

Dispersive and non-dispersive wave runup and some related phenomena

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« The Mathematics of Extreme Sea Waves »



Outline

- 1 Classical Boussinesq systems
- 2 Solitary wave runup
- 3 Wave runup on random bottoms
- 4 Runup amplification phenomena

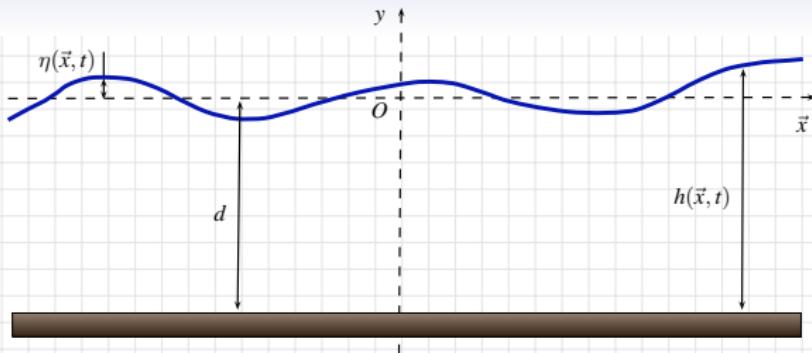
Acknowledgements

To my collaborators

- Dimitrios Mitsotakis: Institute for Mathematics and its Applications, University of Minnesota
- Theodoros Katsaounis: Assistant Professor, University of Crete
- Frédéric Dias: Professor, University College Dublin (on leave from ENS de Cachan)
- Themistoklis Stefanakis: PhD student, UCD & ENS de Cachan
- Céline Labart: Assistant Professor, University of Savoie



Water wave problem



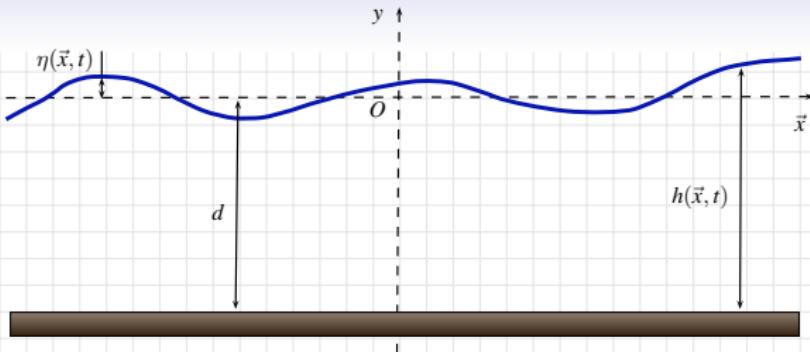
- $\nabla^2 \phi + \partial_y^2 \phi = 0, -d(\vec{x}) < y < \eta(\vec{x}, t)$
- $\partial_t \eta + \nabla \phi \cdot \nabla \eta - \partial_y \phi = 0, y = \eta(\vec{x}, t)$
- $\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} (\partial_y \phi)^2 + g\eta = 0, y = \eta(\vec{x}, t)$
- $\nabla d \cdot \nabla \phi + \partial_y \phi = 0, y = -d(\vec{x})$



Dimensionless parameters:

Nonlinearity: $\varepsilon := \frac{a}{h_0}$; Dispersion: $\mu^2 := \left(\frac{h_0}{\ell}\right)^2$

Water wave problem



- $\nabla^2\phi + \frac{1}{\mu^2}\partial_y^2\phi = 0, -d(\vec{x}) < y < \varepsilon\eta(\vec{x}, t)$
- $\partial_t\eta + \varepsilon\nabla\phi \cdot \nabla\eta - \frac{1}{\mu^2}\partial_y\phi = 0, y = \varepsilon\eta(\vec{x}, t)$
- $\partial_t\phi + \frac{1}{2}\varepsilon|\nabla\phi|^2 + \frac{1}{2}\frac{\varepsilon}{\mu^2}(\partial_y\phi)^2 + \eta = 0, y = \varepsilon\eta(\vec{x}, t)$
- $\varepsilon\nabla d \cdot \nabla\phi + \frac{1}{\mu^2}\partial_y\phi = 0, y = -d(\vec{x})$



Dimensionless parameters:

Nonlinearity: $\varepsilon := \frac{a}{h_0}$; Dispersion: $\mu^2 := \left(\frac{h_0}{\ell}\right)^2$

Dispersive long wave models

Boussinesq-type equations

Long wave (Boussinesq) scaling:

- Nonlinearity: $\varepsilon := \frac{a}{h_0} \ll 1$
- Dispersion: $\mu^2 := \left(\frac{h_0}{\ell}\right)^2 \ll 1$
- Stokes-Ursell number: $S := \frac{\varepsilon}{\mu^2} \sim 1$
- Literature is countless: Peregrine [Per67], Bona-Smith [BS76], Nwogu [Nwo93], Bona & Chen [BC98]



BCS (2002); DL *et al.* (2005): $T = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$

$$\eta_t + u_x + \varepsilon(\eta u)_x + \varepsilon[au_{xxx} - b\eta_{xxt}] = 0$$

$$u_t + \eta_x + \varepsilon uu_x + \varepsilon[c\eta_{xxx} - du_{xxt}] = 0$$

Conservative form of the governing equations

Flat bottom 1D case

Boussinesq equations:

$$(\mathbf{I} - \mathbf{D})\mathbf{v}_t + [\mathbf{F}(\mathbf{v})]_x + [\mathbf{G}(\mathbf{v}_{xx})]_x = 0$$

"Conservative" variables: $\mathbf{v} := (\eta, u)$

Advective flux: $\mathbf{F}(\mathbf{v}) = ((1 + \eta)u, \eta + \frac{1}{2}u^2)$

Elliptic operator: $\mathbf{I} - \mathbf{D} = \text{diag}(1 - b\partial_{xx}^2, 1 - d\partial_{xx}^2)$

Dispersion: $\mathbf{G}(\mathbf{v}_{xx}) = (au_{xx}, c\eta_{xx})$

Reference:

D. Dutykh, T. Katsaounis, D. Mitsotakis. *Finite volume schemes for dispersive wave propagation and runup*. J. Comput. Phys., **230**, 3035–3061, 2011

