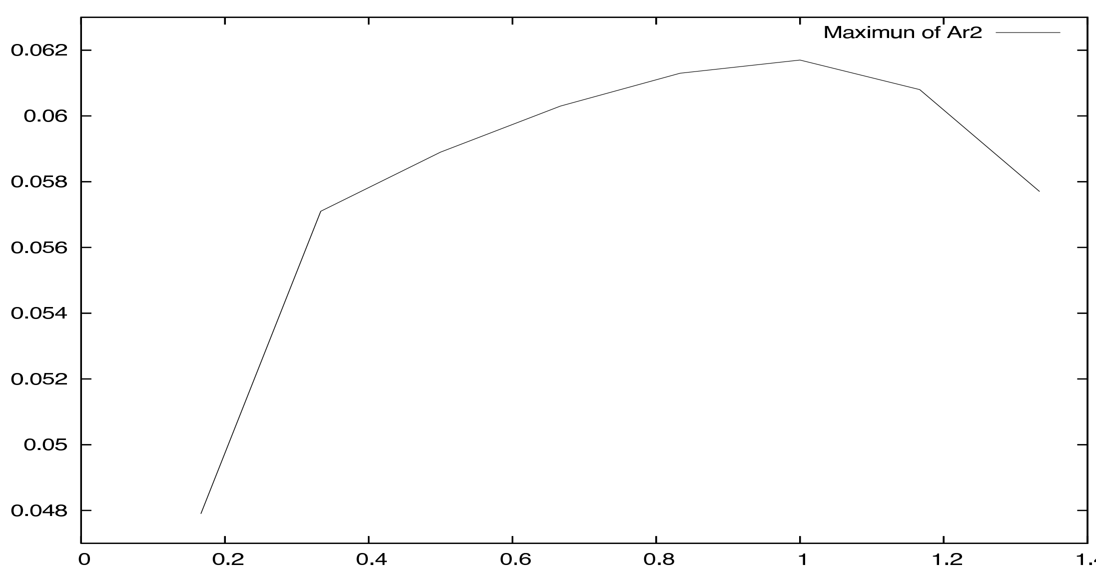
$$A_R^1(0) = 0.03e^{-0.003(x-90)^2}e^{-00.1(y-40)^2}$$

$$A_R^2(0) = 0.03e^{-0.003(x-55)^2}e^{-00.1(y-40)^2}$$

$$E(0) = 0, \quad < n_e >= \partial_t < n_e >= 0$$

2. Three waves interaction model.





3. The Cauchy problem.

Structure of the system:

$$i \left(\partial_t + \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} \partial_z \right) \begin{pmatrix} A_0 \\ AR \\ E_0 \end{pmatrix} + \Delta \begin{pmatrix} A_0 \\ AR \\ E_0 \end{pmatrix}$$

$$= n \begin{pmatrix} A_0 \\ AR \\ E_0 \end{pmatrix} + \begin{pmatrix} -\nabla \cdot E_0 A_R \\ -\nabla \cdot E_0^* A_0 \\ \nabla (A_R^* \cdot A_0) \end{pmatrix}$$

$$\left(\partial_t^2 - \Delta \right) n = \Delta \left(|A_0|^2 + |A_R|^2 + |E_0|^2 \right),$$

3. The Cauchy problem.

The system is an extension of :

$$i\partial_t E + \Delta E = nE,$$

$$\partial_t^2 n - \Delta n = \Delta |E|^2$$

Cauchy problem: Sulem-Sulem (79), Added-Added (88), Ozawa-Tsutsumi (92), Bourgain (96), Ginibre-Tsutsumi-Velo (97).

Finite-time blow-up: Glangetas-Merle (94).

Numerical scheme: Glassey, huge physical litterature

3. The Cauchy problem.

Theorem (Colins) The system is locally well-posed in H^s.

Conservation:

$$\int 2|A_0|^2 + |A_r|^2 + |E_0|^2(t) = Cte$$

Idea: couple Ozawa-Tsutsumi with energy estimates for quasilinear hyperbolic systems.

Problem :

$$i\partial_t \begin{pmatrix} A_0 \ AR \\ E_0 \end{pmatrix} = \begin{pmatrix} -\nabla \cdot E_0 A_R \\ -\nabla \cdot E_0^* A_0 \\ \nabla(A_R^* \cdot A_0) \end{pmatrix}$$

is not hyperbolic!

