Cognitive Science and Semiotics in Mathematics

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U S National Science Foundation

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Background figure from
http://www.ams.org/notices/200206/comm-hofmann.pdf

Bernar Venet

http://www.fields.utoronto.ca/aboutus/

Emergence of Mathematics
Workshop at CWRU, spring 2009




Pre-math

Math

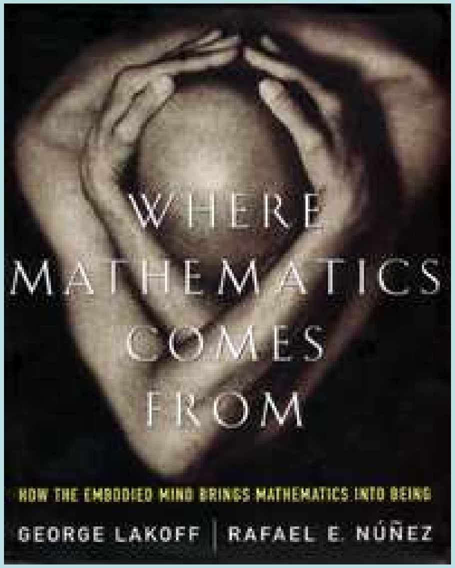
Mathematics

From Rafael's presentation

- Mathematics is made possible by the recruitment of human everyday cognitive mechanisms
- Not directly perceivable through the senses.
- Precise, symbolizable, generalizable

From a formal perspective, much about complex numbers seems arbitrary. From a purely algebraic point of view, i arises as a solution to the equation $x^2 + 1 = 0$. There is nothing geometric about this---no complex *plane* at all. Yet in the complex plane, the i-axis is 90° from the x-axis. Why? Complex numbers have a weird rule of multiplication [Hamilton's rule]: Why? Is this an arbitrary invention of mathematicians? (Lakoff and Núñez 2000: 423)

Where Mathematics Comes From: How the Embodied Mind Brings Mathematics Into Being
George Lakoff, Rafael E. Núñez



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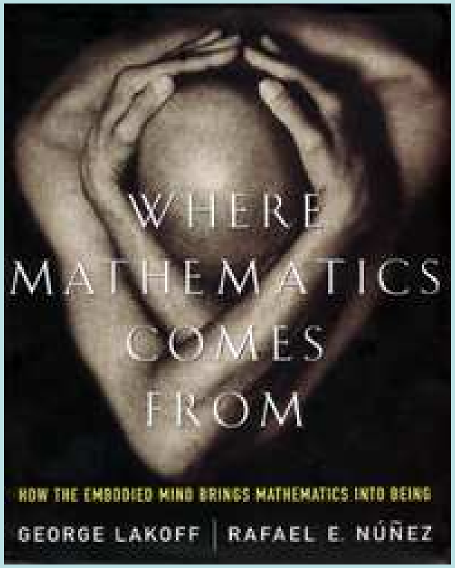

Addition and multiplication of common fractions,

$$\frac{p}{q}+\frac{r}{s}=\frac{ps+rq}{qs}, \qquad \frac{p}{q}\times\frac{r}{s}=\frac{pr}{qs}$$

are operations so basic to arithmetised mathematics that it may be difficult for us today to conceive of a mathematics in which they are unknown or unimportant, . . . [but] I believe that this may indeed have been the case for early Greek mathematics, (Fowler)

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$$(3\,|\,4)\times(2\,|\,3)=?$$

.999999999999999… = 1?

W. H. Auden

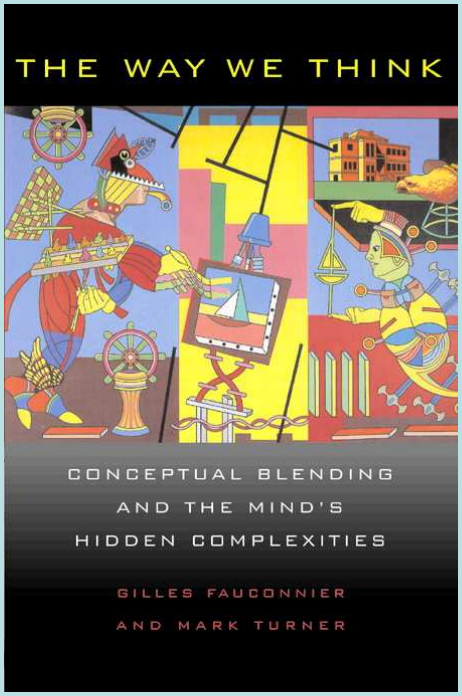
When I was a boy, we were taught the literary languages, like Latin and Greek, extremely well, but mathematics atrociously badly. Beginning with the multiplication table, we learned a series of operations which, if remembered correctly, gave the “right” answer, but about any basic principles, like the concept of number, we were told nothing. Typical of the teaching methods then in vogue is this mnemonic, which I had to learn.