Finite volume schemes for dispersive waves

Schemes implemented in [DKM11]

$$(\mathbf{I} - \mathbf{D})\mathbf{v}_t + [\mathbf{F}(\mathbf{v})]_x + [\mathbf{G}(\mathbf{v}_{xx})]_x = 0$$

Advective fluxes: m-Flux, KT, FVCF

Reconstructions: TVD MUSCL2, UNO2, WENO3

Elliptic operator: 2nd or **4th** order FD:

$$(\mathbf{I} - \mathbf{D})\mathbf{v} \approx \frac{\mathbf{v}_{i-1} + 10\mathbf{v}_i + \mathbf{v}_{i+1}}{12} - (b, d) \frac{\mathbf{v}_{i+1} - 2\mathbf{v}_i + \mathbf{v}_{i-1}}{\Delta x^2}$$

Dispersion: Centered flux:

$$\mathbf{G}_{i+\frac{1}{2}}^m = (a, c) \frac{\mathbf{v}_i^{(2)} + \mathbf{v}_{i+1}^{(2)}}{2}, \quad \mathbf{G}_{i+\frac{1}{2}}^{lm} = (a, c) \frac{\mathbf{v}_{i+\frac{1}{2}}^{(2),L} + \mathbf{v}_{i+\frac{1}{2}}^{(2),R}}{2}$$

Time stepping: Explicit SSP-RK2, SSP-RK3

Bona-Smith system

Some properties of this particular system

Bona-Smith system (we take $\theta^2 = \frac{8}{10}$):

$$\begin{aligned}\eta_t + u_x + (\eta u)_x - \frac{3\theta^2 - 1}{6} \eta_{xxt} &= 0, \\ u_t + \eta_x + uu_x + \frac{2 - 3\theta^2}{3} \eta_{xxx} - \frac{3\theta^2 - 1}{6} u_{xxt} &= 0.\end{aligned}$$

Exact solitary wave solution:

$$\eta(\xi) = \eta_0 \operatorname{sech}^2(\lambda \xi), \quad u(\xi) = B \eta(\xi), \quad \xi := x - c_s t$$

$$\eta_0 = \frac{9}{2} \cdot \frac{\theta^2 - 7/9}{1 - \theta^2}, \quad c_s = \frac{4(\theta^2 - 2/3)}{\sqrt{2(1 - \theta^2)(\theta^2 - 1/3)}}$$

Invariants: the mass conservation $I_0 = \int_{\mathbb{R}} \eta(x, t) dx$

$$I_1 = \int_{\mathbb{R}} (\eta^2(x, t) + (1 + \eta(x, t))u^2(x, t) - c\eta_x^2(x, t) - au_x^2(x, t)) dx$$

Convergence test-cases - I

Numerical results on the solitary wave propagation

$$E_h^2(k) = \|U^k\|_h / \|U^0\|_h, \quad \|U^k\|_h = \left(\sum_i \Delta x |U_i^k|^2 \right)^{1/2},$$

$$E_h^\infty(k) = \|U^k\|_{h,\infty} / \|U^0\|_{h,\infty}, \quad \|U^k\|_{h,\infty} = \max_i |U_i^k|,$$

Δx	Rate(E_h^2)	Rate(E_h^∞)
0.5	1.910	1.978
0.25	1.910	1.954
0.125	1.923	1.937
0.0625	1.936	1.941
0.03125	1.946	1.948

Table: m-Flux

Δx	Rate(E_h^2)	Rate(E_h^∞)
0.5	2.042	2.032
0.25	2.033	2.029
0.125	2.026	2.023
0.0625	2.021	2.019
0.03125	2.017	2.016

Table: MUSCL2 MinMod

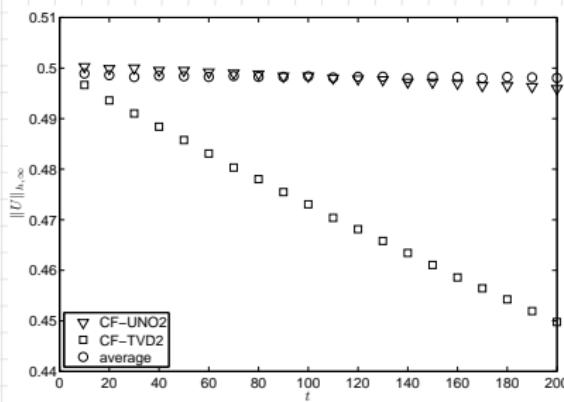
Convergence test-cases - II

Invariants preservation

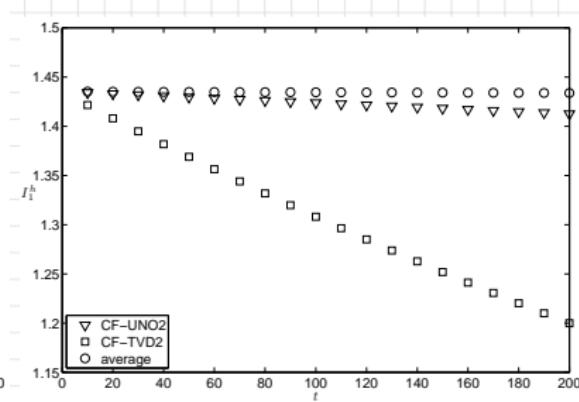
Mass and the generalized energy for the B-S system:

$$I_0 = \int_{\mathbb{R}} \eta(x, t) dx \approx 1.932183566158$$

$$I_1 = \int_{\mathbb{R}} (\eta^2(x, t) + (1 + \eta(x, t))u^2(x, t) - c\eta_x^2(x, t) - au_x^2(x, t)) dx$$



(a) η -amplitude



(b) I_1 invariant

Head-on collision of solitary waves

Comparison with experimental data

Comparison among following models:

- Experimental data
- BBM-BBM system
- Bona-Smith system ($\theta^2 = \frac{9}{11}$)

Numerical scheme: FVCF + UNO2 + SSP-RK3

Numerical parameters:

$$[-300, 300], \quad \Delta x = 0.05, \quad CFL = 0.2$$

Reference:

W. Craig, Ph. Guyenne, J. Hammack, D. Henderson, C. Sulem.
Solitary water wave interactions. Phys. Fluids, 2006

Head-on collision experiment

Comparison with experimental data by D. Henderson

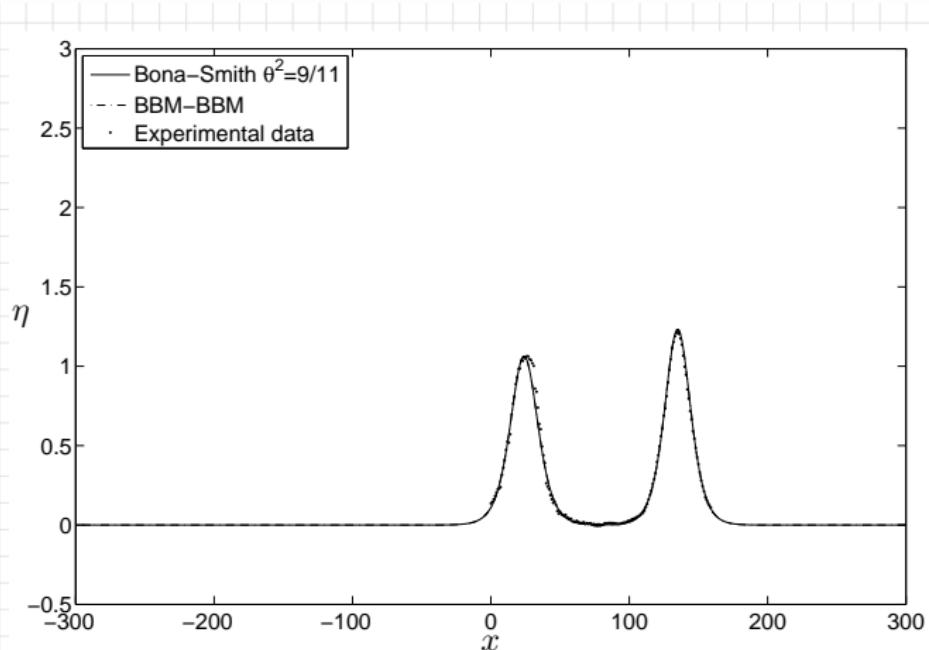


Figure: $t = 18.29993$ s

Head-on collision experiment

Comparison with experimental data by D. Henderson

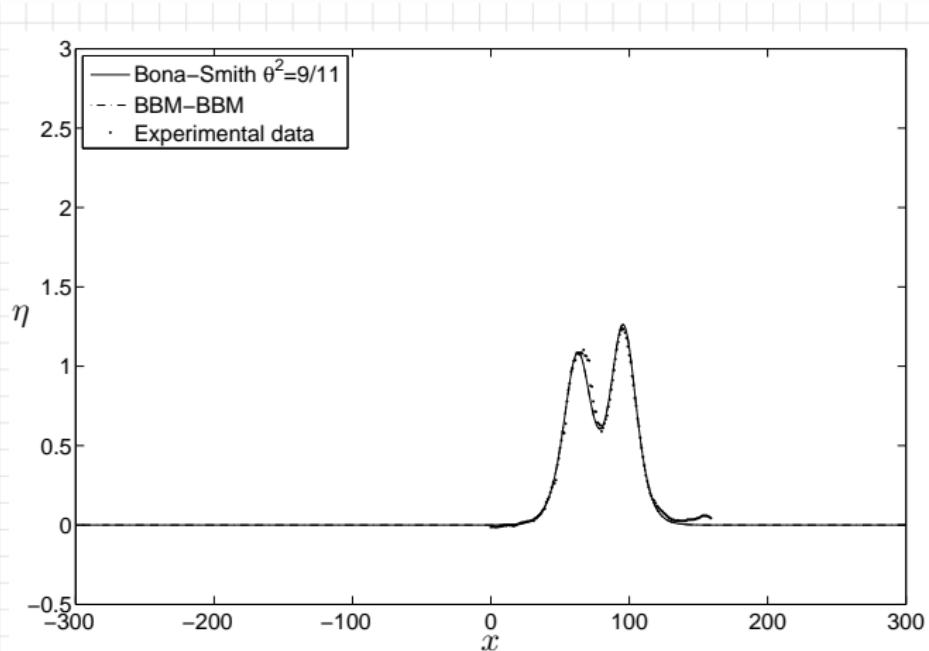


Figure: $t = 18.80067$ s

Head-on collision experiment

Comparison with experimental data by D. Henderson

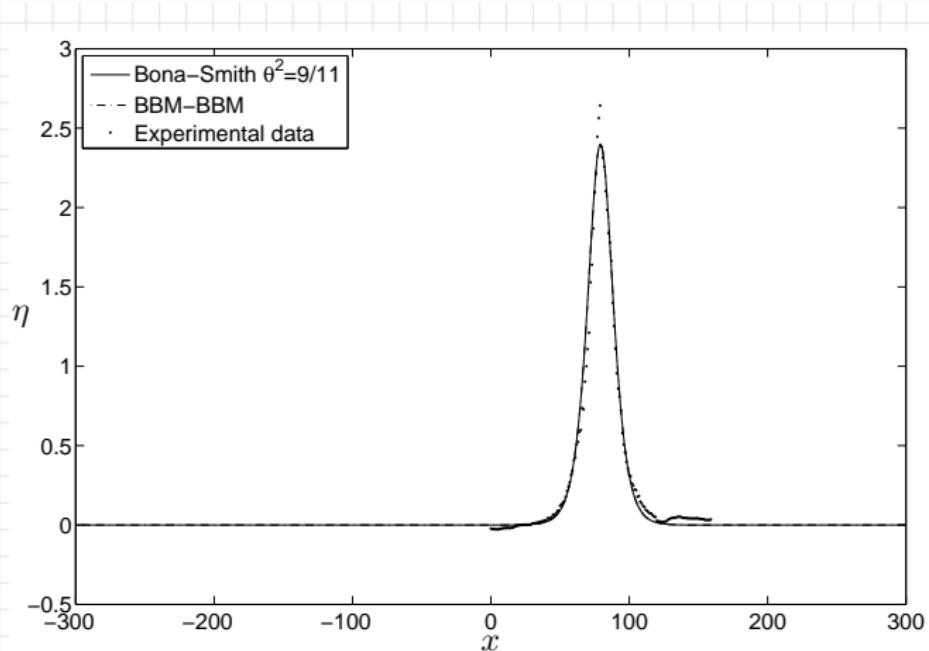


Figure: $t = 19.00956$ s

Head-on collision experiment

Comparison with experimental data by D. Henderson

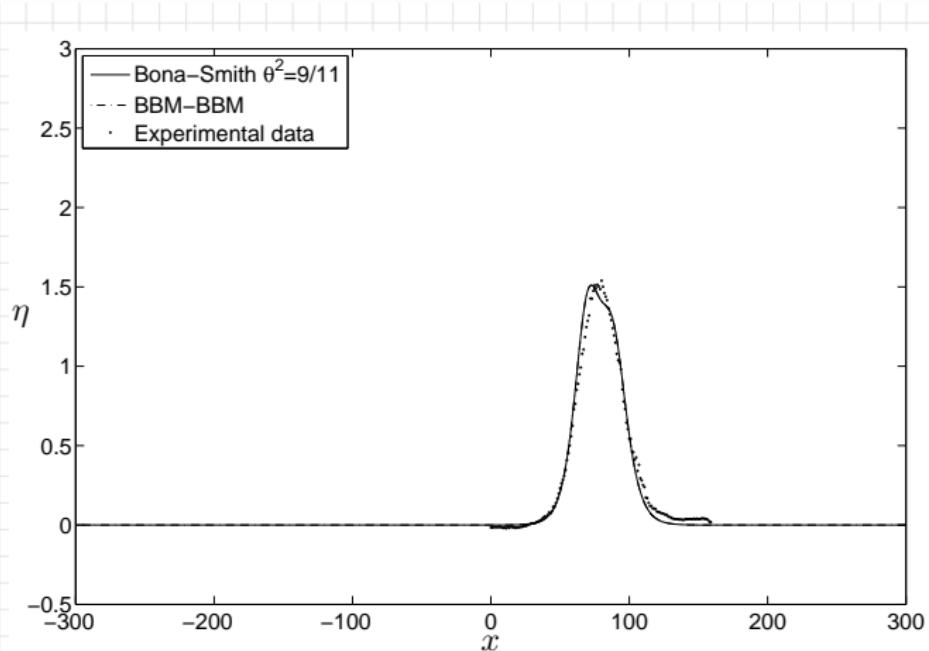


Figure: $t = 19.15087$ s

