Dynamical and statistical explanations of rogue wave occurrence rates

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- motivation
- rogue wave occurrence rates (theory)
- observations
- simulations
- what can we learn from simulations
- wave-current interaction
- conclusion

SAR NEW INITIATIVES FUND

National Search and   Secrétariat national
Rescue Secretariat   recherche et sauvetage

Monsters of the deep

It came from nowhere, snapping giant ships into two….

New Scientist, June 30, 2001

I Rogue waves 'wipe out' spectators at Mavericks surfing competition

THE TIMES Feb. 14, 2010

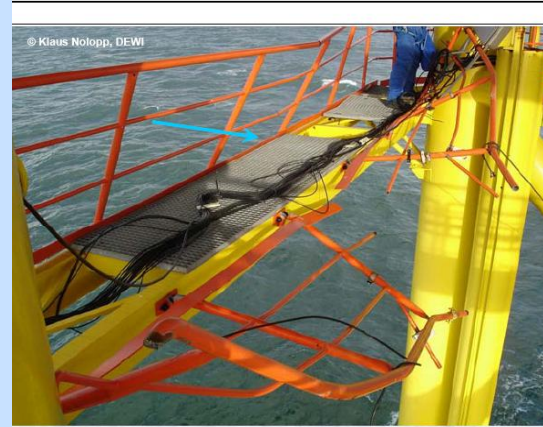**Two walls of water swept dozens of people off a concrete seawall …
At least 13 spectators received significant injuries…**
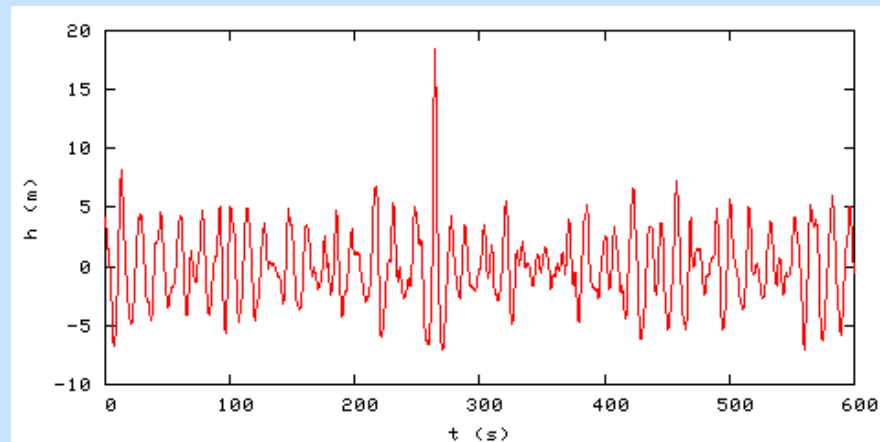
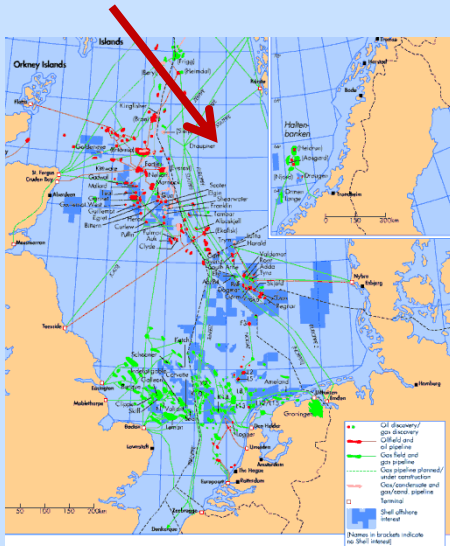## II Wave watching, Vancouver Island

Examples of deep water rogue waves

I. FINO research platform North Sea, Nov 1, 2006

Storm event of the 1st Nov. 2006

BUNDESAMT FÜR SEESCHIFFFAHRT UND HYDROGRAPHIE

SKN + 15 m
Mean HW + 12,5 m

© Klaus Nolopp, DEWI

II. DRAUPNER oil platform North Sea, January 1, 1995

H_s = 11.9m , η_{max} =18.5 m

Courtesy of W. Rosenthal

Is the topic of rogue waves even appropriate for a physicist?

"In her new book
***The Wave: In Pursuit of the Rogues, Freaks and Giants of the Ocean*,**
award-winning Toronto-born journalist Susan Casey describes
<u>**walls of water that defy the laws of physics**</u>

●●●
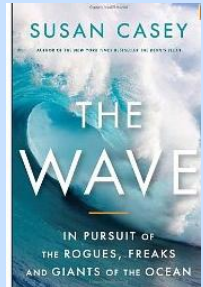
Rogue waves <u>operate outside the rules of physics</u> and pop up in unlikely conditions. Do you feel like you understand how they are formed?

I can explain it in so far as science can explain it, but there are many circumstances under which science still can't explain them. ..."

(Maybe we should leave this topic for the media to deal with)

Terminology

Monsters of the deep

rogue wave

freak wave

giant wave

extreme wave
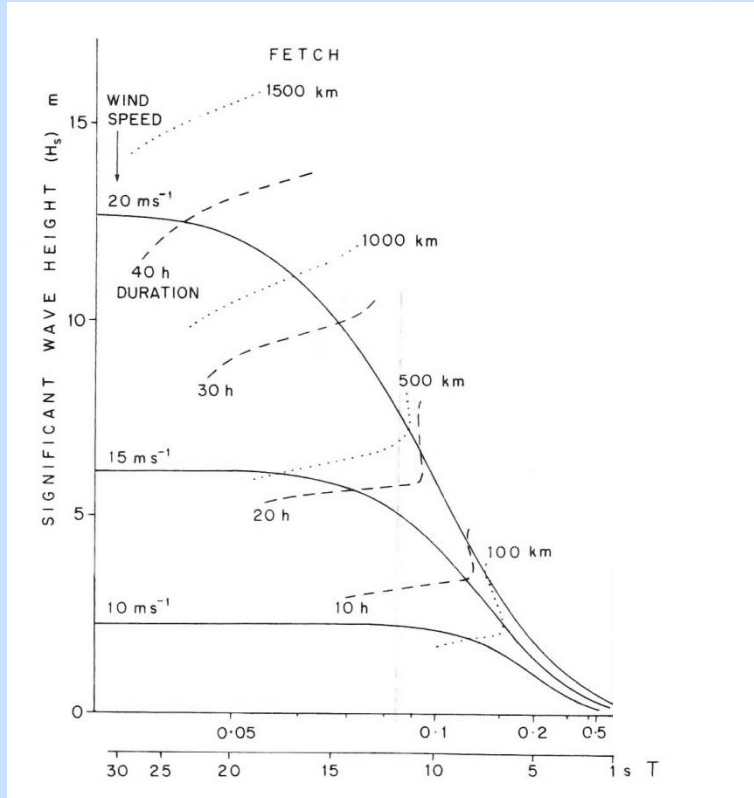
wall of water

holes in the surface of the sea

…

waves in the tail of the probability distribution

Large waves

Wind wave growth:

H = f(wind speed, duration, fetch)

large wave ≠ rogue wave

	T = 10h	T = 20h	T = 40h
Wind = 15 m/s	H = 4m	H = 5.5m	H = 6m
Wind = 20 m/s	H = 4.5m	H = 7m	H = 13m

Also dependent on fetch

Rogue wave definition

Individual wave:

> **H: wave height (trough-crest)**

> η**: crest height (mean water level - crest)**

> **(linear theory, narrow-banded spectrum: H = 2η)**
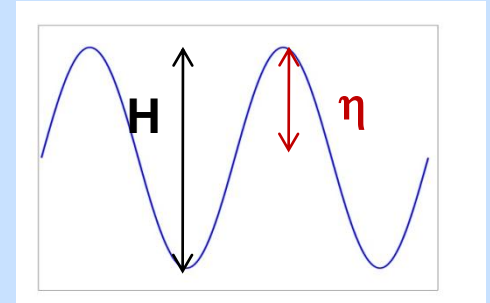
Wave record:

$H_s = 4\sigma$**: significant wave height (average of 1/3 highest waves)**
σ**: standard deviation of surface elevation**
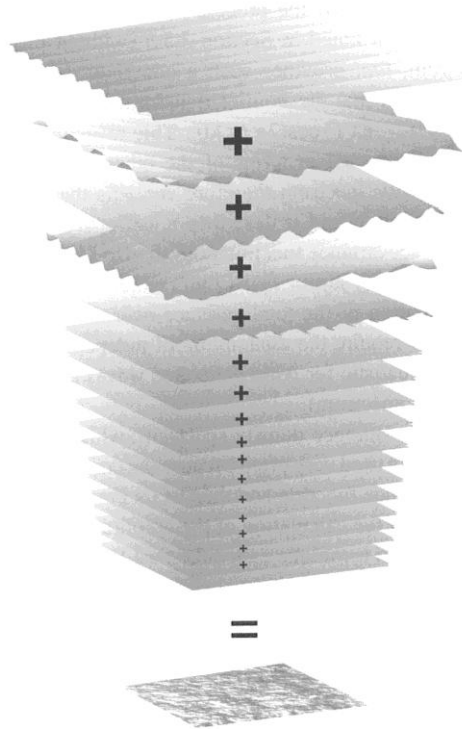
common *rogue wave* definition:

> $H_{rogue} \geq 2.2\ H_s$

or

> $\eta_{rogue} \geq 1.25\ H_s$

The basics: linear theory

Linear superposition

$$\zeta(x, y, t) = \sum_{i=1}^{N} \sum_{j=1}^{M} a_{i,j} \cos\left(\omega_i t - k_i x \cos\theta_j - k_i y \sin\theta_j + \varphi_{i,j}\right)$$
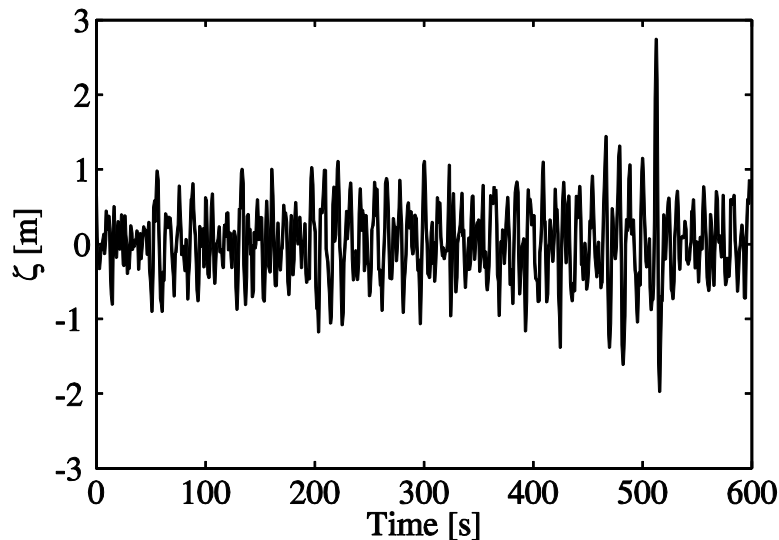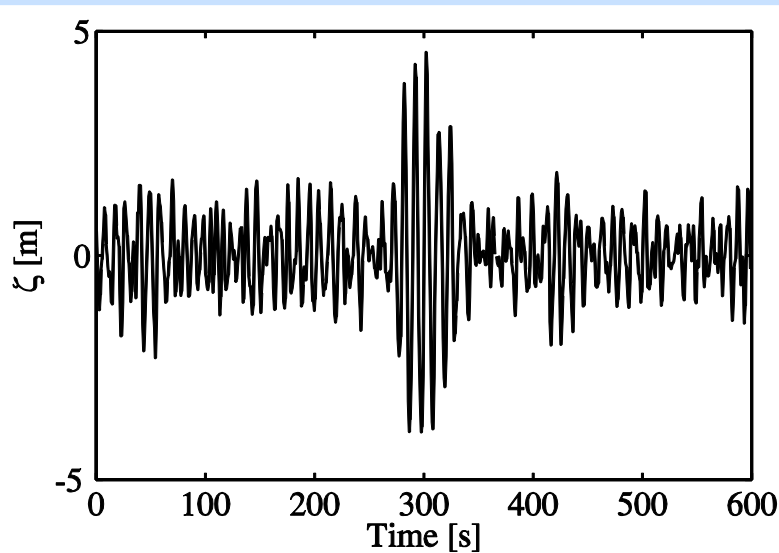
Random amplitude $a_{i,j}$, random phase $\varphi_{i,j}$

Figure 3.10 The random waves moving in time, i.e., the sum of a large number of harmonic wave components, travelling across the ocean surface with different periods, directions, amplitudes and phases (after Pierson *et al.*, 1955).

Dispersion relation

$$\omega^2 = gk \tanh(kd)$$

From Holthuijsen 2007

The basics

surface elevation at a fixed point

Examples:
section of wave buoy record off Tofino, BC

H_s = 3.53m
H_{max} = 8.47m
η_{max} = 4.50m

H_s = 1.96m
H_{max} = 3.62m
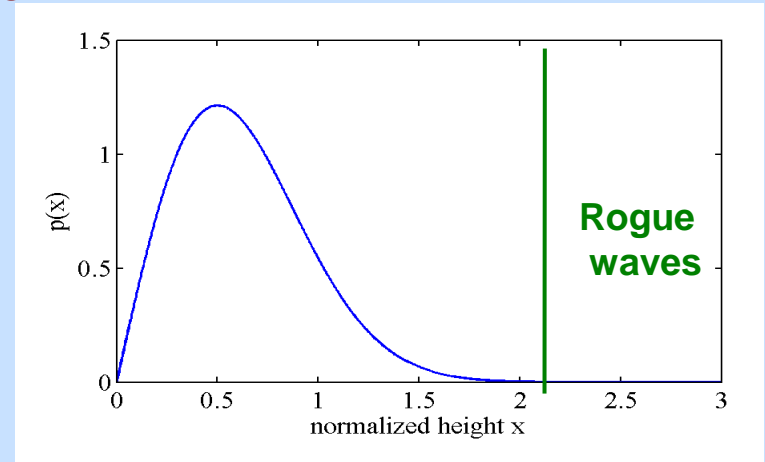η_{max} = 2.75m

Real data !
Consistent with linear theory ?