> Minus times Minus equals Plus;
> The reason for this we need not discuss.

Mathematics as formal system

Formal systems are not the same thing as meaning systems, nor are they small translation modules that sit on top of meaning systems to encode work that is done independently by the meaning systems. Like the warrior and the armor, meaning systems and formal systems are inseparable. They co-evolve in the species, the culture, and the individual.

The Way We Think, Conceptual Blending
and the Mind's Hidden Complexities
Fauconnier and Turner, 2002



Sandbox

Ordered pairs

- Melds a single identity from two.
- Simply a pairing of an element of one set with an element of a second.
- A common cognitive blend.
  - Example: Married couples (in an ordered pair, there is a first-named and a second-named; there is also the underlying unordered pair, in which the ordering is immaterial).
  - Example: Ordered pairs (x, y) of real numbers, on horizontal and vertical axis, respectively, marking points in the Cartesian plane or on a map (hence often called Cartesian product).

Quotient sets

- Amount to taking collective nouns seriously.
  - Humans collectivize all the time.   Families, cohorts, companies, troops, brigades, lots (of goods), gardens, forests, …
  - Thinking of the forest, as opposed to the trees.
- What drives the use of quotient sets is the concept of “sameness.”
  - Things that are the same, in some way or another, are collected together.

Fractions (rational numbers)

Fouconnier and Turner

If in one [mental] space we have whole numbers and in the other space we have proportions of objects, then in the blend we have all the proportions, all categorized as numbers. Those proportions that had whole-number counterparts are fused with those counterparts, so that, for example, 6:3, 12:6, and 500:250 are fused in the blend with 2. But now 3:4, 256:711, and 5:9, which had no whole-number counterparts, are now also numbers in the blend.

Blending

As long as … mathematical conceptions are based in small … stories at human scale, that is, fitting the kinds of scenes for which human cognition is evolved, mathematics can seem straightforward, even natural. The same is true of physics. If mathematics and physics stayed within these familiar story worlds, they might as disciplines have the cultural status of something like carpentry: very complicated and clever, and useful, too, but fitting human understanding. The problem comes when mathematical work runs up against structures that do not fit our basic stories. In that case, the way we think begins to fail to grasp the mathematical structures. The mathematician is someone who is trained to use conceptual blending to achieve new blends that bring what is not at human scale, not natural for human stories, back into human scale, so it can be grasped. Turner

Blends, metaphorical and nonmetaphorical, occur throughout mathematics. Many of the most important ideas in mathematics are metaphorical conceptual blends. Lakoff and Núñez

Category metamorphosis

Category metamorphosis can change fundamentally the structure of the category. We have already seen in Chapter 11 how the category *number* changed to include zero and fractions, and, in the case of fractions, how complicated was the blending that produced the new version of that category. After the fact, it looks as if new elements have simply been added to the old ones, because we still use the same words for them. But in fact, in the metamorphosis of the category, the entire structure and organizing principles have been dramatically altered. It is an illusion that the old input is simply transferred wholesale as a subset of the new category.

Fouconnier & Turner

Generalization and Abstraction

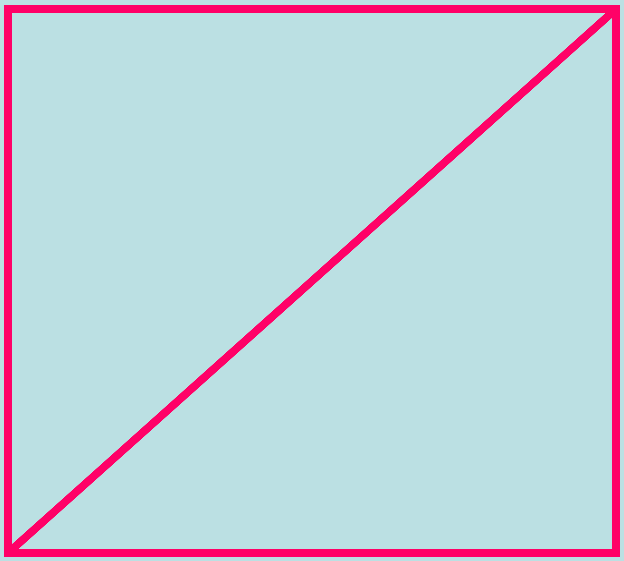
- Abstraction
  - Framing
  - Creation of category
- Generalization
  - Blending
  - Adjoint of forgetful functor