3. The Cauchy problem.

If $A_0 = u_1 + iu_2$, $A_R = u_3 + iu_4$, $E_0 = u_5 + iu_6$, the system reads:

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = M \partial_x \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}$$

3. The Cauchy problem.

with

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & -u_4 & -u_3 \\ 0 & 0 & 0 & 0 & u_3 & -u_4 \\ 0 & 0 & 0 & 0 & -u_2 & u_1 \\ 0 & 0 & 0 & 0 & u_1 & u_2 \\ -u_4 & u_3 & u_2 & -u_1 & 0 & 0 \\ -u_3 & -u_4 & -u_1 & -u_2 & 0 & 0 \end{pmatrix}$$

Idea : $i\frac{\partial\cdot}{dt}$ behaves like $-\Delta\cdot$

4. Solitary waves

a) The semi-trivial waves.

$$
\begin{aligned}
i\partial_t u_1 &= -\Delta u_1 - |u_1|^{p-1}u_1 + \gamma \overline{u_2} u_3, \\
i\partial_t u_2 &= -\Delta u_2 - |u_2|^{p-1}u_2 - \gamma u_3 \overline{u_1}, \\
i\partial_t u_3 &= -\Delta u_3 - |u_3|^{p-1}u_3 - \gamma u_1 u_2,
\end{aligned}
$$

4. Solitary waves

a) The semi-trivial waves.

$$
\begin{aligned}
i\partial_t u_1 &= -\Delta u_1 - |u_1|^{p-1} u_1 + \gamma \overline{u_2} u_3,\\
i\partial_t u_2 &= -\Delta u_2 - |u_2|^{p-1} u_2 - \gamma u_3 \overline{u_1},\\
i\partial_t u_3 &= -\Delta u_3 - |u_3|^{p-1} u_3 - \gamma u_1 u_2,
\end{aligned}
$$

$$
(e^{2i\omega t}\varphi, 0, 0), \quad (0, e^{2i\omega t}\varphi, 0), \quad (0, 0, e^{2i\omega t}\varphi)
$$

4. Solitary waves

a) The semi-trivial waves.

$$i\partial_t u_1 = -\Delta u_1 - |u_1|^{p-1} u_1 + \gamma \overline{u_2} u_3,$$
$$i\partial_t u_2 = -\Delta u_2 - |u_2|^{p-1} u_2 - \gamma u_3 \overline{u_1},$$
$$i\partial_t u_3 = -\Delta u_3 - |u_3|^{p-1} u_3 - \gamma u_1 u_2,$$

$$(e^{2i\omega t}\varphi, 0, 0), \quad (0, e^{2i\omega t}\varphi, 0), \quad (0, 0, e^{2i\omega t}\varphi)$$

$$-\Delta\varphi + 2\omega\varphi - |\varphi|^{p-1}\varphi = 0$$

4. Solitary waves

- Conserved Quantities :

$$E(\vec{u}) = \sum_{j=1}^{3} \left(\frac{1}{2} \|\nabla u_j\|_{L^2}^2 - \frac{1}{p+1} \|u_j\|_{L^{p+1}}^{p+1} \right) - \gamma \operatorname{Re} \int_{\mathbb{R}^N} u_1 u_2 \overline{u_3}\, dx,$$

$$Q_1(\vec{u}) = \|u_1\|_{L^2}^2 + \|u_3\|_{L^2}^2 \quad \text{and} \quad Q_2(\vec{u}) = \|u_2\|_{L^2}^2 + \|u_3\|_{L^2}^2,$$

4. Solitary waves

- Conserved Quantities :

$$E(\vec{u}) = \sum_{j=1}^{3} \left(\frac{1}{2} \|\nabla u_j\|_{L^2}^2 - \frac{1}{p+1} \|u_j\|_{L^{p+1}}^{p+1} \right) - \gamma \,\mathrm{Re} \int_{\mathbb{R}^N} u_1 u_2 \overline{u_3}\, dx,$$

$$Q_1(\vec{u}) = \|u_1\|_{L^2}^2 + \|u_3\|_{L^2}^2 \quad \text{and} \quad Q_2(\vec{u}) = \|u_2\|_{L^2}^2 + \|u_3\|_{L^2}^2,$$

- Global Well-Posedness : in H^1 for N=1,2,3, $1 < p < 1 + 4/N$, $\gamma > 0$.