Head-on collision experiment

Comparison with experimental data by D. Henderson

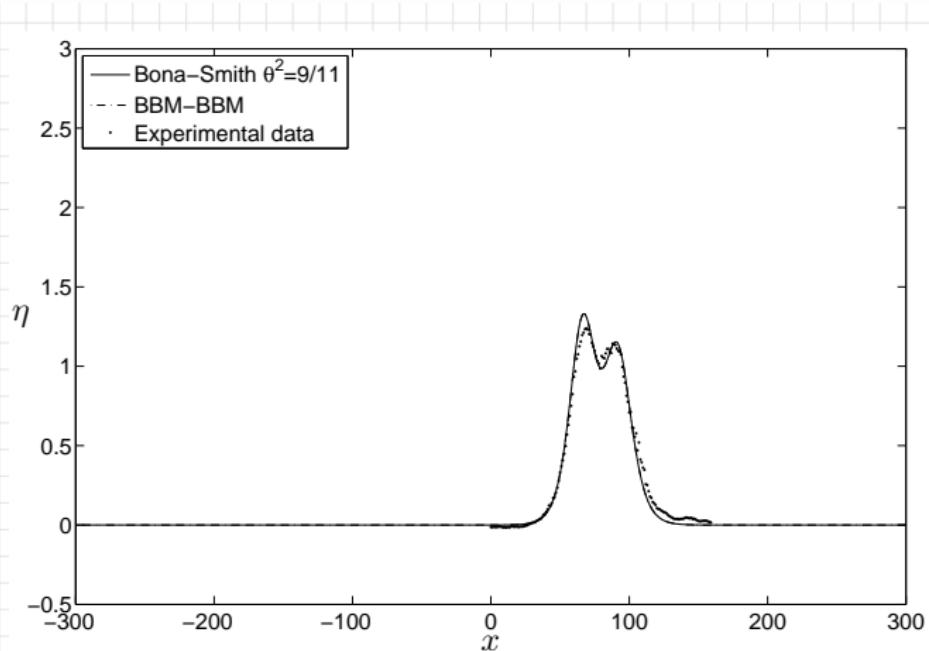


Figure: $t = 19.19388$ s

Head-on collision experiment

Comparison with experimental data by D. Henderson

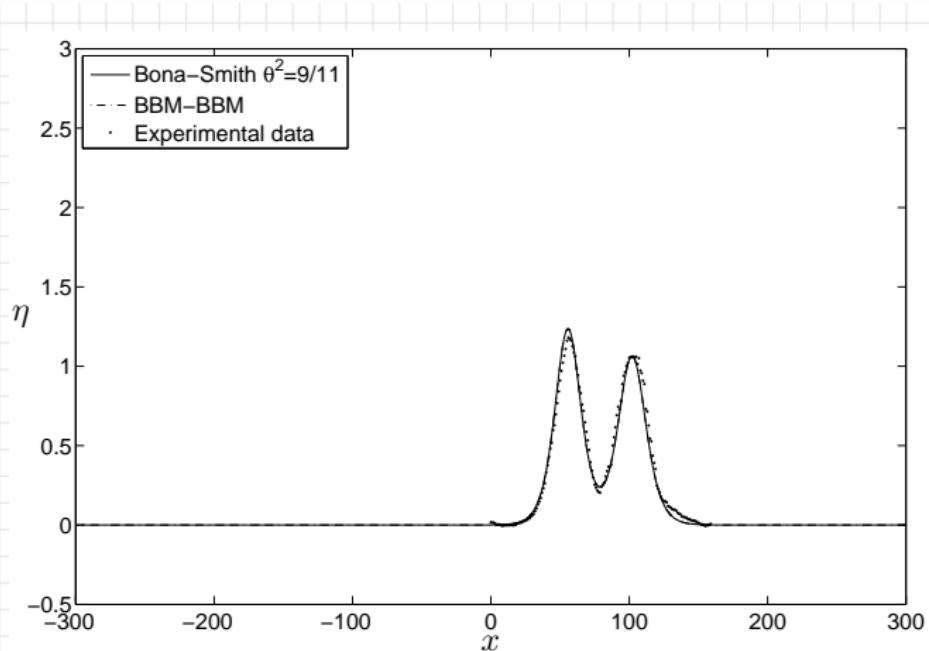


Figure: $t = 19.32904$ s

Head-on collision experiment

Comparison with experimental data by D. Henderson

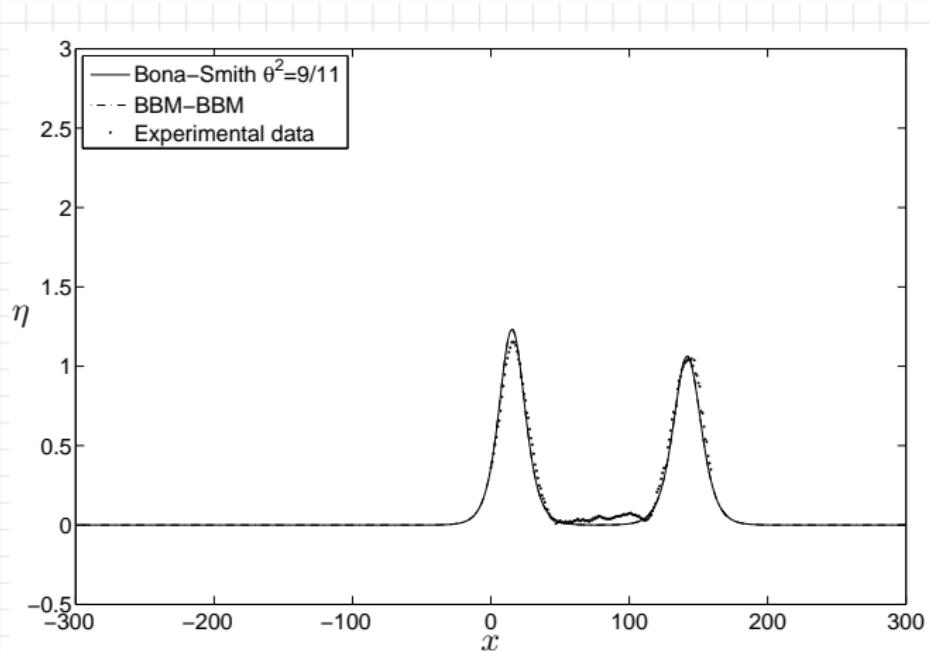


Figure: $t = 19.84514$ s

Head-on collision experiment

Comparison with experimental data by D. Henderson

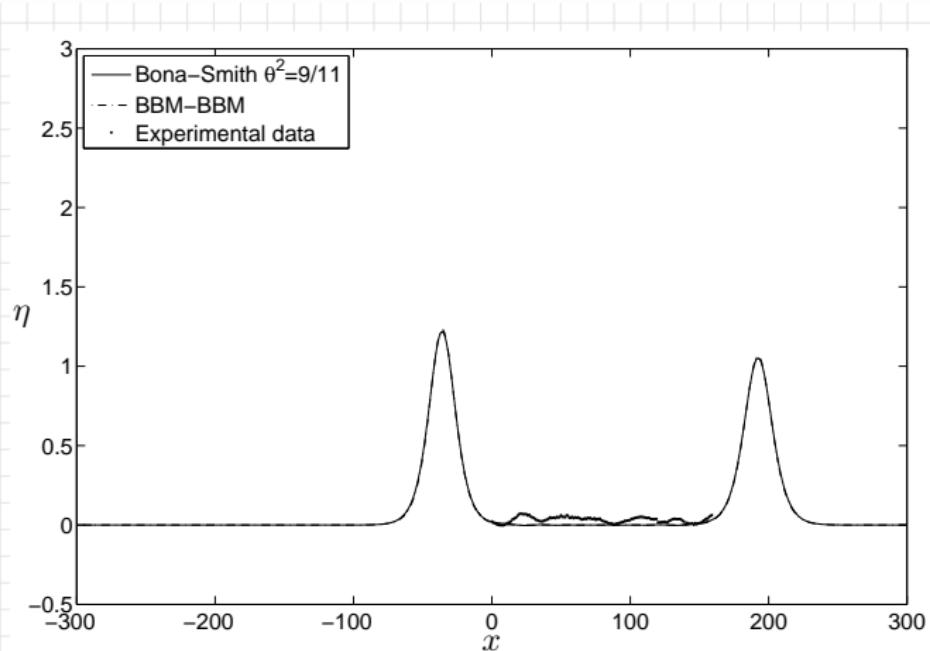
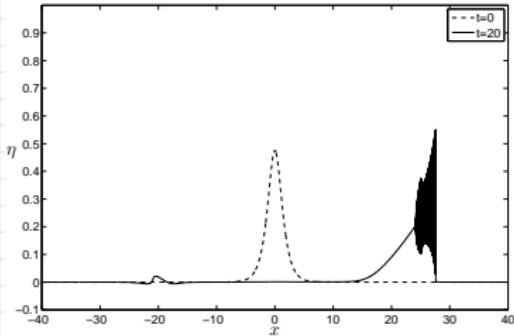


Figure: $t = 20.49949$ s

Further numerical experiments

Additional test-cases:

- Overtaking solitary waves collision
- Dispersive shock wave formation (conservation of invariants, etc.)



Reference:

D. Dutykh, T. Katsaounis, D. Mitsotakis. *Finite volume schemes for dispersive wave propagation and runup*. J. Comput. Phys., **230**, 3035–3061, 2011

Dispersive wave runup computation

Numerically stiff and tricky problem

P. Madsen *et al.* (1997), [MSS97]:

"However, to make this technique [slot technique] operational in connection with Boussinesq type models a couple of problems call for special attention. [...] Firstly the Boussinesq terms are switched off at the still water shoreline, where their relative importance is extremely small anyway. Hence in this region the equations simplify to the nonlinear shallow water equations."