Rogue wave occurrence

Need to know probability distribution for crest height η (wave height H)

Rogue waves are waves in the tail of the probability distribution

Theoretical probability distributions:

Linear theory:

sea surface height ζ is made up of a large number of independent sinusoids → its probability density function p(ζ) is Gaussian:

$$p(\zeta) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{\zeta^2}{2\sigma^2}\right)$$

→ wave height distribution is the Rayleigh distribution

(if narrow band):

$$p(H) = \frac{H}{4\sigma^2} \exp\left[-\frac{H^2}{8\sigma^2}\right]$$

Longuet-Higgins, 1952

Rogue wave occurrence

Wave height distribution:
(linear superposition, narrow-band frequency spectrum)

Rayleigh distribution:
$$p(H) = \frac{H}{4\sigma^2} \exp\left[-\frac{H^2}{8\sigma^2}\right]$$

Exceedance probability:
$$P(H/H_s > z) = \exp(-2z^2)$$ **wave height**
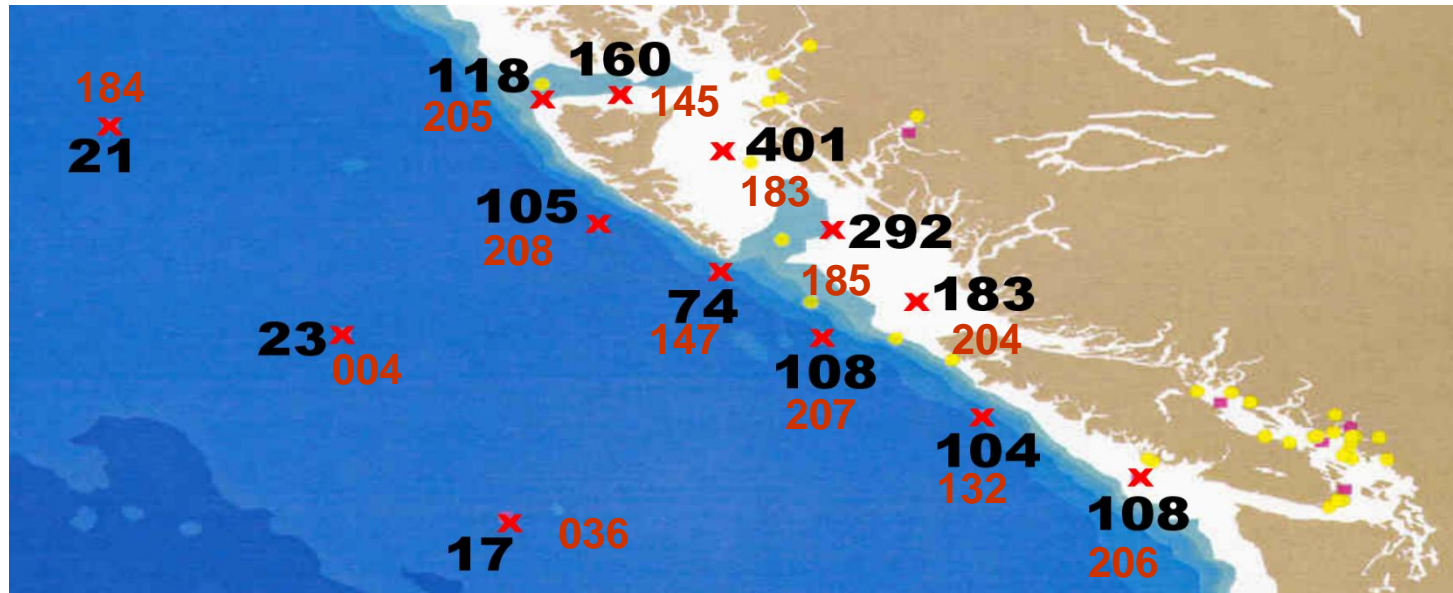
$$P(\eta/H_s > z) = \exp(-8z^2)$$ **crest height**

→ **P(H>2.2H$_s$) ≈ 1/16800**   **1 "rogue wave" every 1-2 days**

Problem solved !   (?)

Data analysis: Extreme maximum wave height

12 – 20 year wave buoy records (operational), Meteorological Service Canada

Average "rogue" wave occurrence

184
×
21

118 **160**
205 × × 145

× **401**
183

105 ×
208

× **292**

185
× **183**
204

23 ×
004

74
147

×
108
207

×
104
132

×
108
206

17 ×
036

x: Locations of operational wave buoys (report hourly statistics only). C46xxx
Black number: average number of $H_{max} \geq 2.2 H_s$ occurrences / year
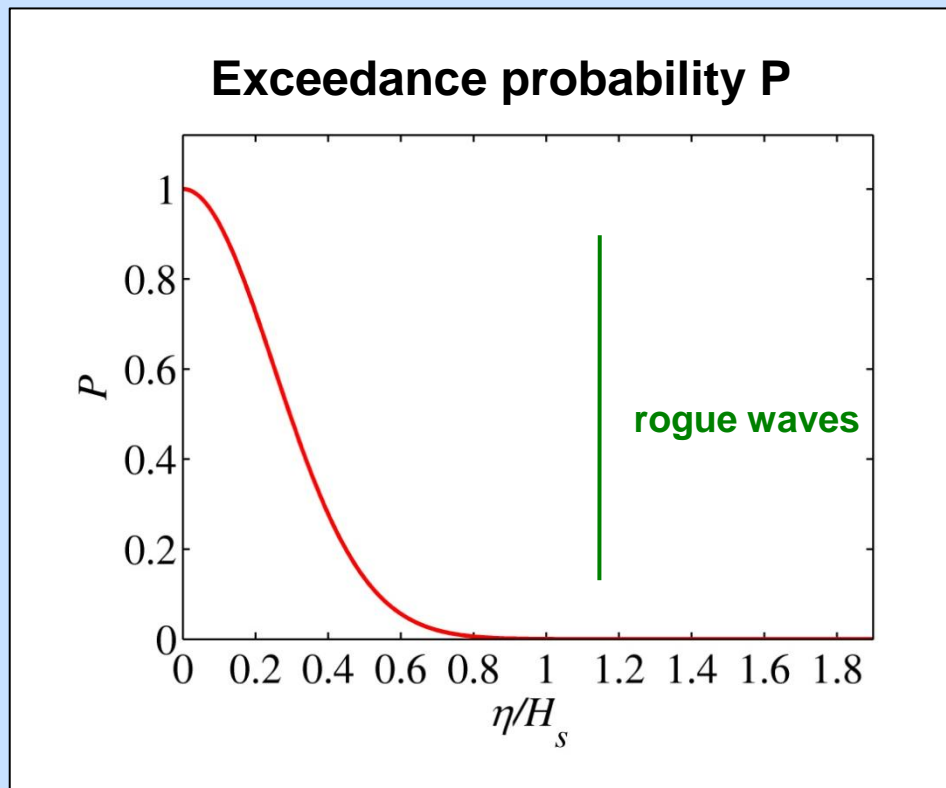**(high sea states only)**

→ **Rogue waves more frequent on continental shelf**
→ **data not consistent with simple Rayleigh distribution**

data quality ?

Presentation

Data may be compared with formulae for P(η/H_s), P(H/H_s)

However, large waves → small P

Presentation

Better presentation: $\ln(-\ln P)$

$$P(\eta_l / H_s > z) = \exp(-8z^2) \quad \rightarrow \quad \boxed{\ln[-\ln P(\eta_l / H_s > z)] = 2\ln z + \ln 8}$$

(straight line if plotted against $ln\ z$)

Presentation

Better presentation: $\quad \ln(-\ln P)$

$$P(\eta_l / H_s > z) = \exp(-8z^2) \quad \rightarrow \quad \boxed{\ln[-\ln P(\eta_l / H_s > z)] = 2\ln z + \ln 8}$$

(straight line if plotted against $ln\ z$)

More general: **Weibull distribution**

$$P(\eta / H_s > z) = \exp\left(-\frac{z^\alpha}{\beta}\right) \quad \rightarrow \quad \boxed{\ln[-\ln P(\eta / H_s > z)] = \alpha \ln z - \ln \beta}$$

(straight line if plotted against $ln\ z$)

Predicting rogue wave occurrence rates:

    find α, β from data   (e.g. Forristall, 2000)

Presentation

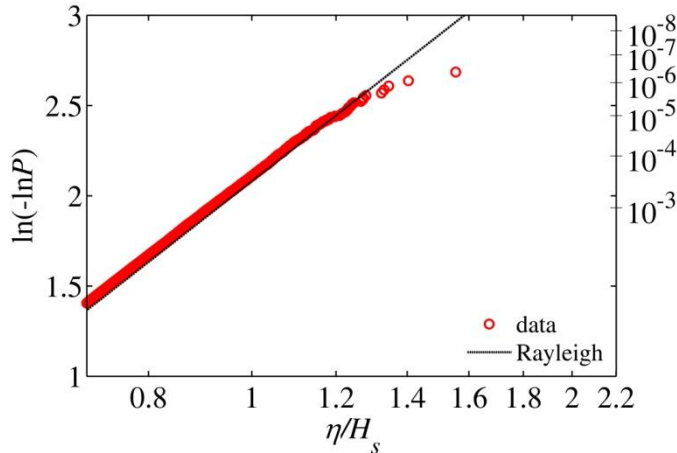$$P(\eta_l / H_s > z) = \exp(-8z^2) \quad \Rightarrow \quad \ln[-\ln P(\eta_l / H_s > z)] = 2\ln z + \ln 8$$
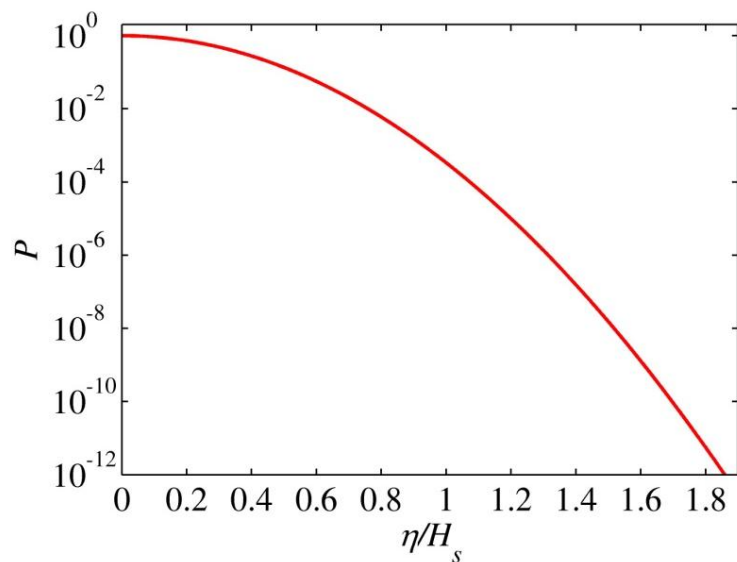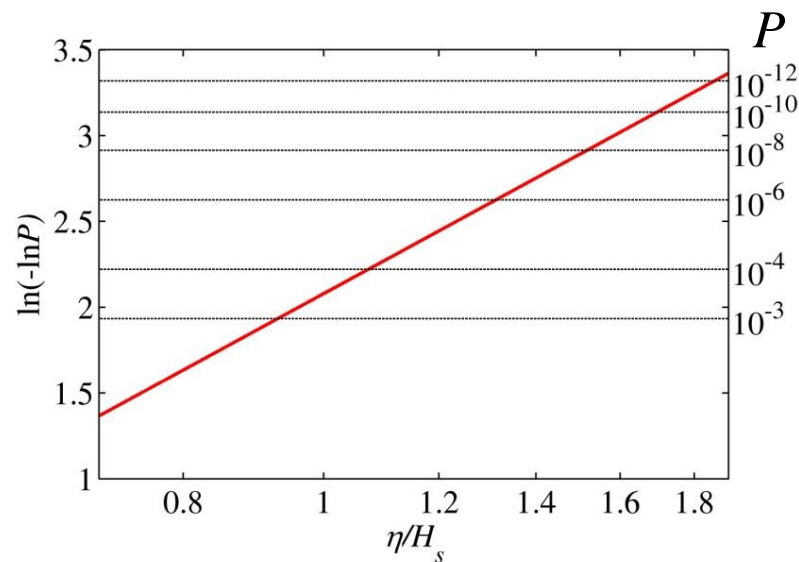
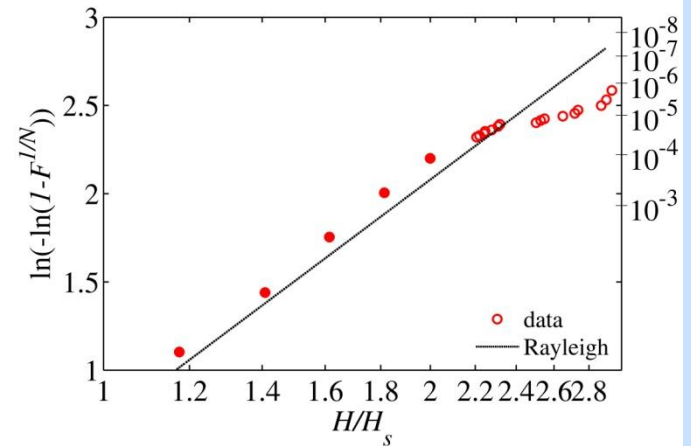# Observed crest (wave) height distributions