see T. Cazenave

4. Solitary waves

- Conserved Quantities :

$$E(\vec{u}) = \sum_{j=1}^{3} \left(\frac{1}{2} \|\nabla u_j\|_{L^2}^2 - \frac{1}{p+1} \|u_j\|_{L^{p+1}}^{p+1} \right) - \gamma \,\mathrm{Re} \int_{\mathbb{R}^N} u_1 u_2 \overline{u_3}\, dx,$$

$$Q_1(\vec{u}) = \|u_1\|_{L^2}^2 + \|u_3\|_{L^2}^2 \quad \text{and} \quad Q_2(\vec{u}) = \|u_2\|_{L^2}^2 + \|u_3\|_{L^2}^2,$$

- Global Well-Posedness : in H^1 for N=1,2,3, $1 < p < 1 + 4/N$, $\gamma > 0$.

see T. Cazenave

- $u_\omega(t) = e^{2i\omega t}\varphi$ solution to

$$i\partial_t u = -\Delta u - |u|^{p-1} u$$

  – stable if $1 < p < 4/N$

  – unstable if $1 + 4/N \leq p < 1 + 4/(N-2)$

4. Solitary waves

- What about

$$(e^{2i\omega t}\varphi, 0, 0), \quad (0, e^{2i\omega t}\varphi, 0), \quad (0, 0, e^{2i\omega t}\varphi)?$$

4. Solitary waves

- What about

$$(e^{2i\omega t}\varphi, 0, 0), \quad (0, e^{2i\omega t}\varphi, 0), \quad (0, 0, e^{2i\omega t}\varphi)?$$

- Def : $(e^{2i\omega t}\varphi, 0, 0)$ is stale if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^N, \mathbb{C}^3)$ and $\|u_0 - (\varphi, 0, 0)\|_{H^1} < \delta$, then the solution $u(t)$ with $u(0) = u_0$ satisfies

$$\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^N} \|u(t) - (e^{i\theta}\varphi(\cdot + y), 0, 0)\|_{H^1} < \varepsilon.$$

4. Solitary waves

- What about

$$(e^{2i\omega t}\varphi, 0, 0), \quad (0, e^{2i\omega t}\varphi, 0), \quad (0, 0, e^{2i\omega t}\varphi)?$$

- Def : $(e^{2i\omega t}\varphi, 0, 0)$ is stale if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^N, \mathbb{C}^3)$ and $\|u_0 - (\varphi, 0, 0)\|_{H^1} < \delta$, then the solution $u(t)$ with $u(0) = u_0$ satisfies

$$\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^N} \|u(t) - (e^{i\theta}\varphi(\cdot + y), 0, 0)\|_{H^1} < \varepsilon.$$

- $1 + 4/N \leq p < 1 + 4/(N-2) \Longrightarrow$ No

4. Solitary waves

Theorem 1: (MC, T. Colin and M. Ohta, **Ann. IHP, 09'**)) Let $1 \leq N \leq 3$, $1 < p < 1 + 4/N$, $\gamma > 0$, $\omega > 0$, Then, the solitary wave solutions $(e^{2i\omega t}\varphi, 0, 0)$ and $(0, e^{2i\omega t}\varphi, 0)$ are orbitally stable.

$\rightsquigarrow$ Variational method

4. Solitary waves

Theorem 1: (MC, T. Colin and M. Ohta, **Ann. IHP, 09'**)) Let $1 \leq N \leq 3$, $1 < p < 1 + 4/N$, $\gamma > 0$, $\omega > 0$, Then, the solitary wave solutions $(e^{2i\omega t}\varphi, 0, 0)$ and $(0, e^{2i\omega t}\varphi, 0)$ are orbitally stable.

$\rightsquigarrow$ Variational method

Theorem 2 : (MC, T. Colin and M. Ohta, **Ann. IHP, 09'**) Let $1 \leq N \leq 3$, $1 < p < 1 + 4/N$, $\omega > 0$. Then, there exists a positive constant γ^* satisfying the following.

(i) If $0 < \gamma < \gamma^*$, then the solitary wave solution $(0, 0, e^{2i\omega t}\varphi)$ is orbitally stable.

(ii) Assume further that $N \leq 2$ and $p > 2$. If $\gamma > \gamma^*$, then the solitary wave solution $(0, 0, e^{2i\omega t}\varphi)$ is orbitally unstable.