G. Belotti & M. Brocchini (2002), [BB02]:

"In our attempt to use these equations from intermediate waters up to the shoreline we run into numerical troubles when reaching the run-up region, i.e. $x > 0$. These problems were essentially related to numerical instabilities due to the uncontrolled growth of the dispersive contributions (i.e. $\mathcal{O}(\mu^2)$ -terms)."

Boussinesq-type equations for non-flat bottom

The celebrated Peregrine system (1967)

Peregrine system in its original form:

$$\eta_t + ((h + \eta)u)_x = 0,$$

$$u_t + uu_x + g\eta_x - \frac{h}{2}(hu)_{xxt} + \frac{h^2}{6}u_{xxt} = 0$$

Reference [Per67]:

D.H. Peregrine. *Long waves on a beach.* JFM, 1967

Vertical translations (group G_5 , Benjamin & Olver [BO82]):

$$z \leftarrow z + d, \quad \eta \leftarrow \eta - d, \quad h \leftarrow h + d, \quad u \leftarrow u,$$

This symmetry is **broken**, only $H = h + \eta$ remains invariant

Peregrine system invariantization

Invariantization under vertical translations + conservative form of equations

Pre-conservative form:

$$H_t + (Hu)_x = 0,$$

$$(Hu)_t + \left(\varepsilon Hu^2 + \frac{1}{2\varepsilon} H^2\right)_x - \mu^2 \left(\frac{Hh}{2}(hu)_{xxt} - \frac{Hh^2}{6} u_{xxt} \right) = \frac{1}{\varepsilon} H h_x$$

Using $h = H + \mathcal{O}(\varepsilon)$, $H_t = \mathcal{O}(\varepsilon)$, $Q = Hu$:

$$H_t + Q_x = 0$$

$$\left(\left(1 + \frac{1}{3} H_x^2 - \frac{1}{6} HH_{xx} \right) Q - \frac{1}{3} H^2 Q_{xx} - \frac{1}{3} HH_x Q_x \right)_t + \left(\frac{Q^2}{H} + \frac{g}{2} H^2 \right)_x = g H h_x$$

- New system is asymptotically equivalent to the original one
- Symmetry G_5 is recovered
- Dispersive terms naturally $\rightarrow 0$ near the shoreline

m-Peregrine system discretization

With finite volumes method

$$\mathbf{D}(\mathbf{v}_t) + [\mathbf{F}(\mathbf{v})]_x = \mathbf{S}(\mathbf{v})$$

Advection: KT, FVCF + MUSCL2, UNO2

Dispersion: 2nd order FD for the elliptic operator

$$\mathbf{D}(\mathbf{v}) = \begin{pmatrix} H \\ (1 + \frac{1}{3}H_x^2 - \frac{1}{6}HH_{xx})Q - \frac{1}{3}HH_xQ_x - \frac{H^2}{3}Q_{xx} \end{pmatrix}$$

Source terms: Well-balanced hydrostatic reconstruction

Time stepping: Explicit SSP-RK3

For more details [DKM11]:

D. Dutykh, T. Katsaounis, D. Mitsotakis. *Finite volume schemes for dispersive wave propagation and runup*. J. Comput. Phys., **230**, 3035–3061, 2011

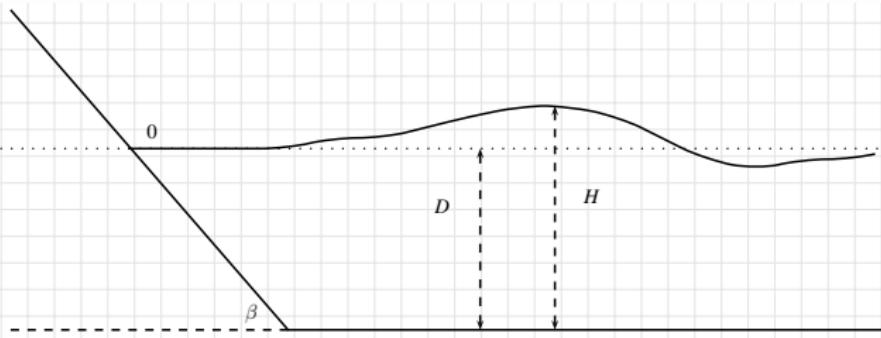
W.M. Keck Laboratory of Hydraulics, Caltech

PhD thesis of Costas Synolakis



Solitary wave runup on a plane beach

Experiments by C. Synolakis [Syn87], Caltech



Bottom shape:

$$-h(x) = \begin{cases} -x \tan \beta, & x \leq \cot \beta, \\ -1, & x > \cot \beta, \end{cases}$$

Initial condition:

$$\eta_0(x) = A_s \operatorname{sech}^2(\lambda(x - X_0)), \quad u_0(x) = -c_s \frac{\eta_0(x)}{D_0 + \eta_0(x)}$$

Solitary wave runup: $A_s = 0.04$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

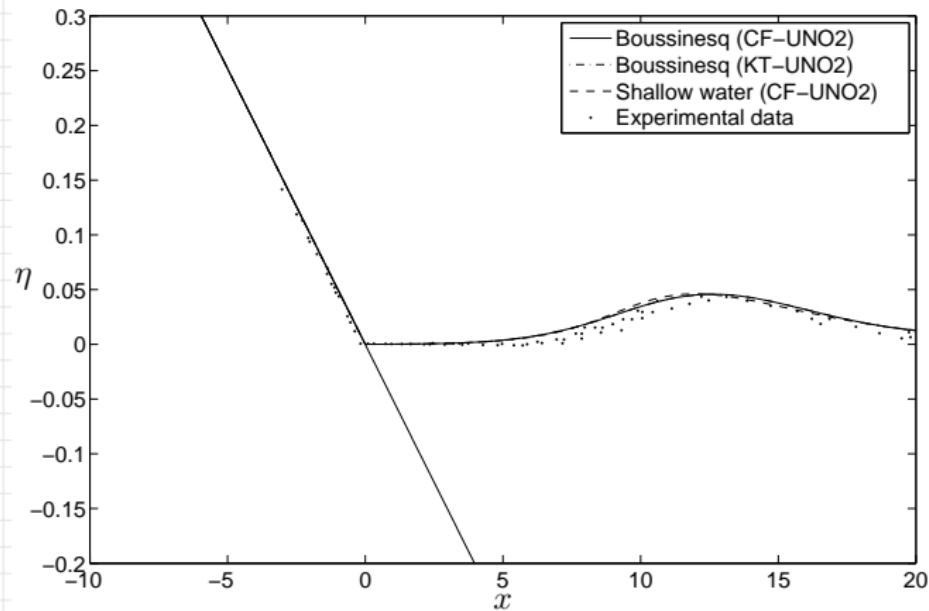


Figure: $t = 20$ s

Solitary wave runup: $A_s = 0.04$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