From an English textbook ~1800

A number may be greater or less than another number: it may be added to, taken from, multiplied into, or divided by, another number; but in other respects it is very intractable; though the whole world should be destroyed, one will be one, and three will be three, and no art whatever can change their nature. You may put a mark before one, which it will obey; it submits to be taken away from another number greater than itself, but to attempt to take it away from a number less than itself is ridiculous. Yet this is attempted by algebraists, who talk of a number less than nothing, of multiplying a negative number into a negative number, and thus producing a positive number, of a number being imaginary… This is all jargon, at which common sense recoils; but from its having been once adopted, like many other figments it finds the most strenuous supporters among those who love to take things up on trust and hate the colour of serious thought. (Frend 1796-1799)

Irrational numbers

- Ratio of diagonal of square to side not rational ($\sqrt{2}$)
  - Pythagoreans
  - Separation of geometry and number
  - $\sqrt{2}$ a length, but not a number
- Rationals have holes



Dedekind cuts

Of the greatest importance, however, is the fact that in the straight line L there are infinitely many points which correspond to no rational number. The straight line L is infinitely richer in point-individuals than the domain R of rational numbers in number-individuals. If now, as is our desire, we try to follow up arithmetically all phenomena in the straight line, the domain of rational numbers is insufficient and it becomes absolutely necessary that the instrument R constructed by the creation of the rational numbers be essentially improved by the creation of new numbers such that the domain of numbers shall gain the same completeness, or as we may say at once, the same continuity, as the straight line.

Richard Dedekind (1872)

Emphasizes order

Cauchy sequences

- Agustin-Louis Cauchy (1789-1857)
- Limits
    - Internal and external
    - Decimal expansion
- Cauchy did not complete generalization
    - Define numbers as limits
- Blend
    - Input: internal limit
    - Output: external limit
    - Emergent structure: order, arithmetic

Since, in proving geometrical figures, when rational numbers fail us, irrational numbers take their place and prove exactly those things which rational numbers could not prove …, we are moved and compelled to assert that they truly are numbers, compelled that is, by the results which follow from their use—results which we perceive to be real, certain, and constant. On the other hand, other considerations compel us to deny that irrational numbers are numbers at all. To wit, when we seek to determine their decimal expansion …, we find they flee away perpetually, so that not one of them can be apprehended precisely in itself … Now that cannot be called a true number which is of such a nature that it lacks precision … Therefore, just as an infinite number is not a number, so an irrational number is not a true number, but lies hidden in a kind of cloud of infinity.

Michael Stifel (1544)

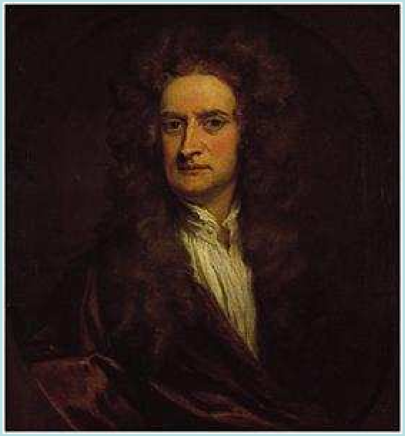
Imaginary numbers

… neither the true roots nor the false are always real; sometimes they are, however, imaginary [imaginaire]; namely, whereas we can always imagine as many roots for each equation as I indicated, there is still not always a quantity which corresponds to each root so imagined. Thus, while we may think of the equation $x^3 - 6xx + 13 - 10 = 0$ as having three roots, yet there is just one real root, which is 2, and the other two, however, increased, decreased, or multiplied as just explained, never are other than imaginary.

René Descartes (1637)

But it is just that the Roots of Equations should be often impossible, lest they should exhibit the cases of Problems that are impossible as if they were possible.

Isaac Newton (1720)



We must finally abandon the premise that the concept of impossible [unmöglich] numbers might be viewed as an idle whim. This opinion is groundless. The premise of impossible numbers is in fact of greatest importance, since problems often arise in which one cannot know immediately whether what is asked for is possible or impossible. Whenever their solution leads to such impossible numbers, one has a sure sign that the problem asks for something impossible.