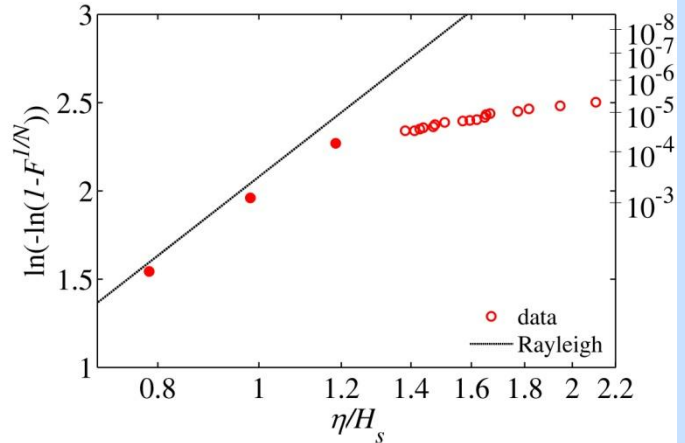**Wave buoy record off Tofino, BC**

Laser wave gauge, North Sea oil platform Gorm (Dysthe et al. 2008)
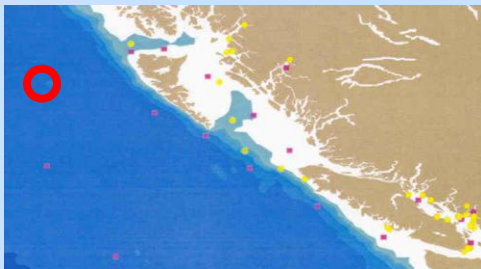
crest
height

wave
height

- **finite bandwidth (H-distribution not Rayleigh)**
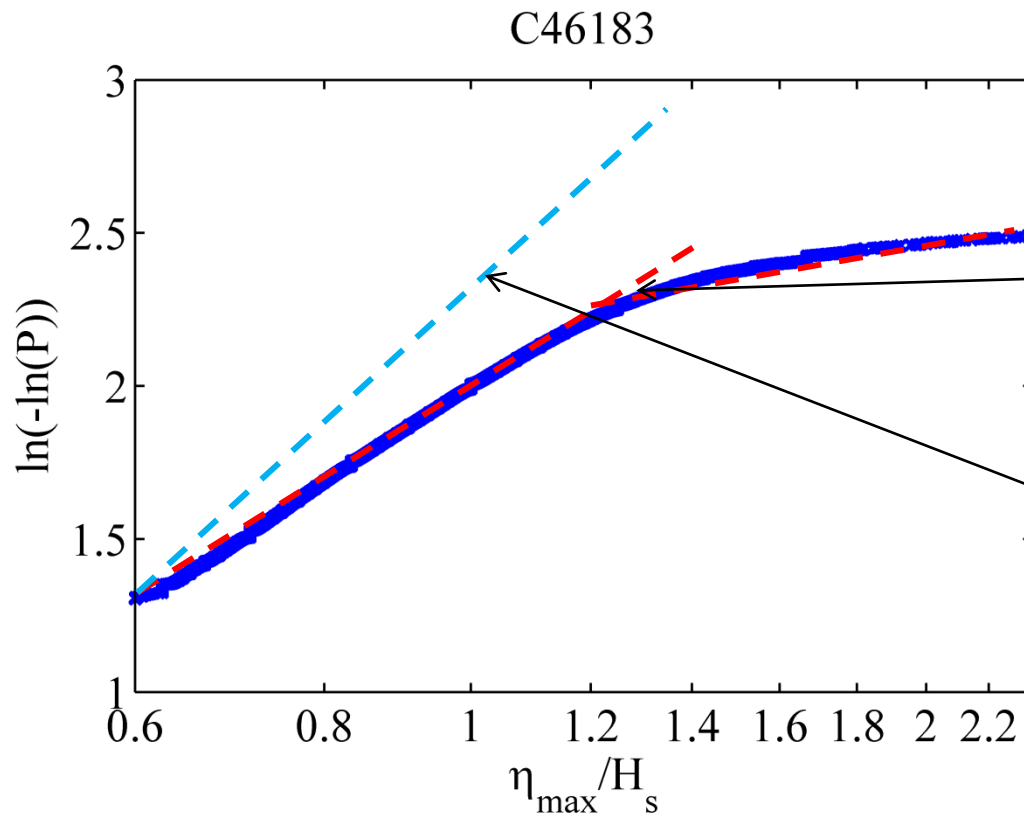- **large crests more frequent than in Rayleigh distribution**

Extreme maximum crest height occurrence
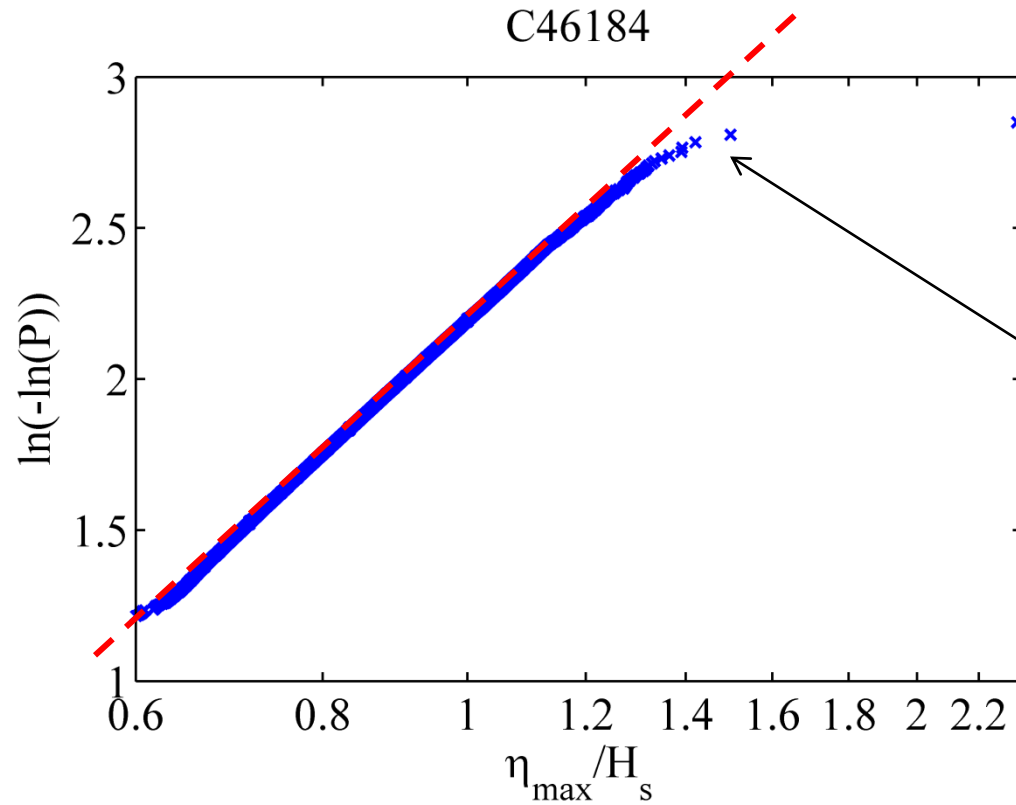
C46184

large waves too frequent (small sample volume?)

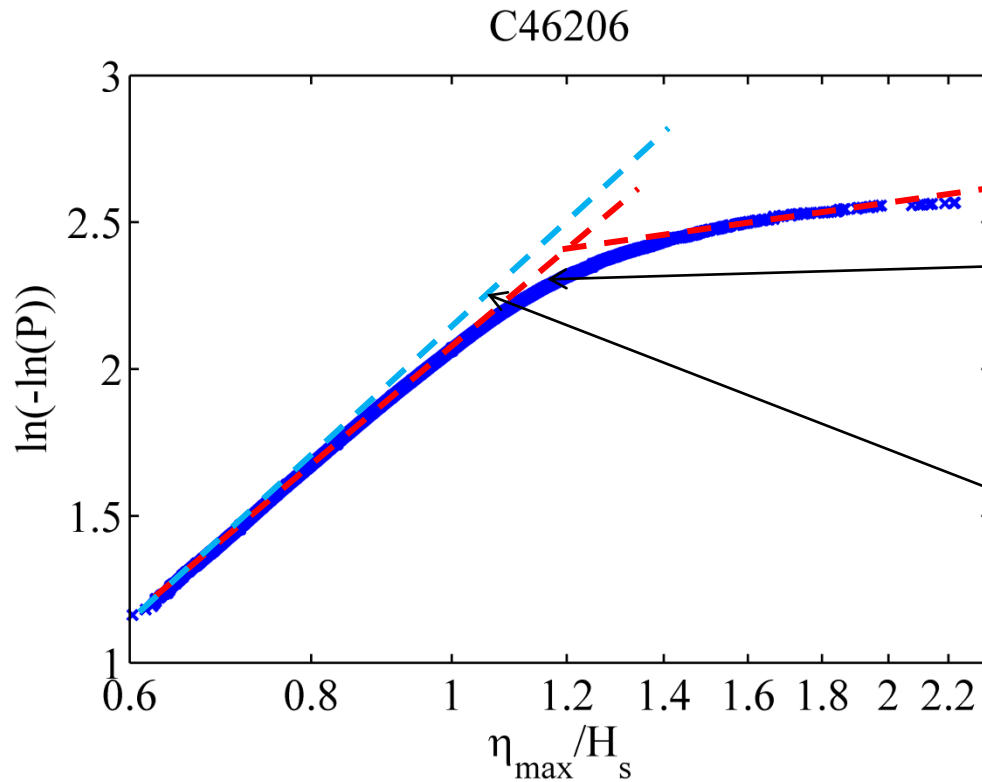# Extreme maximum crest height occurrence

changing slope $\eta/H_s > 1.2$
(different population?)

smaller slope than C46184

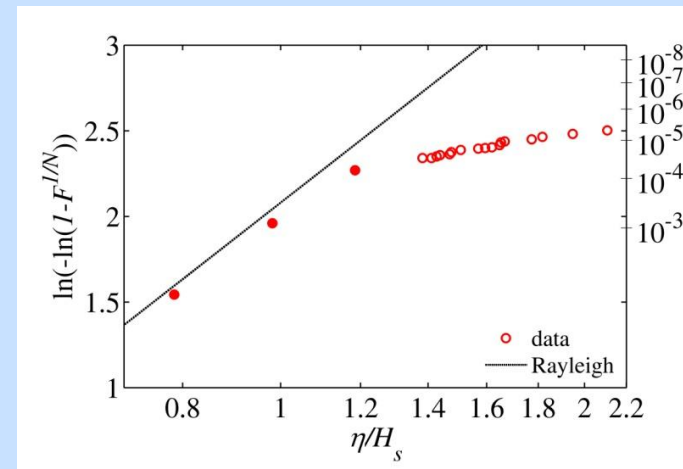# Extreme maximum crest height occurrence

C46206

changing slope $\eta/H_s > 1.2$
(different population?)

similar slope to C46184

# Potential causes for deviation from standard model (Weibull distribution)

- **non-stationarity**

- **higher harmonics**

Deviations from Weibull distribution

Test with

- **analytical solutions (where available)**

- **simulated surface elevation time series (Monte-Carlo simulation)**
  - **linear, random superposition of wave Fourier components**
  - **Fourier components based on JONSWAP spectrum**

JONSWAP wave spectrum
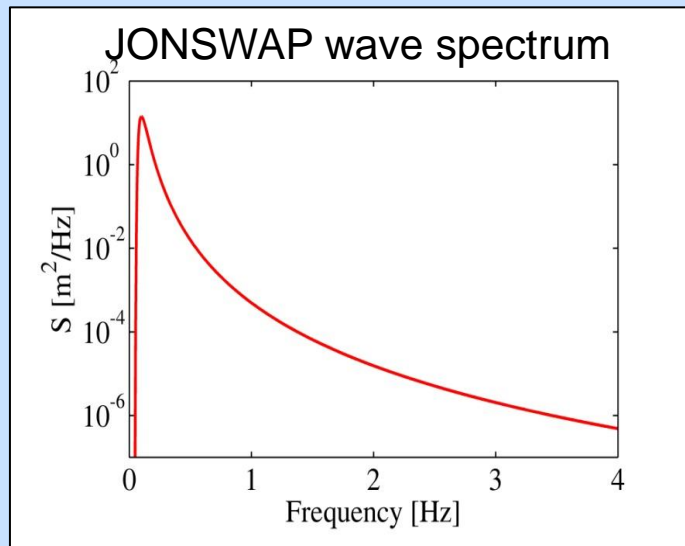
60 day simulation, 10 Hz sampling (51,840,000 data points), 0-6Hz frequency band
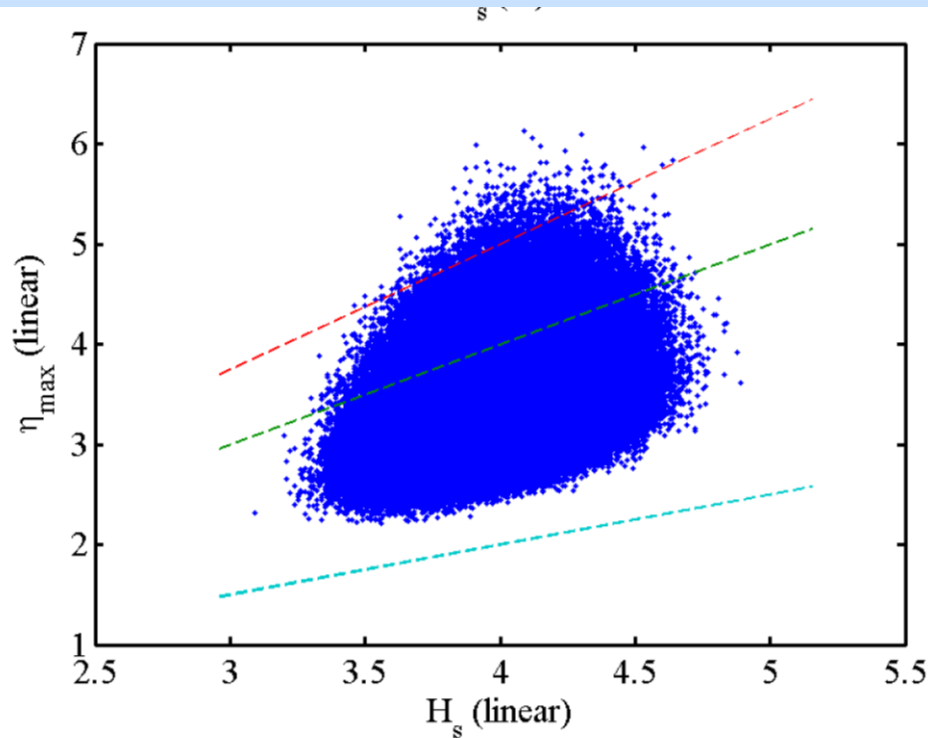
275 runs