$\rightsquigarrow$ Spectral method

4. Solitary waves

$$\gamma^* = \inf \left\{ \frac{\|\nabla v\|_{L^2}^2 + \omega \|v\|_{L^2}^2}{\int_{\mathbb{R}^N} \varphi_\omega(x) |v(x)|^2 \, dx} : v \in H^1(\mathbb{R}^N) \setminus \{0\} \right\}.$$

4. Solitary waves

$$\gamma^* = \inf\left\{ \frac{\|\nabla v\|_{L^2}^2 + \omega\|v\|_{L^2}^2}{\int_{\mathbb{R}^N} \varphi_\omega(x)|v(x)|^2\,dx} : v \in H^1(\mathbb{R}^N)\setminus\{0\}\right\}.$$

Theorem 3 : (MC, T. Colin and M. Ohta, **Funkcialaj Ekvacioj, 09'**) Let $1 \le N \le 3$, $1 < p < 1 + 4/N$, $\omega > 0$. If $\gamma > \gamma^*$, then the standing wave solution $(0, 0, e^{2i\omega t}\varphi)$ is orbitally unstable.

4. Solitary waves.

b) Bifurcation from Semi-Trivial solitary waves

$$
\begin{aligned}
i\partial_t u_1 &= -\Delta u_1 - |u_1|^{p-1}u_1 + \gamma \overline{u_2} u_3, \\
i\partial_t u_2 &= -\Delta u_2 - |u_2|^{p-1}u_2 - \gamma u_3 \overline{u_1}, \\
i\partial_t u_3 &= -\Delta u_3 - |u_3|^{p-1}u_3 - \gamma u_1 u_2,
\end{aligned}
$$

4. Solitary waves.

b) Bifurcation from Semi-Trivial solitary waves

$$i\partial_t u_1 = -\Delta u_1 - |u_1|^{p-1} u_1 + \gamma \overline{u_2} u_3,$$
$$i\partial_t u_2 = -\Delta u_2 - |u_2|^{p-1} u_2 - \gamma u_3 \overline{u_1},$$
$$i\partial_t u_3 = -\Delta u_3 - |u_3|^{p-1} u_3 - \gamma u_1 u_2,$$

$\rightsquigarrow \gamma^*$ is a bifurcation point (Crandall and Rabinowitz)

4. Solitary waves.

b) Bifurcation from Semi-Trivial solitary waves

$$
\begin{aligned}
i\partial_t u_1 &= -\Delta u_1 - |u_1|^{p-1}u_1 + \gamma \overline{u_2} u_3,\\
i\partial_t u_2 &= -\Delta u_2 - |u_2|^{p-1}u_2 - \gamma u_3 \overline{u_1},\\
i\partial_t u_3 &= -\Delta u_3 - |u_3|^{p-1}u_3 - \gamma u_1 u_2,
\end{aligned}
$$

$\rightsquigarrow$ γ^* is a bifurcation point (Crandall and Rabinowitz)

$\rightsquigarrow$ What about the structure and stability properties?

b) Bifurcation from Semi-Trivial solitary waves

$$
\begin{aligned}
i\partial_t u_1 &= -\Delta u_1 - |u_1|^{p-1} u_1 + \gamma \overline{u_2} u_3, \\
i\partial_t u_2 &= -\Delta u_2 - |u_2|^{p-1} u_2 - \gamma u_3 \overline{u_1}, \\
i\partial_t u_3 &= -\Delta u_3 - |u_3|^{p-1} u_3 - \gamma u_1 u_2,
\end{aligned}
$$

$\rightsquigarrow$ γ^* is a bifurcation point (Crandall and Rabinowitz)

$\rightsquigarrow$ What about the structure and stability properties?

$\rightsquigarrow$ Difficult to answer in the general setting!

$\rightsquigarrow$ $p = 2$ and $u_1 = u_2$

4. Solitary waves.

$$\begin{cases} i\partial_t u_1 = -\Delta u_1 - \kappa |u_1| u_1 - \gamma \overline{u_1} u_2 \\ i\partial_t u_2 = -2\Delta u_2 - 2|u_2| u_2 - \gamma {u_1}^2 \end{cases}$$

$$\rightsquigarrow \gamma^* = 1$$

$$\rightsquigarrow (0, e^{2i\omega t}\varphi)$$

Theorem 4 : Let $N \leq 3$, $\kappa \in \mathbb{R}$, $\gamma > 0$, $\omega > 0$. Then, the semi-trivial standing wave solution $(0, e^{2i\omega t}\varphi)$ is stable if $0 < \gamma < 1$, and it is unstable if $\gamma > 1$.