Leonard Euler (1770)

Verum enim vero tenacior est varietatis suae pulcherrimae Natura rerum, aeternarum varietatum parens, vel potius Divina Mens, quam ut omnia sub unum genus compingi patiatur. Itaque elegans et mirabile effugium reperit in illo Analyseos miraculo, idealis munid monstro, pene inter Ens et non-Ens Amphibio, quo radicem imaginariam apellamus.

Gottfried Wilhelm Leibniz (1702)

Difficulties, one has believed, that surround the theory of imaginary magnitudes, is based in large part to that not so appropriate designation (it has even been discordantly called an impossible quantity). If, at the beginning, one proffered a manifold of two dimensions (which presents the intuition of space with greater clarity), the positive magnitudes would have been called direct, the negative inverse, and the imaginary lateral, there would be simplicity instead of confusion, clarity instead of darkness.

Carl Friedrich Gauss (1831)

From presentation of Kalevi Kull

Mathematical world is semiotic.

1.25 = 1.2500000000000…

3.3333333333333333…

3.1459265358979323…

12.349857123409873…

1.25 = 1.25000000000000…

3.33333333333333333…

3.14592653589793238…

12.3498571234098734…

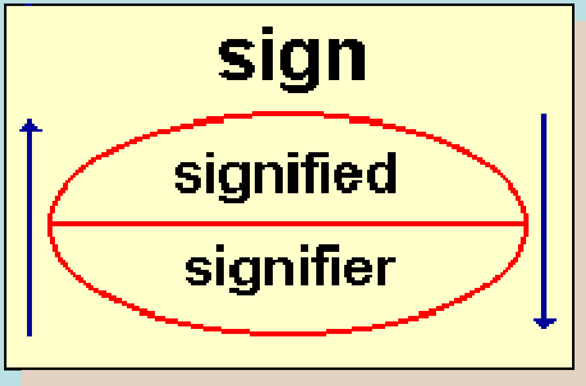
.999999999999999… = 1?

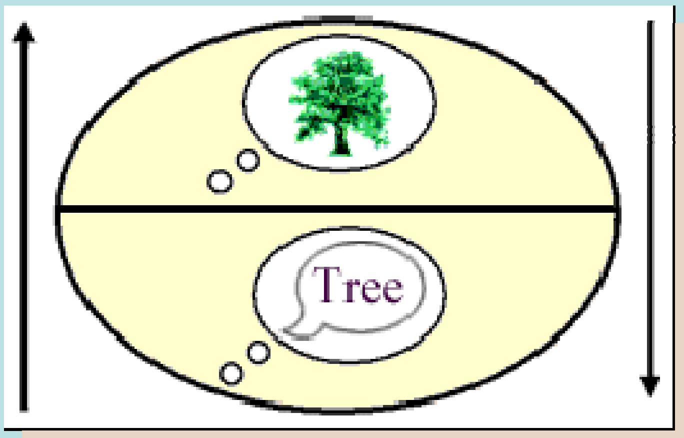
$C_8H_{10}N_4O_2$

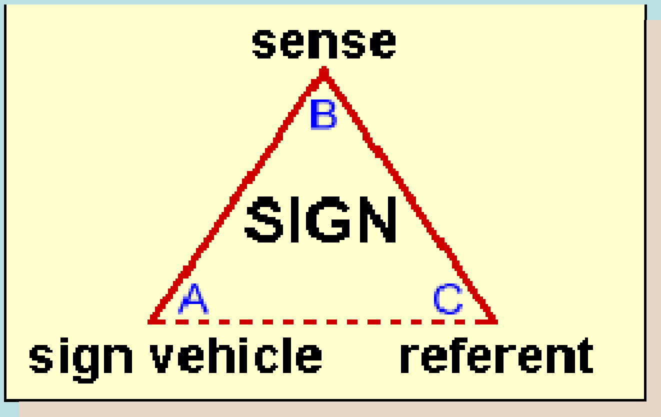
Field equations of general relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\mathbf{R} - \frac{1}{2} \mathbf{g} R = \frac{8\pi G}{c^4} \mathbf{T}$$







Semiotics for Beginners   Daniel Chandler

Mathematics is a rich lode, primed for mining
by cognitive scientists, semioticians, …