→ 45 year time series

Monte-Carlo simulations, non-stationarity

Observed H_s values are obtained from 40 minute records
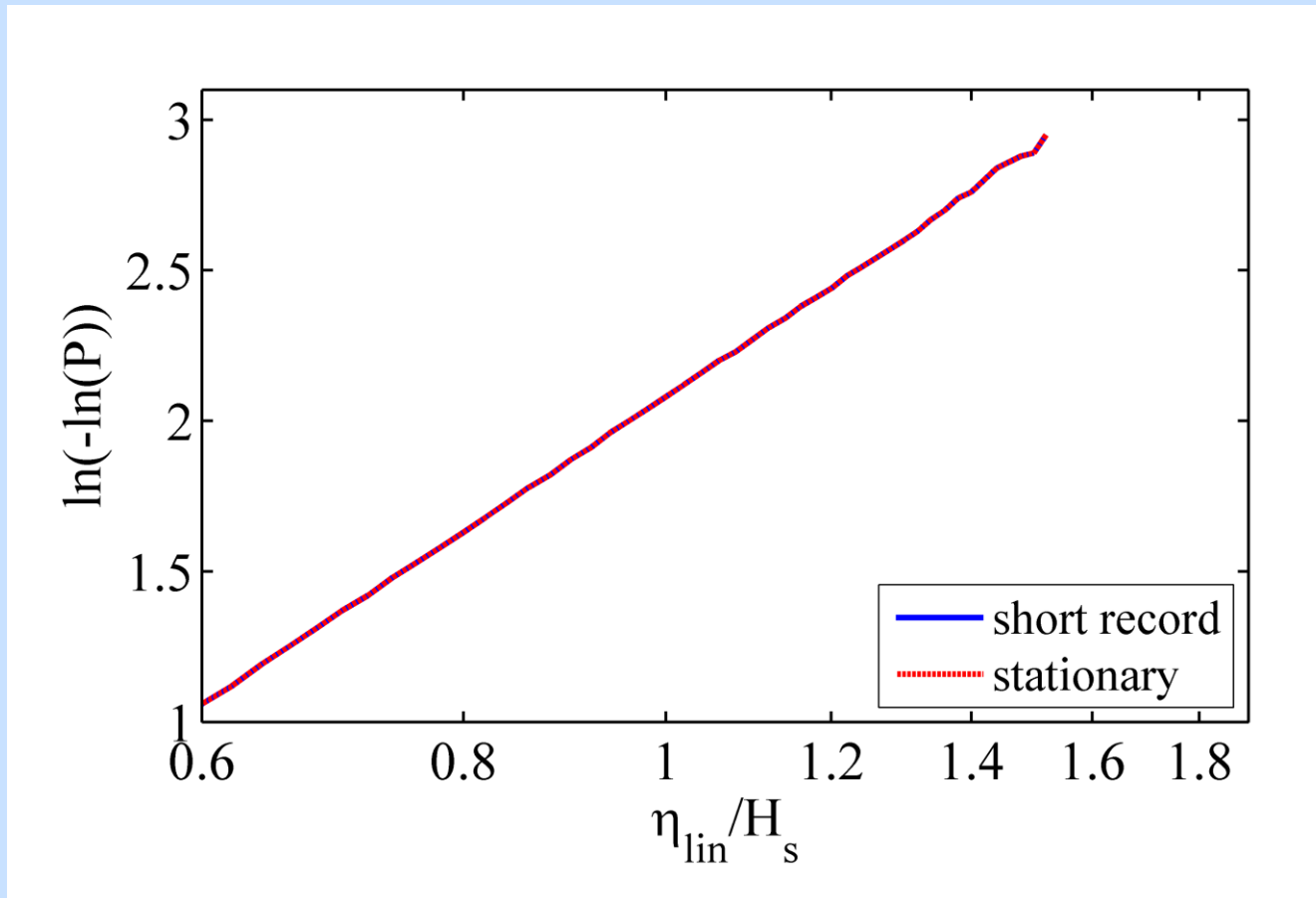**[$H_s = 4\,\sigma(\zeta(t))$]**

Partition simulated surface elevation (45 year stationary) into 40 minute segments

stationary $H_s = 4$m
short-record H_s variability: ±20%

η_{max}/H_s **largest at moderate H_s**

Monte-Carlo simulations, non-stationarity

Long record stationary data and short record segmentation have same crest-height probability distributions (Weibull)
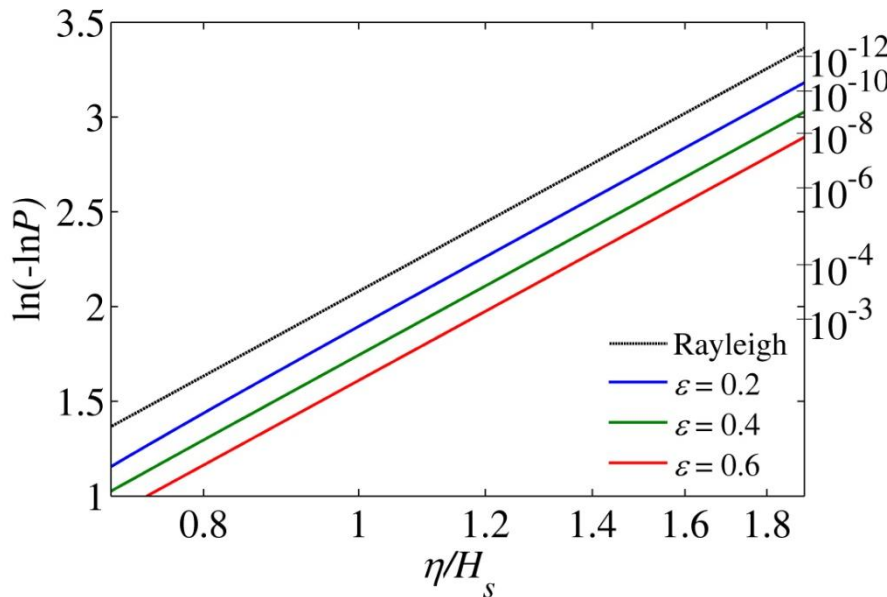
Non-stationarity

Assume wave height time series with significant wave height H$_s$ but 2 stationary halves:

(H$_s$ is always calculated from entire record length)

$$H_1^2 = H_s^2(1+\epsilon)$$

$$H_2^2 = H_s^2(1-\epsilon)$$

$$P(\eta/H_s > z) = \frac{1}{2}\left[\exp\left(-\frac{8z^2}{1+\epsilon}\right) + \exp\left(-\frac{8z^2}{1-\epsilon}\right)\right].$$

Rogue wave occurrence in a non-stationary record of two equal length parts (coloured lines) is much higher than if the record were treated as 2 stationary parts (black).

Monte-Carlo simulations, non-stationarity
Analytical approach

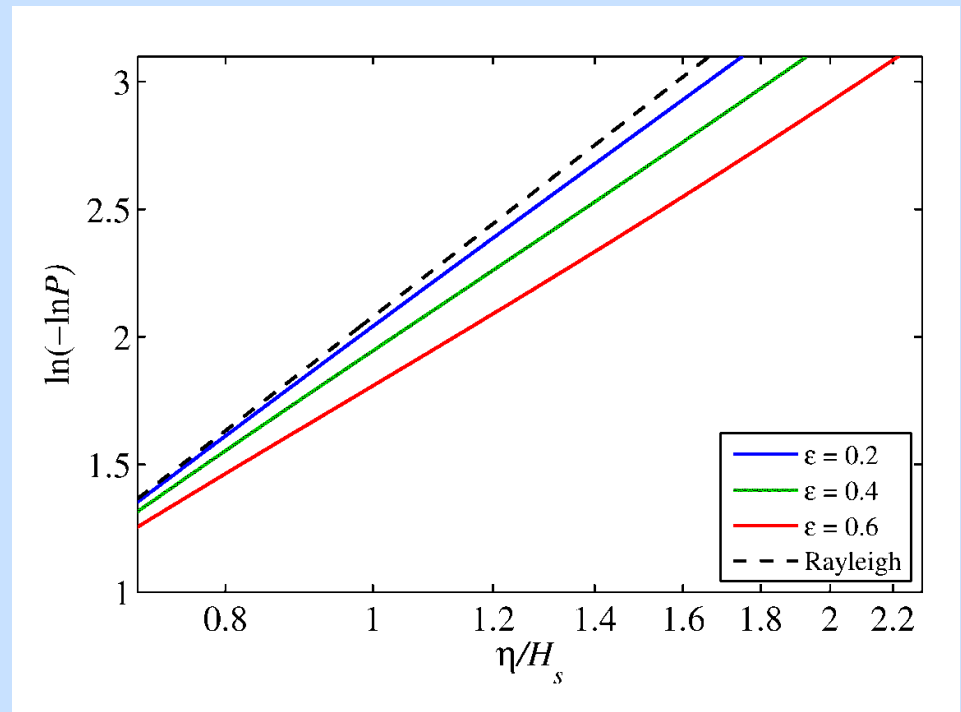$$H_s^2(t) = \frac{\overline{H_s}^2}{(1+\alpha t)^{-1}}$$

variance changing slowly with time
(e.g. ship steaming into region of wave current interactions)

$$P(\eta / H_s > z) = \exp(-Bz^2)\frac{\sinh(B\epsilon z^2)}{B\epsilon z^2}$$

$$\varepsilon = \alpha T, \quad B = \frac{4}{\epsilon}\ln\left(\frac{1+\epsilon}{1-\epsilon}\right).$$

$$\epsilon \to 0, B \to 8$$

Increased probabilities, but constant slope (for fixed ε)

Higher harmonics (Stokes correction)

5.6 Nonlinear, permanent waves   141

linear wave

+

second-order Stokes correction
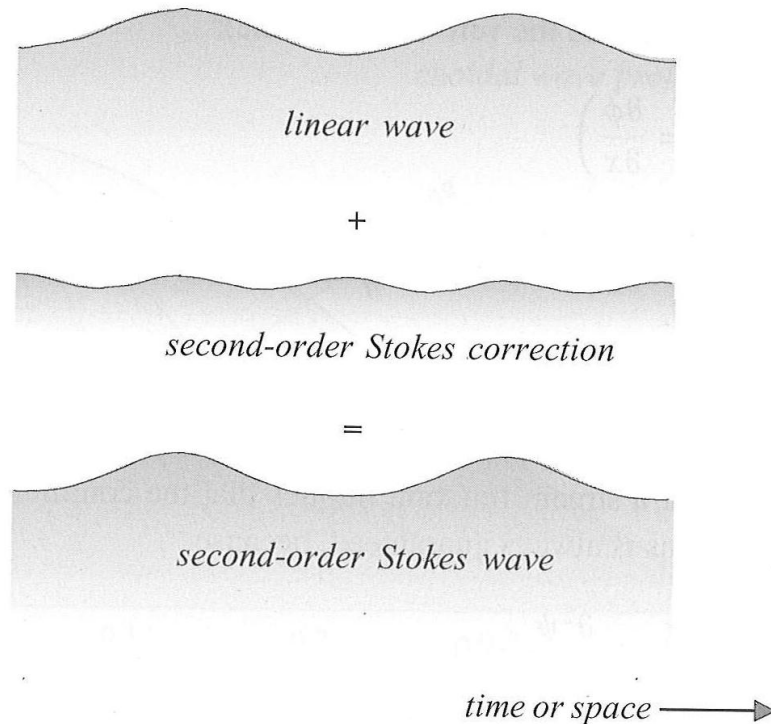
=

second-order Stokes wave

time or space ⟶

Figure 5.13 The surface profile of a second-order Stokes wave. From Holthuijsen 2007

higher harmonics:
- **same phase speed as primary wave**
- **multiple frequency (2ω, 3ω,…)**
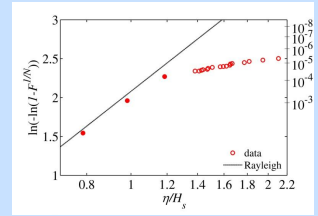- **multiple wave number (2k, 3k,…)**

➔

bound wave

$$\omega^2 = gk$$
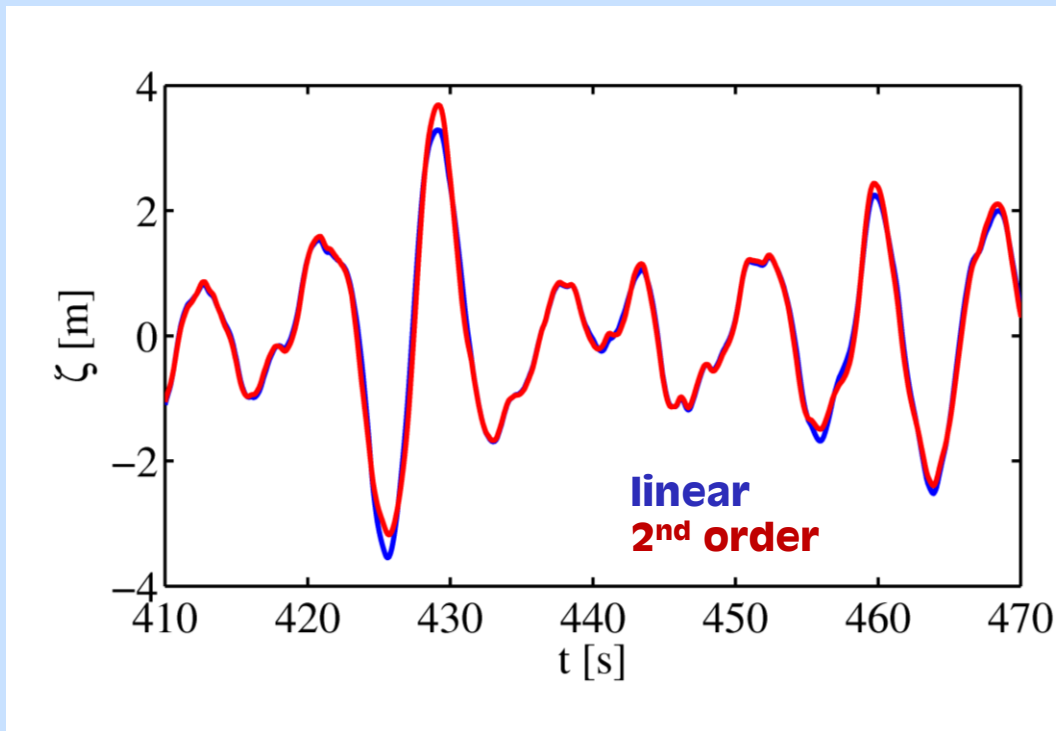
$$(2\omega)^2 \neq g2k$$