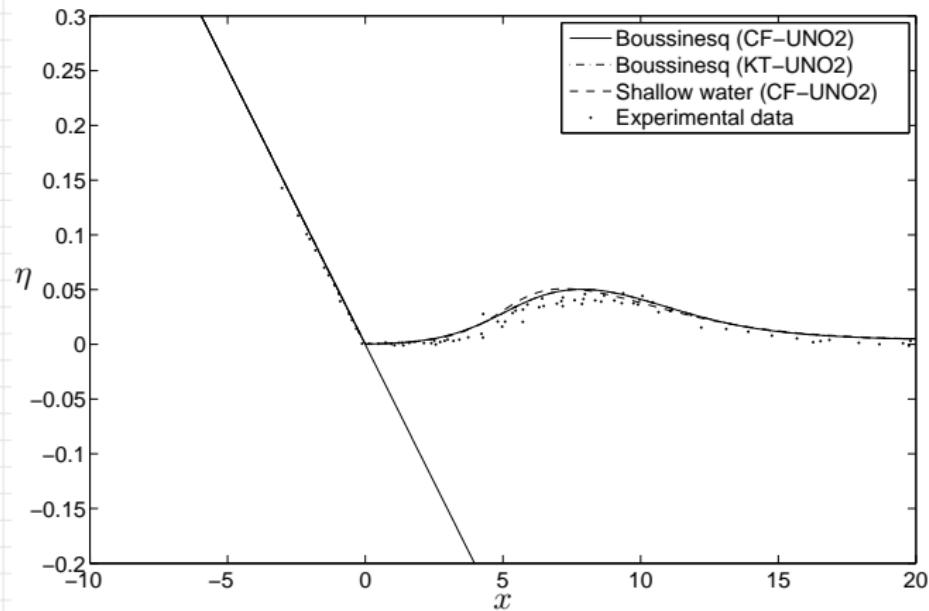


Figure: $t = 26$ s

Solitary wave runup: $A_s = 0.04$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

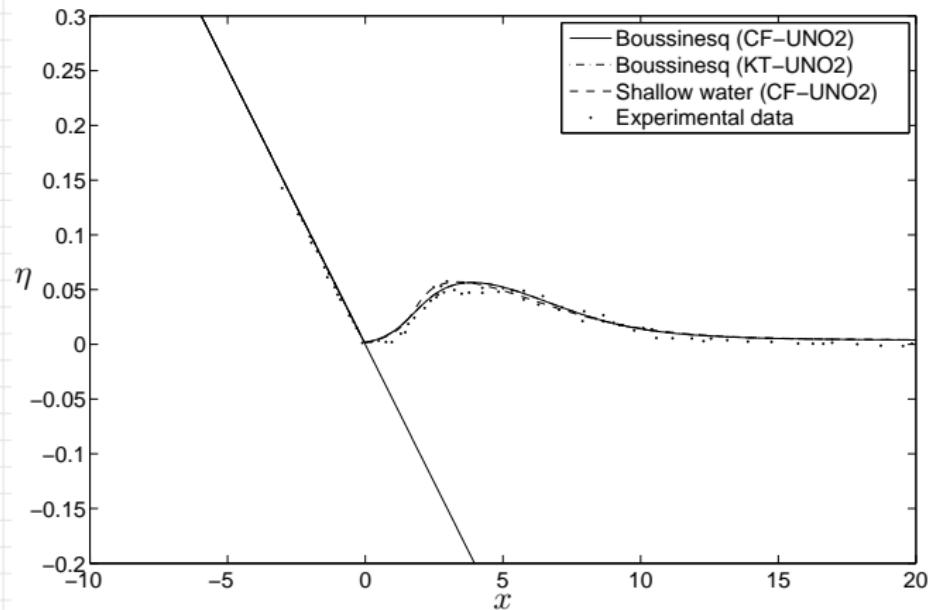


Figure: $t = 32$ s

Solitary wave runup: $A_s = 0.04$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

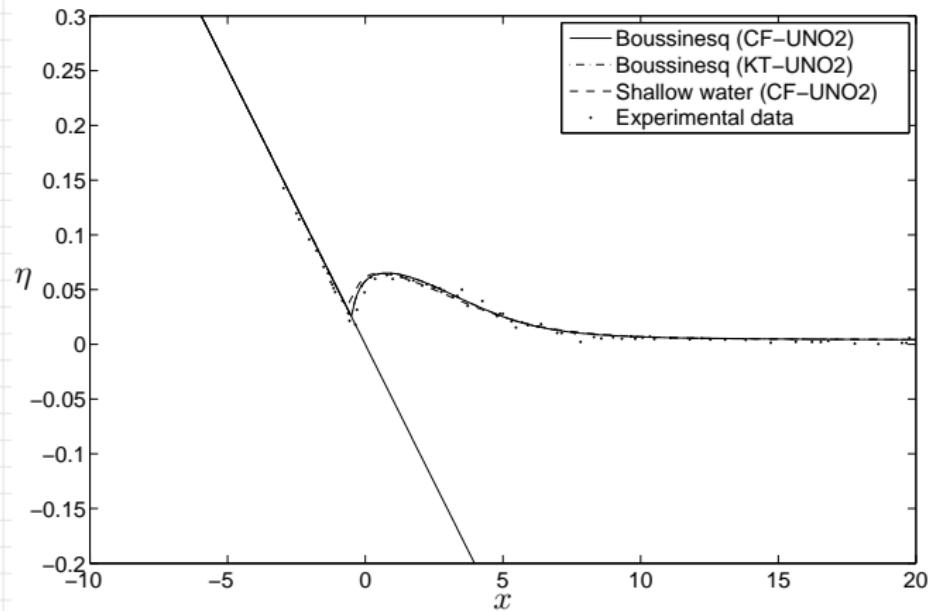


Figure: $t = 38$ s

Solitary wave runup: $A_s = 0.04$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