4. Solitary waves.

Bifurcation Branch : $(\alpha\varphi, \beta\varphi), \quad (\alpha, \beta) \in]0, \infty[^2$

$$-\Delta\varphi + 2\omega\varphi - \varphi^2 = 0$$

$$\begin{cases} -\Delta\varphi_1 + \omega\varphi_1 = \kappa|\varphi_1|\varphi_1 + \gamma\overline{\varphi_1}\varphi_2 \\ -\Delta\varphi_2 + \omega\varphi_2 = |\varphi_2|\varphi_2 + (\gamma/2)\varphi_1^2 \end{cases}$$

$$\kappa\alpha + \gamma\beta = 1, \quad \gamma\alpha^2 + 2\beta^2 = 2\beta$$

4. Solitary waves.

$$\mathcal{S}_{\kappa,\gamma} = \{(x,y) \in]0,\infty[^2 \colon \kappa x + \gamma y = 1,\ \gamma x^2 + 2y^2 = 2y\}.$$

$$\alpha_{\pm} = \frac{(2-\gamma)\kappa \pm \gamma\sqrt{\kappa^2 + 2\gamma(\gamma-1)}}{2\kappa^2+\gamma^3}$$

$$\beta_{\pm} = \frac{\kappa^2 + \gamma^2 \pm \kappa\sqrt{\kappa^2 + 2\gamma(\gamma-1)}}{2\kappa^2+\gamma^3},$$

$$\alpha_0 = \frac{(2-\gamma)\kappa}{2\kappa^2+\gamma^3}, \quad \beta_0 = \frac{\kappa^2+\gamma^2}{2\kappa^2+\gamma^3}$$

4. Solitary waves.

$$\mathcal{S}_{\kappa,\gamma} = \{(x,y) \in]0,\infty[^2 \colon \kappa x + \gamma y = 1,\ \gamma x^2 + 2y^2 = 2y\}.$$

$$\alpha_\pm = \frac{(2-\gamma)\kappa \pm \gamma\sqrt{\kappa^2 + 2\gamma(\gamma-1)}}{2\kappa^2+\gamma^3}$$

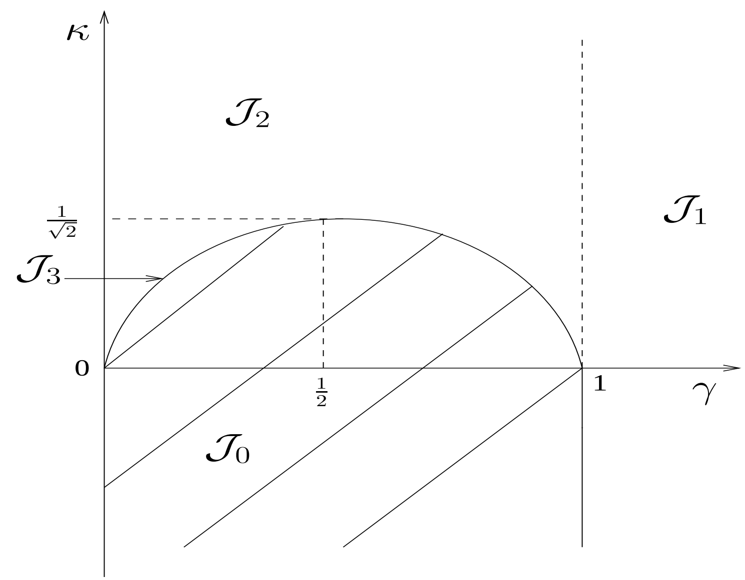
$$\beta_\pm = \frac{\kappa^2 + \gamma^2 \pm \kappa\sqrt{\kappa^2 + 2\gamma(\gamma-1)}}{2\kappa^2+\gamma^3},$$

$$\alpha_0 = \frac{(2-\gamma)\kappa}{2\kappa^2+\gamma^3}, \quad \beta_0 = \frac{\kappa^2+\gamma^2}{2\kappa^2+\gamma^3}$$

- $\kappa \le 0$, $(\alpha_+, \beta_-) \to (0,1)$ as $\gamma \to 1+0$: $\{(\alpha_+\varphi, \beta_-\varphi) : \gamma > 1\}$ bifurcate from $(0,\varphi)$ at $\gamma = 1$
- $\kappa > 0$, $(\alpha_-, \beta_+) \to (0,1)$ as $\gamma \to 1-0$: $\{(\alpha_-\varphi, \beta_+\varphi) : \gamma_m < \gamma < 1\}$

4. Solitary waves.

$$\mathcal{J}_3 = \{(\kappa,\gamma) : 0 < \gamma < 1,\ \kappa = \sqrt{2\gamma(1-\gamma)}\}$$



,

Proposition 1 : (0) If $(\kappa,\gamma) \in \mathcal{J}_0$, then $\mathcal{S}_{\kappa,\gamma}$ is empty.