Higher harmonics, simulations

2nd
$$\zeta = a\cos\theta + \frac{1}{2}ka^2\cos(2\theta)$$

Up to 4th
$$\zeta = a\cos\theta + \left(\frac{1}{2}ka^2 + \frac{17}{24}k^3a^4\right)\cos(2\theta) + \frac{3}{8}k^2a^3\cos(3\theta) + \frac{1}{8}k^3a^4\cos(4\theta)$$

linear
2nd order

wave phase $\theta = kx - \omega t$

wave number $k = 2\pi/\lambda$

wave amplitude a

wave steepness ak

Exceedance probability, 2nd order Stokes waves

Recall:

Linear crest height
(normalized):
$$\frac{\eta_l}{H_s} \rightarrow \qquad P(\eta_l / H_s > z) = \exp(-8z^2)$$

2nd order crest height:

$$\zeta = a\cos\theta + \frac{1}{2}ka^2\cos(2\theta) \rightarrow \frac{\eta}{H_s} = \frac{\eta_l}{H_s} + \frac{1}{2}R\left(\frac{\eta_l}{H_s}\right)^2, \quad \textit{R=kH}_s : \textbf{wave steepness}$$
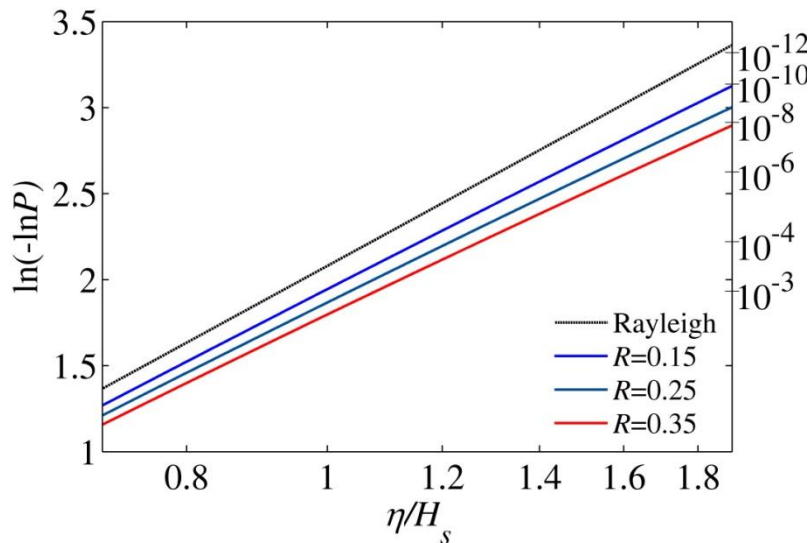
$$\rightarrow \quad \frac{\eta_l}{H_s} = \frac{(1+2R\eta/H_s)^{1/2}-1}{R}$$

$$\boxed{P(\eta/H_s > z) = \exp\left\{-\frac{8}{R^2}\left[(1+2Rz)^{1/2}-1\right]^2\right\}}$$
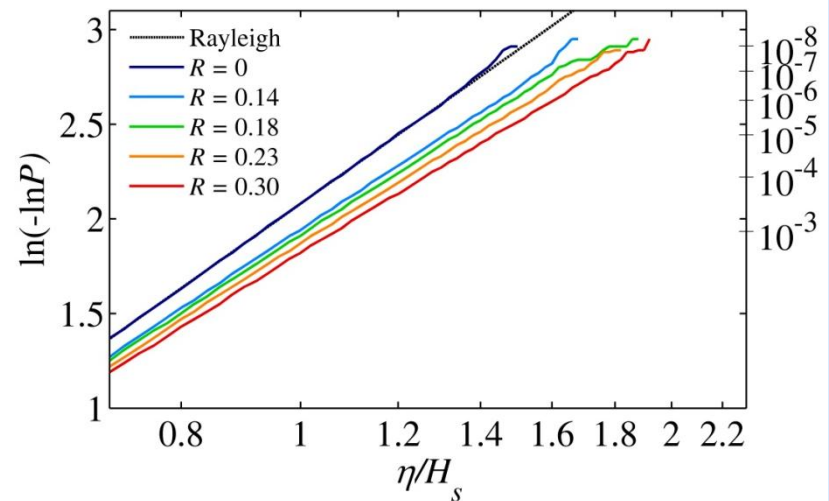
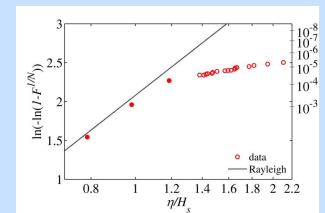(Tayfun, 1980)

Exceedance probability, 2nd order Stokes waves

Theory

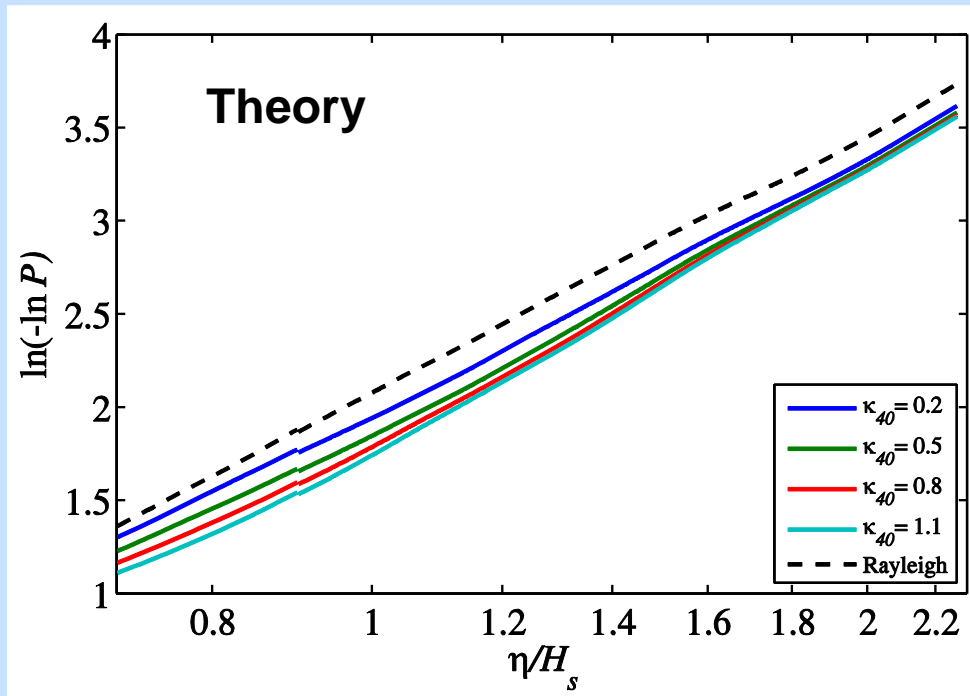

From Monte-Carlo simulations

- **Occurrence rate of large crests increased overall (compared to linear waves)**
- **no "extra" increase of extreme crests (still straight line)**
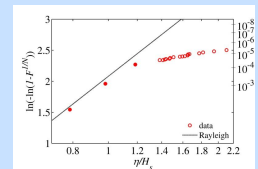
Exceedance probability, modulational instability

$$P(\eta / H_s > z) = \exp(-8z^2)\left[1 + \frac{8}{3}\kappa_{40}z^2\left(4z^2 - 1\right)\right].$$

From Mori & Janssen, JPO 2006
(with H=2η)

$$\text{kurtosis} = 3 + \kappa_{40}$$

Theory

Legend:
- $\kappa_{40} = 0.2$
- $\kappa_{40} = 0.5$
- $\kappa_{40} = 0.8$
- $\kappa_{40} = 1.1$
- Rayleigh

Axes: $\ln(-\ln P)$ vs η/H_s

- **Occurrence rate of large crests increased overall (compared to linear waves)**
- **no "extra" increase of extreme crests (still straight line)**

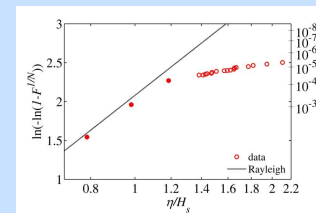# Exceedance probability, 4th order Stokes waves
from Monte-Carlo simulations
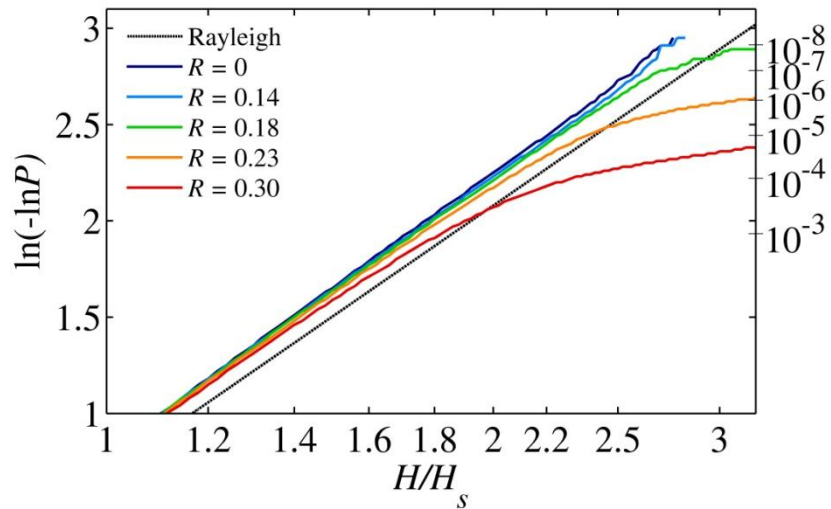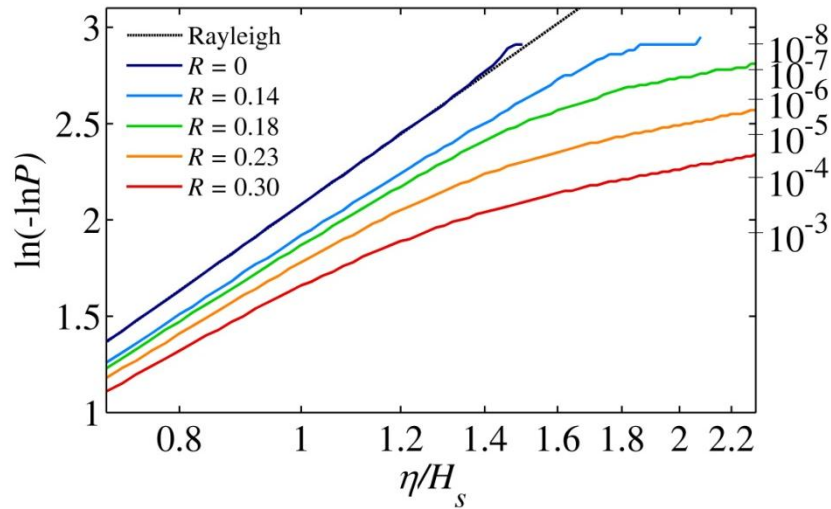
Increased occurrence of extreme crests and of extreme wave heights (curved lines, consistent with data)

Higher order harmonics may play a role in the generation of rogue waves

Linear superposition
→ **large waves (= steep)**
→ **increased 4th order correction**
→ **extra large wave**

Generalized extreme value theory GEV

Asymptotic formulae for the exceedance probability for the maximum in a block of data containing a large number of individual events

Starting point: exceedance probability for individual waves

$$P(\eta / H_s > z) = \exp\left(-\frac{z^\alpha}{\beta} \right)$$ **(Weibull distribution)**

M_n is the maximum of $z = \eta/H_s$ in a block of N waves

➔ $$P(M_n < z) = \left[1 - \exp\left(-\frac{z^\alpha}{\beta} \right) \right]^N \equiv F$$

define $$a_N = (\beta \ln N)^{1/\alpha}, \qquad b_N = \frac{a_N}{\alpha \ln N},$$

➔ $$P\left(\frac{M_N - a_N}{b_N} < z \right) = \left\{ 1 - \exp\left[-\frac{(b_N z + a_N)^\alpha}{\beta} \right] \right\}^N.$$

Generalized Extreme Value theory, GEV

$$\rightarrow \quad \exp\left[-\frac{(b_N z + a_N)^\alpha}{\beta}\right] = \exp\left[-\ln N\left(1+\frac{z}{\alpha \ln N}\right)^\alpha\right]$$

$$\simeq \exp(-\ln N - z) \quad \boxed{\text{if} \quad z \ll \alpha \ln N}$$

$$= N^{-1}e^{-z}.$$

Then it follows

$$P\left(\frac{M_N - a_N}{b_N} < z\right) = \left\{1 - \exp\left[-\frac{(b_N z + a_N)^\alpha}{\beta}\right]\right\}^N$$

$$\simeq \left(1 - N^{-1}e^{-z}\right)^N \rightarrow \exp\left(e^{-z}\right) \quad \text{as} \quad N \rightarrow \infty$$

$$\rightarrow \quad \boxed{F \equiv P(M_N < z) \simeq \exp\left[-\exp\left(-\frac{z-a_N}{b_N}\right)\right]. \quad \textbf{Gumbel distribution}}$$

Probability that block maximum does not exceed z

Generalized extreme value theory GEV

i.e. the exceedance probability of the block maximum may be
approximated by the Gumbel distribution

$$F \equiv P(M_N < z) \simeq \exp\left[-\exp\left(-\frac{z - a_N}{b_N}\right)\right].$$

Note that $\qquad P(\eta / H_s > z) \equiv 1 - F^{1/N}$

Idea: find a_N, b_N from data records → predict occurrence rates for large waves

Generalized extreme value theory GEV

i.e. the exceedance probability of the block maximum may be approximated by the Gumbel distribution

$$F \equiv P(M_N < z) \simeq \exp\left[-\exp\left(-\frac{z - a_N}{b_N}\right)\right].$$

Note that $\qquad P(\eta \, / \, H_s > z) \equiv 1 - F^{1/N}$

Idea: find a_N, b_N from data records → predict occurrence rates for large waves
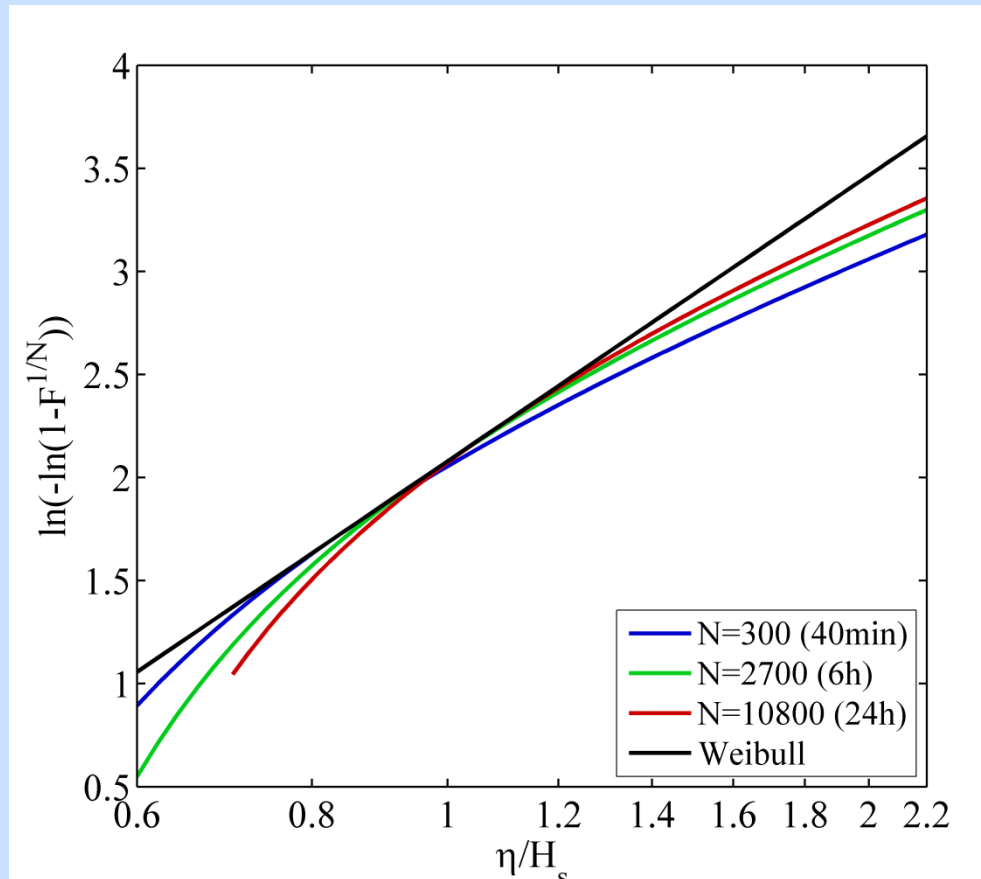
Approximation may be tested by comparing

$$\ln[-\ln P(\eta_l \, / \, H_s > z)] = \alpha \ln z + \ln \beta \qquad \textbf{with} \qquad \ln[-\ln(1 - F^{1/N})]$$

→ $\qquad \alpha \ln z + \ln \beta \simeq \ln\left[-\ln\left(1 - \left\{\exp\left[-\exp\left(-\frac{z - a_N}{b_N}\right)\right]\right\}^{1/N}\right)\right]$

?

Generalized Extreme Value theory GEV

Does the Gumbel distribution provide a reliable way for extrapolating the distribution for rare events (large z)?


Fitting GEVs to block maxima obtained from data, without any a priori assumption about the pdf of the individual crest heights, does not seem appropriate. (Derivation required lnN to be large, note only N)

Other "extreme wave" phenomena

- unexpected waves
- wave-current interaction (may increase wave height, but not statistics)

II. Unexpected waves

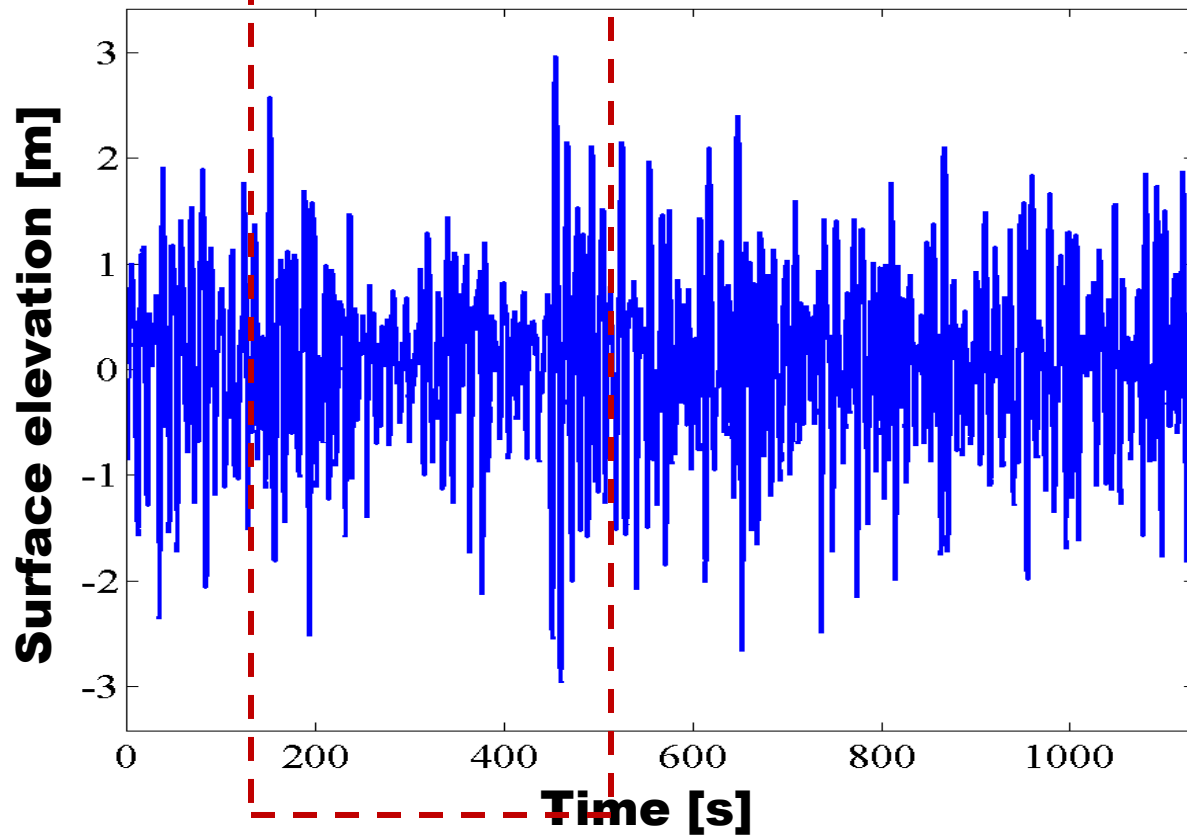

Observations off Cape Scott, BC, Canada

$H_s = 3.6m$

Rogue wave: H>8m

$H_{max} < 6m$
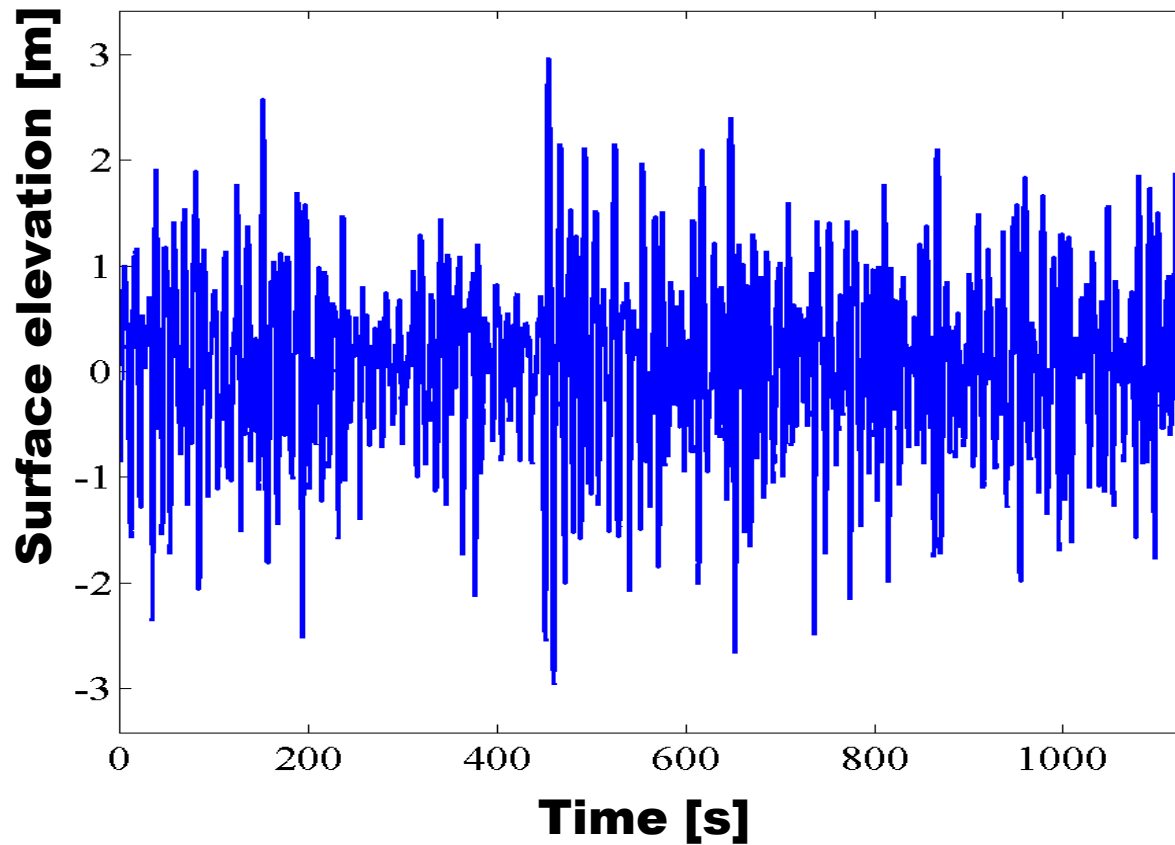
No rogue wave in record

Unexpected waves

H_s = 3.6m

Rogue wave: H>8m

H_{max} < 6m

No rogue wave
in record

MEDS226, Cape Scott, BC, Canada

Unexpected waves – data example, simulations
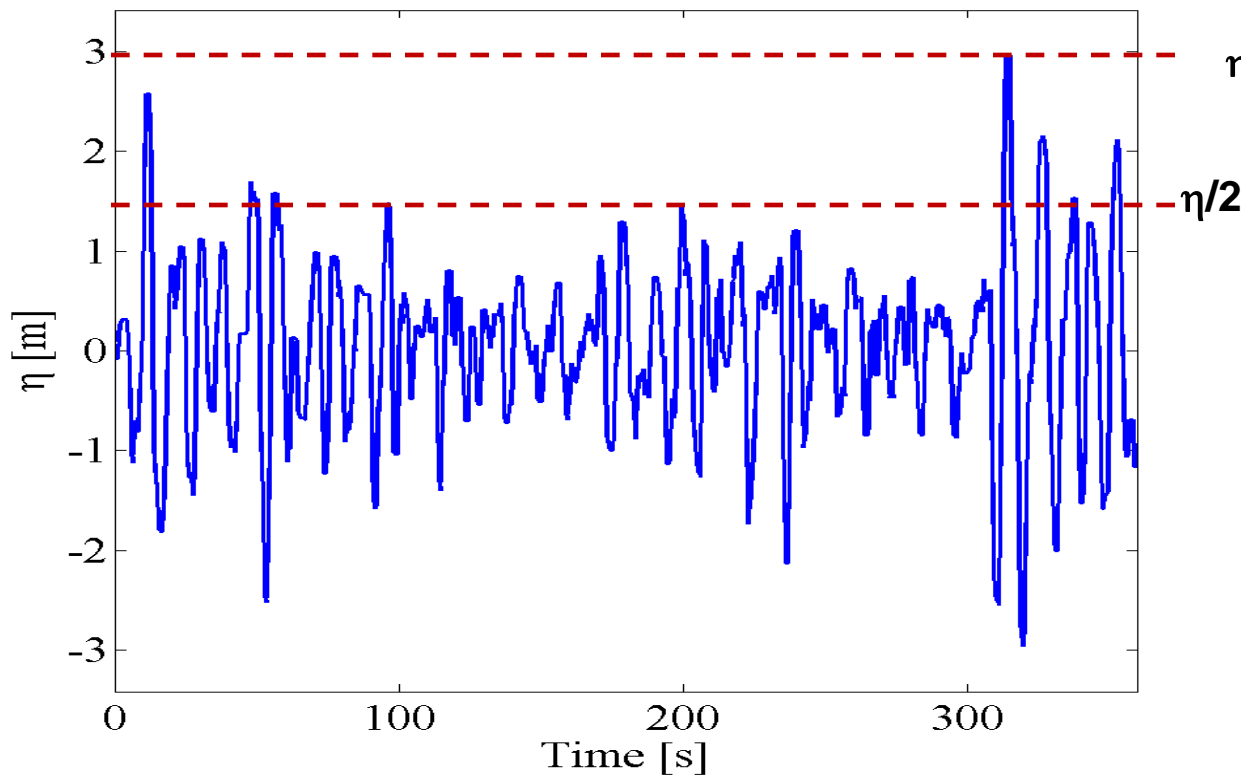
A wave crest twice as high as any in the preceding 30 waves.