(1) If $(\kappa,\gamma) \in \mathcal{J}_1$, then $\mathcal{S}_{\kappa,\gamma} = \{(\alpha_+, \beta_-)\}$.

(2) If $(\kappa,\gamma) \in \mathcal{J}_2$, then $\mathcal{S}_{\kappa,\gamma} = \{(\alpha_+, \beta_-),\ (\alpha_-, \beta_+)\}$.

(3) If $(\kappa,\gamma) \in \mathcal{J}_3$, then $\mathcal{S}_{\kappa,\gamma} = \{(\alpha_0, \beta_0)\}$.

4. Solitary waves.

Theorem 5 : (MC and M. Ohta, 11') Let $N \geq 3$. Then

1) If $(\kappa, \gamma) \in \mathcal{J}_1 \cup \mathcal{J}_2$, the standing wave $(e^{i\omega t}\alpha_+\varphi, e^{2i\omega t}\beta_-\varphi)$ is stable.

2) If $(\kappa, \gamma) \in \mathcal{J}_2$, the standing wave $(e^{i\omega t}\alpha_-\varphi, e^{2i\omega t}\beta_+\varphi)$ is unstable.

3) If $\kappa > 0$ and $\gamma = 1$, then for any $\omega > 0$, the standing wave $(0, e^{2i\omega t}\varphi)$ is unstable.

4) If $\kappa \leq 0$ and $\gamma = 1$, then for any $\omega > 0$, the standing wave $(0, e^{2i\omega t}\varphi)$ is stable.

4. Solitary waves.

$$\langle S''_\omega(\Phi)\vec{u},\vec{u}\rangle = \langle \mathcal{L}_R \mathrm{Re}(\vec{u}), \mathrm{Re}(\vec{u})\rangle + \langle \mathcal{L}_I \mathrm{Im}(\vec{u}), \mathrm{Im}(\vec{u})\rangle$$

$$\mathcal{L}_R = \begin{bmatrix} -\Delta+\omega & 0 \\ 0 & -\Delta+\omega \end{bmatrix} - \begin{bmatrix} (2\alpha+\gamma\beta)\varphi & \gamma\alpha\varphi \\ \gamma\alpha\varphi & 2\beta\varphi \end{bmatrix},$$

$$\mathcal{L}_I = \begin{bmatrix} -\Delta+\omega & 0 \\ 0 & -\Delta+\omega \end{bmatrix} - \begin{bmatrix} (\alpha-\gamma\beta)\varphi & \gamma\alpha\varphi \\ \gamma\alpha\varphi & \beta\varphi \end{bmatrix}.$$

4. Solitary waves.

c) The Ground State problem.

$$\begin{cases} -\Delta\varphi_1 + \omega\varphi_1 = \kappa|\varphi_1|\varphi_1 + \gamma\overline{\varphi_1}\varphi_2 \\ -\Delta\varphi_2 + \omega\varphi_2 = |\varphi_2|\varphi_2 + (\gamma/2)\varphi_1^2 \end{cases}$$

$$\begin{aligned} S_\omega(\vec{u}) &= \tfrac{1}{2}\left(\|\nabla u_1\|_{L^2}^2 + \|\nabla u_2\|_{L^2}^2\right) - \tfrac{\kappa}{3}\|u_1\|_{L^3}^3 - \tfrac{1}{3}\|u_2\|_{L^3}^3 \\ &\quad -\tfrac{\gamma}{2}\mathrm{Re}\int_{\mathbb{R}^N} u_1^2\overline{u_2}dx + \omega\tfrac{1}{2}\left(\|\vec{u}_1\|_{L^2}^2 + \|u_2\|_{L^2}^2\right) \end{aligned}$$

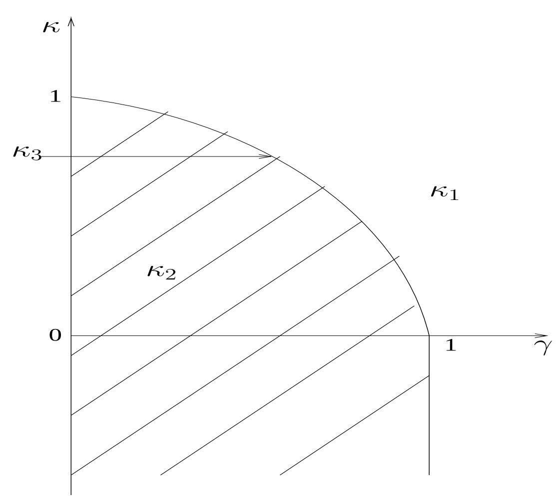
Ground state : solution which minimize the action S_ω among all the solutions.