=

unexpected wave

Monte Carlo simulations:
- linear, random superposition,
- 2nd order Stokes correction
- intermediate water depth correction

Occurrence rate of *unexpected* waves (deep water):

about 1 in 14,000 (daily)
more frequent in shallow water

Gemmrich & Garrett: Unexpected Waves, JPO, 2008, Ocean Eng. 2009

III. Wave intensification by currents

Motivated by hourly time series of significant wave heights H_s from northeast Pacific wave buoys (up to 30 year records)

Wave intensification by currents

Wave breaking due to wave-current interaction in a tidal front (Haro Strait, BC). *Photo: B. Baschek*

Strait of Gibraltar: Internal waves → inhomogeneous surface currents → modify steepness of surface waves → modified sun glint → observable from space

The effect of currents (casual observations)

waves with an initial phase speed *c* in still water propagating into an opposing current *u* will steepen
→ increased wave amplitude (e.g. river estuaries, tidal fronts, Agulhas current)

Waves will be stopped completely by a current of $u \geq \alpha c$.

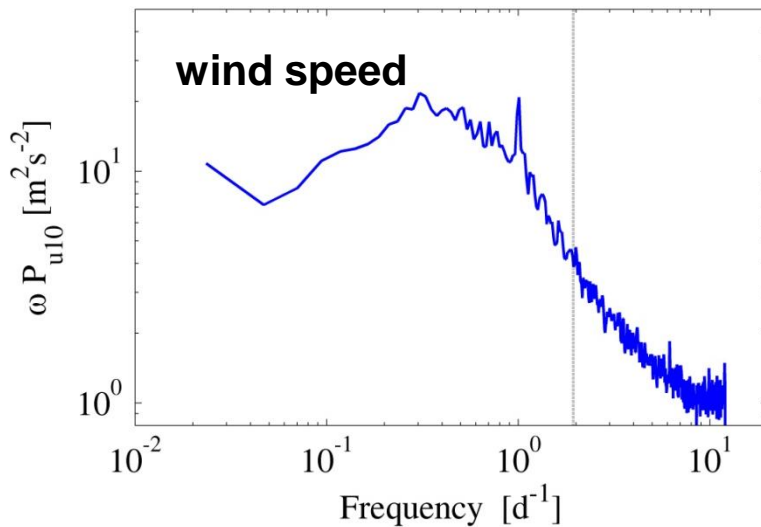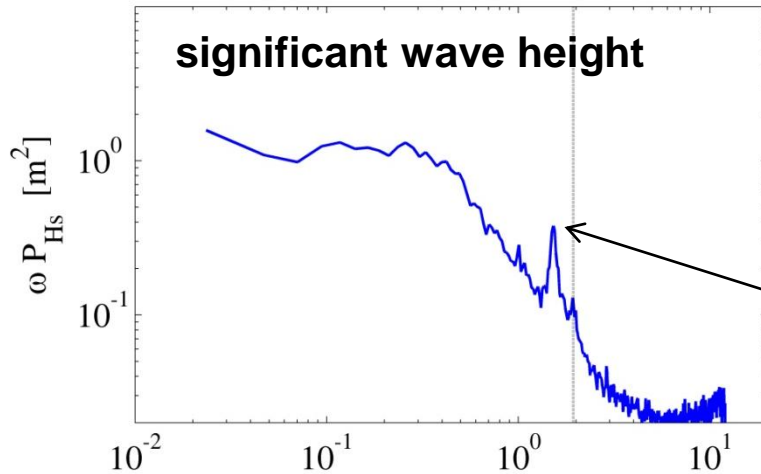What is the value of α?

$\alpha = 1/4$!
(factor 2: $c_g = \frac{1}{2} c_p$
factor 2: waves shorten → decrease in c_p)

Spectral content of wave and wind fluctuations

C46206
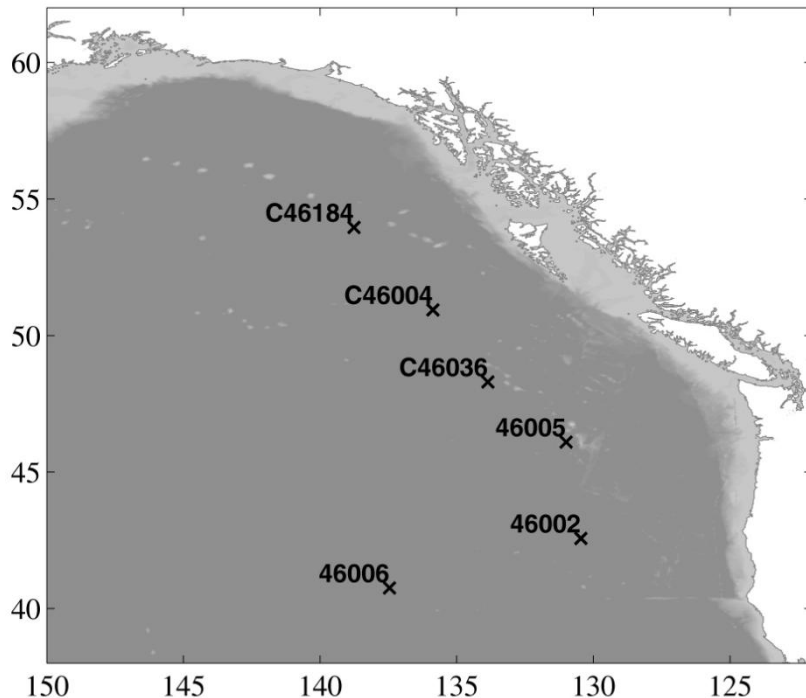
significant wave height

wind speed

inertial period
(~16h, wind-induced currents)

no inertial peak in wind forcing
→
wave - inertial current interaction

Spectral content of wave fluctuations -- latitude

Offshore wave buoy stations

Wind-induced "inertial" current:

$$\mathbf{v} = v_o \begin{pmatrix} \sin \omega t \\ \cos \omega t \end{pmatrix}, \quad \boxed{\omega = 2\Omega \sin \phi}$$
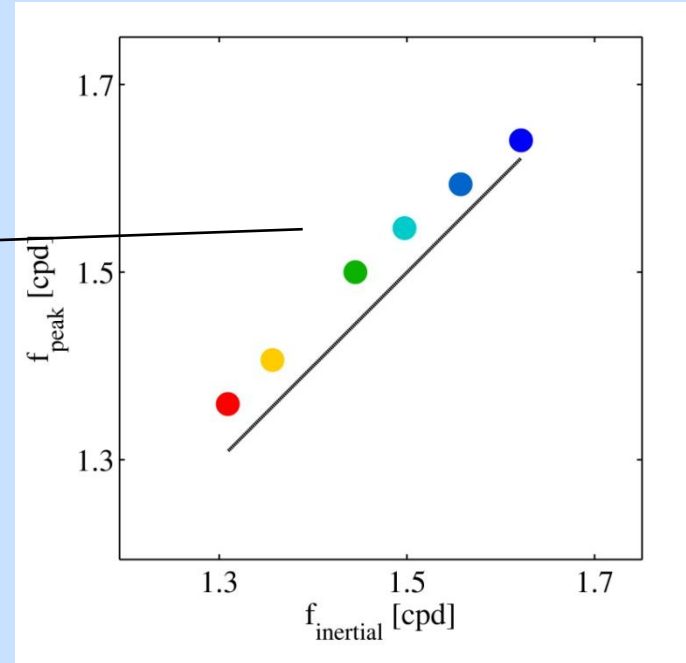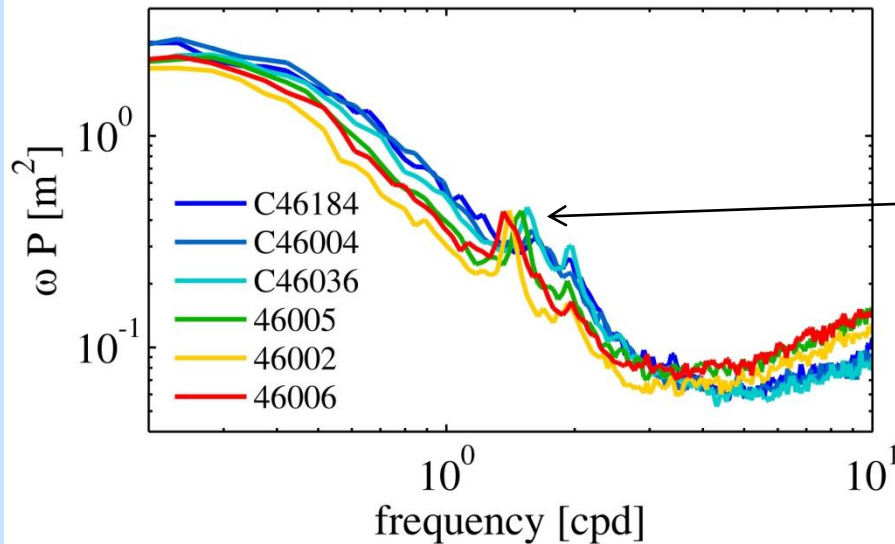
ϕ: latitude

Increasing frequency with latitude
(Foucault pendulum)

Can this be seen in the wave height as well ?

Spectral content of wave fluctuations -- latitude

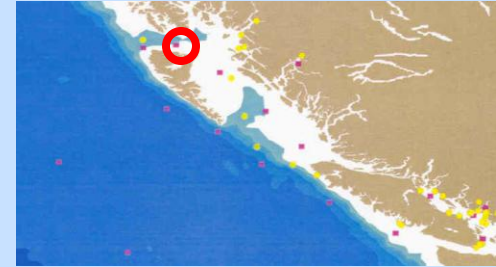Significant wave heights, offshore NE Pacific
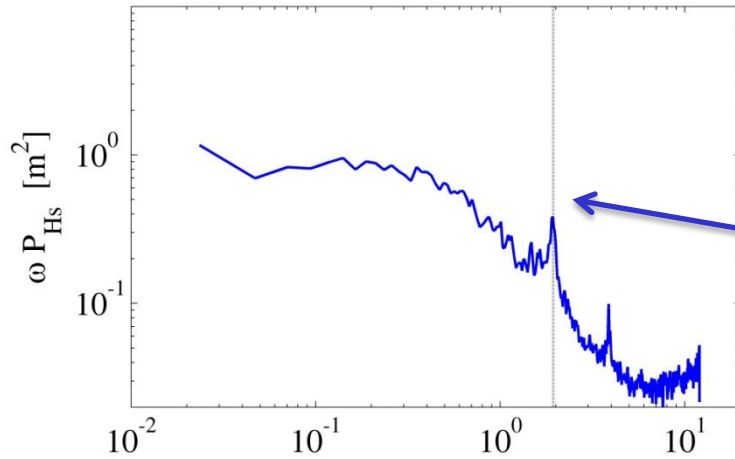

Wave height modulation:

- **frequency increases with latitude**
- **same as inertial currents**

Wave heights are modulated by inertial currents

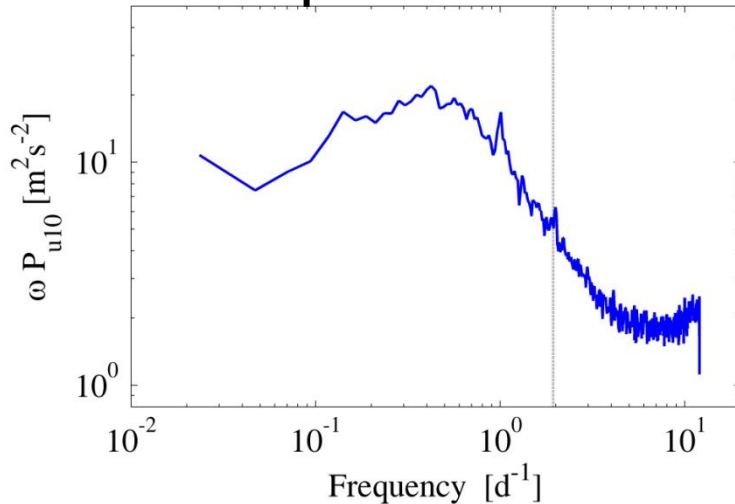# Spectral content of wave and wind fluctuations, Dixon Entrance
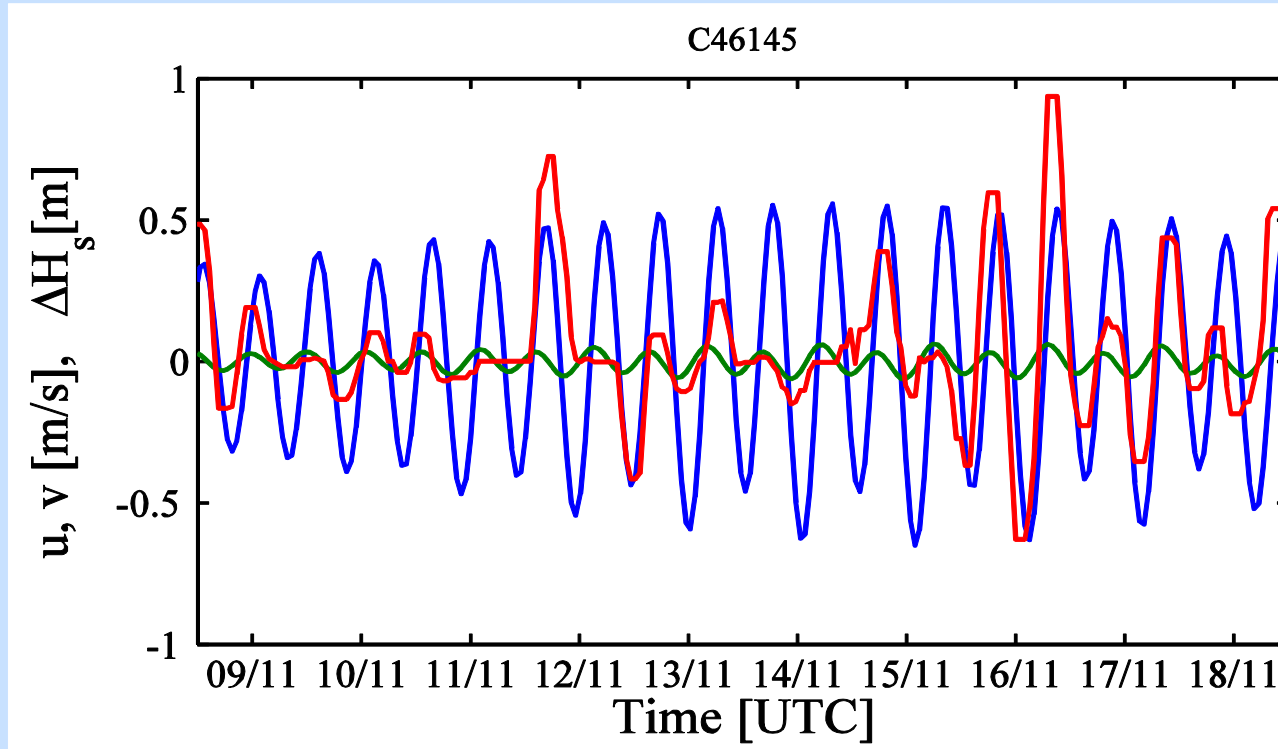