4. Solitary waves.

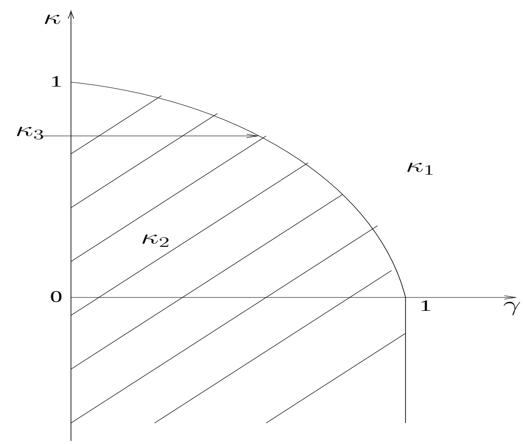
$$\kappa(\gamma) = \frac{1}{2}(\gamma + 2)\sqrt{1-\gamma}, \quad 0 < \gamma < 1$$

$$\rightsquigarrow \; \gamma(\kappa)$$

$$\mathcal{K}_3 = \{(\kappa, \gamma) : 0 < \kappa < 1, \; \gamma = \gamma(\kappa)\}$$



4. Solitary waves.



$$\mathcal{G}_\omega^0 = \{G(\theta)\tau_y(0,\varphi) : \theta \in \mathbb{R},\ y \in \mathbb{R}^N\},$$

$$\mathcal{G}_\omega^1 = \{G(\theta)\tau_y(\alpha_+\varphi, \beta_-\varphi) : \theta \in \mathbb{R},\ y \in \mathbb{R}^N\}$$

Theorem 6 : (MC and M. Ohta, 11') Let $N \leq 3$ and $\omega > 0$.

(1) If $(\kappa,\gamma) \in \mathcal{K}_1$, then $\mathcal{G}_\omega = \mathcal{G}_\omega^1$

(2) If $(\kappa,\gamma) \in \mathcal{K}_2$, then $\mathcal{G}_\omega = \mathcal{G}_\omega^0$

(3) If $(\kappa,\gamma) \in \mathcal{K}_3$, then $\mathcal{G}_\omega = \mathcal{G}_\omega^0 \cup \mathcal{G}_\omega^1$.

4. Solitary waves.

- First step : Variational characterization of φ.

$$\|\varphi\|_{L^3} = \inf \left\{ \|v\|^2_{H^1_\omega} / \|v\|^2_{L^3} : v \in H^1(\mathbb{R}^N) \setminus \{0\} \right\}.$$

$$\{v \in H^1(\mathbb{R}^N) : \|v\|^2_{H^1_\omega} = \|v\|^3_{L^3} = \|\varphi\|^3_{L^3}\} = \{e^{i\theta}\varphi(\cdot + y) : \theta \in \mathbb{R},\ y \in \mathbb{R}^N\}.$$

4. Solitary waves.

- First step : Variational characterization of φ.

$$\|\varphi\|_{L^3} = \inf\left\{\|v\|_{H^1_\omega}^2/\|v\|_{L^3}^2 : v \in H^1(\mathbb{R}^N)\setminus\{0\}\right\}.$$

$$\{v \in H^1(\mathbb{R}^N) : \|v\|_{H^1_\omega}^2 = \|v\|_{L^3}^3 = \|\varphi\|_{L^3}^3\} = \{e^{i\theta}\varphi(\cdot + y) : \theta \in \mathbb{R},\ y \in \mathbb{R}^N\}.$$

- Second step : Let $(u_1, u_2) \in \mathcal{G}_\omega$ and put

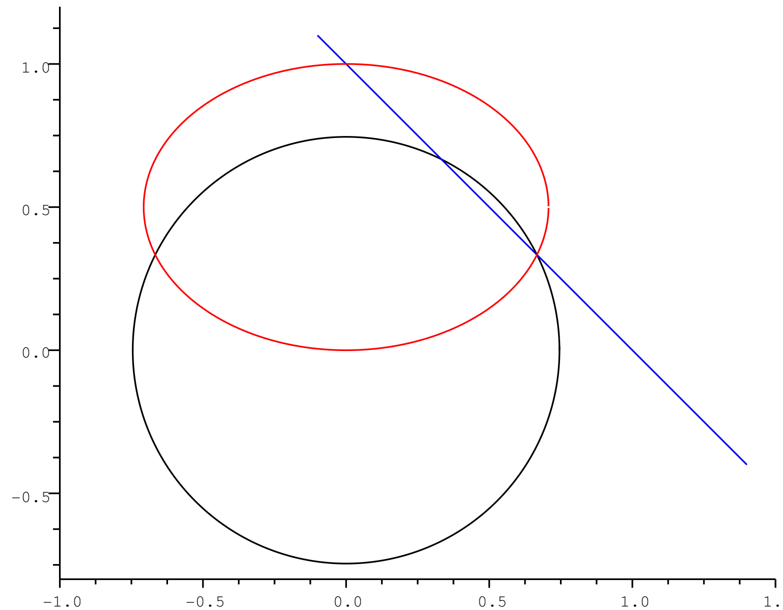
$$a = \|u_1\|_{L^3}/\|\varphi_\omega\|_{L^3}, \quad b = \|u_2\|_{L^3}/\|\varphi_\omega\|_{L^3}.$$

Then, $a \geq 0$, $b > 0$

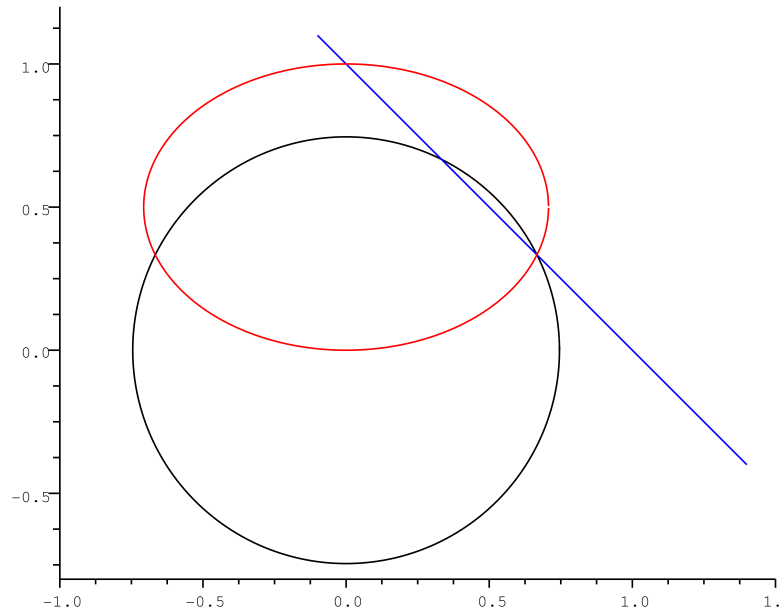
$$a^2 \leq a^2(\kappa a + \gamma b), \quad 2b \leq 2b^2 + \gamma a^2, \quad a^2 + b^2 \leq \ell.$$

$$\ell = \begin{cases} \alpha_+^2 + \beta_-^2 & \text{if } (\kappa,\gamma) \in \mathcal{K}_1, \\ 1 & \text{if } (\kappa,\gamma) \in \mathcal{K}_2 \cup \mathcal{K}_3, \end{cases}$$

4. Solitary waves.



4. Solitary waves.



- Third step :

(1) If $(\kappa, \gamma) \in \mathcal{K}_1$, then $(a, b) = (\alpha_+, \beta_-)$.

(2) If $(\kappa, \gamma) \in \mathcal{K}_2$, then $(a, b) = (0, 1)$.

(3) If $(\kappa, \gamma) \in \mathcal{K}_3$, then $(a, b) \in \{(\alpha_+, \beta_-), (0, 1)\}$.

4. Solitary waves.

$$\|u_1/\alpha_+\|_{H^1_\omega}^2 = \|\varphi_\omega\|_{L^3}^3 = \|u_1/\alpha_+\|_{L^3}^3$$

$$\|u_2/\beta_-\|_{H^1_\omega}^2 = \|\varphi_\omega\|_{L^3}^3 = \|u_2/\beta_-\|_{L^3}^3$$

$$\int_{\mathbb{R}^N} u_1^2\overline{u_2}\,dx = \|u_1\|_{L^3}^2\|u_2\|_{L^3}$$