significant wave height

wind speed

Semi-diurnal peak

no semi-diurnal peak in wind forcing
→
wave - tidal current interaction

Significant wave height – tidal current
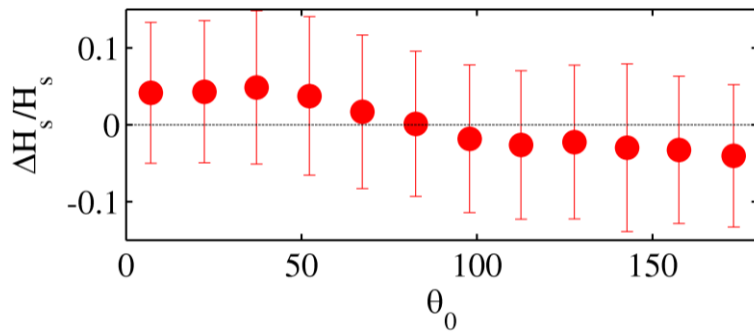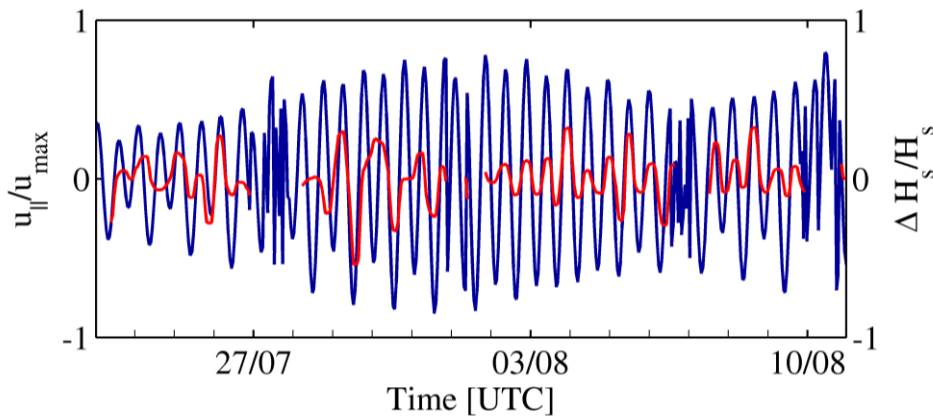

C46145

ΔH$_s$: wave height fluctuations
(12h median band-pass filter)
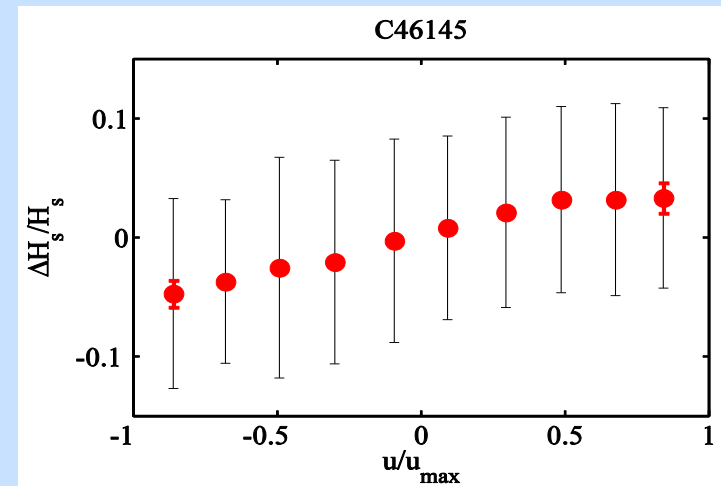u: E-W barotropic tidal current
(positive towards E)
v: N-S barotropic tidal current
(positive towards N)

wave height fluctuations – current are in phase !

Average significant wave height fluctuations– tidal current

10-year record, hourly observations

ΔH$_s$: wave height fluctuations
(12h median band-pass filter)
u$_{max}$: maximum amplitude of tidal current

wave height fluctuations – current are in phase !

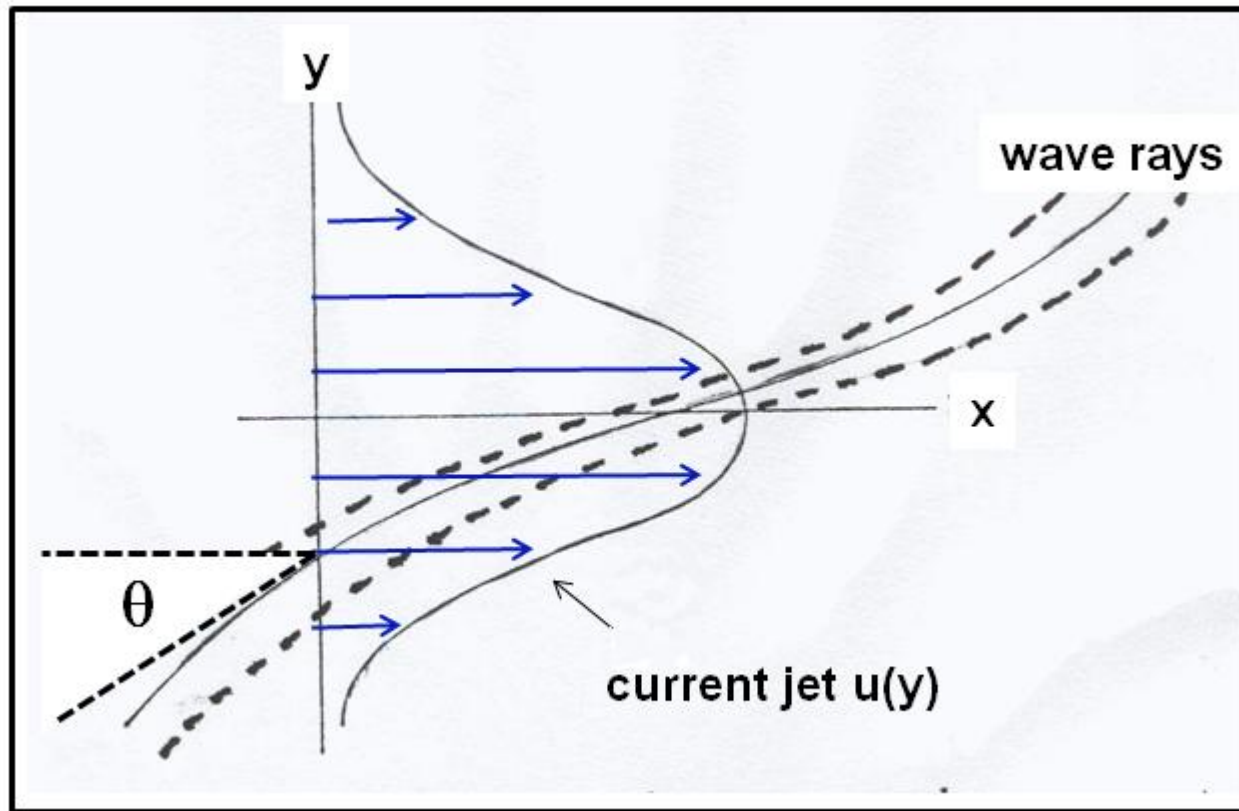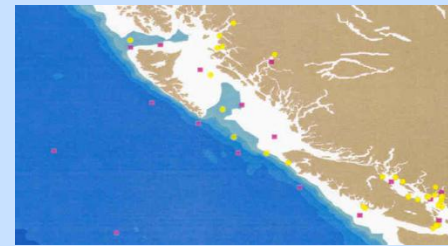# Significant wave height fluctuations– tidal current

> **wave height fluctuations – current are in phase !**

i.e. wave height **increases** when waves **follow** the current

wave height **decreases** in **opposing** currents

Contrary to
casual observations: wave height increases when waves oppose the current
(wave blocking at opposing current $u = c_p/4$)
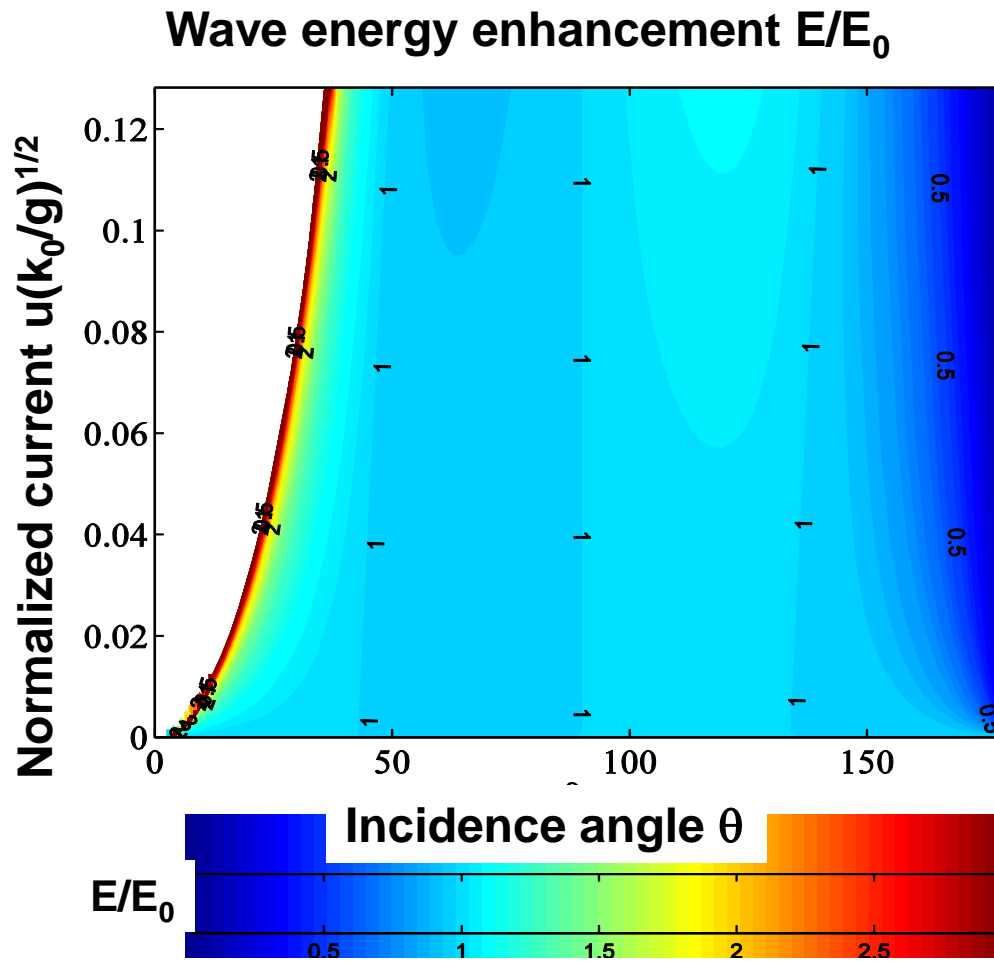
Wave – current interaction (2-d)

**Wave propagating obliquely into current jet
(conservation of wave action flux)**

→ **wave refraction** → **narrowing of ray tube** → **wave intensification**

from: Garrett, 1976, J. Mar. Res.

Wave – current interaction (conservation of wave action)

$$\frac{\partial A}{\partial t} + \nabla \cdot \left[(\mathbf{c}_g' + \mathbf{U})\, A \right] = 0, \qquad \frac{E}{E_0} = B \sin\theta \left[1 - \left(\frac{\cos\theta}{B} \right)^2 \right]^{-1/2}, B = \left(1 - U^* \cos\theta \right)^2, U^* = U(k_0 / g)^{1/2}$$

Wave energy enhancement E/E$_0$

Enhancement:
$0 \le \theta \le 45$
(waves following current)

Attenuation:
$135 \le \theta \le 180$
(waves opposing current)

Consistent
with observations

Conclusions

- **Simple probability distributions models underpredict rogue wave occurrences**
- **Monte-Carlo simulations :**
    - **4th order Stokes wave corrections give best agreement with observations**
        - **→ non-resonant interactions**

- **"Unexpected waves" , may be relevant to recreational boating**

- **Wave height (H_s) modifications due to tidal and inertial currents are significant (up to 45%, not included in wave forecast models)**
- **GEV analysis does not extend statistical prediction range**

not every large wave is a rogue wave !