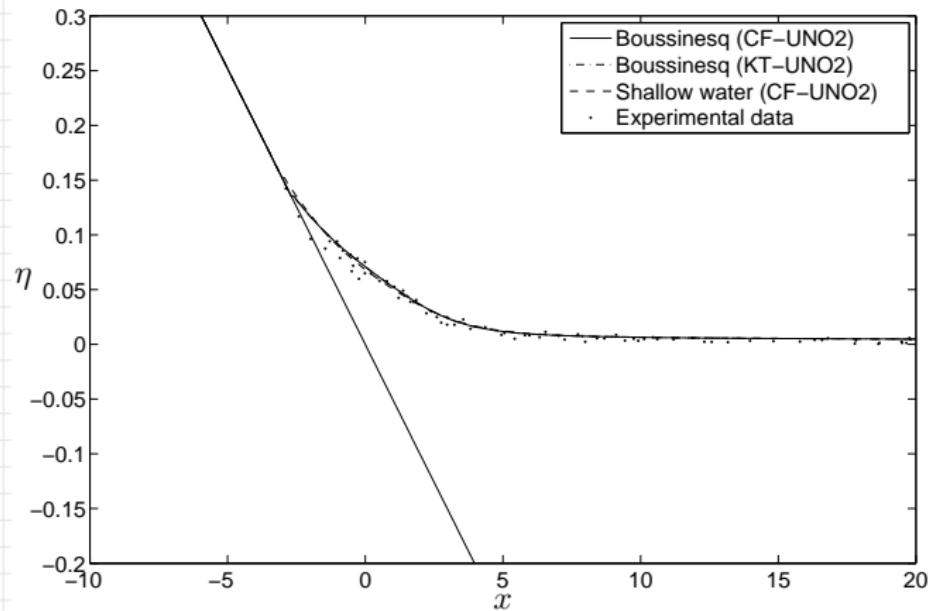


Figure: $t = 44$ s

Solitary wave runup: $A_s = 0.04$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

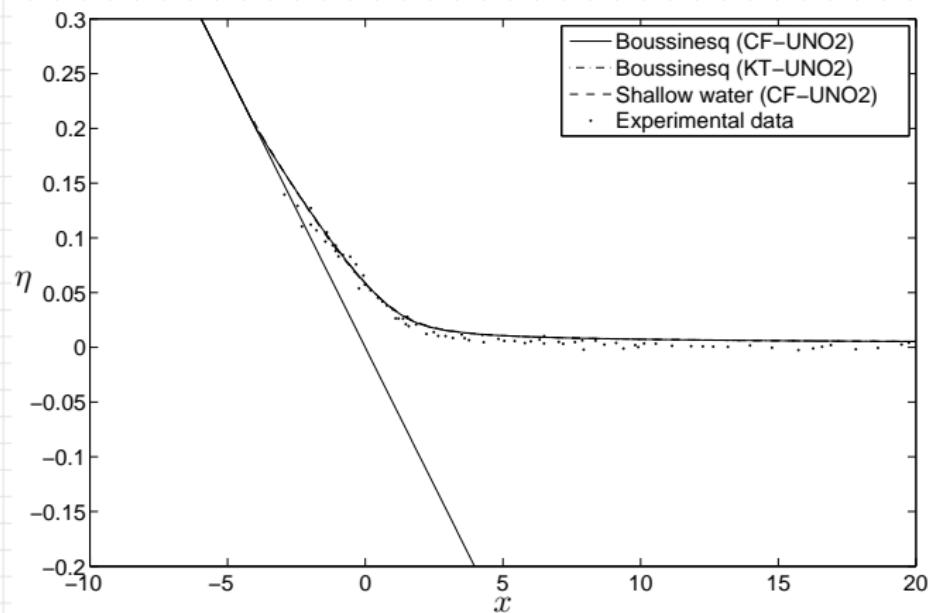


Figure: $t = 50$ s

Solitary wave runup: $A_s = 0.04$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

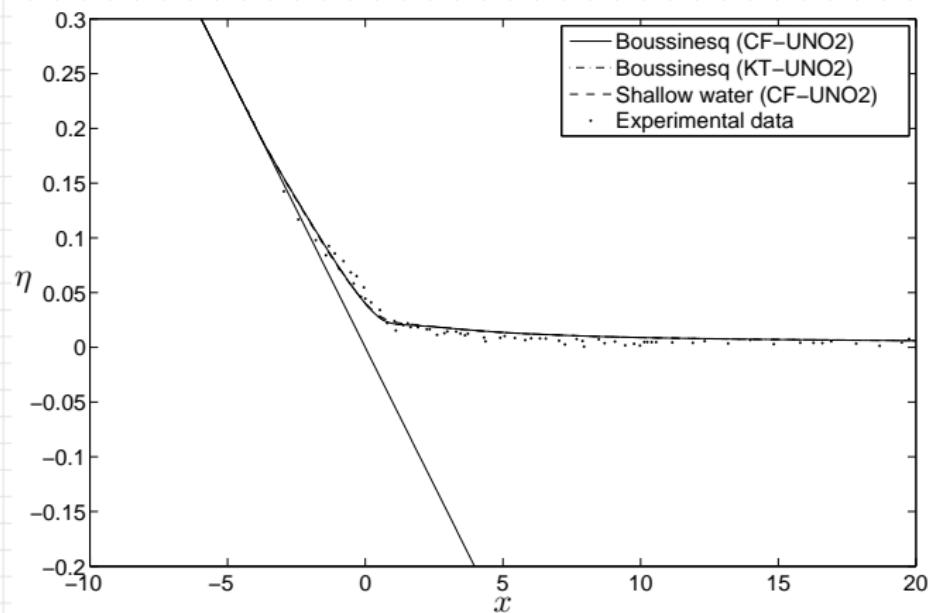


Figure: $t = 56$ s

Solitary wave runup: $A_s = 0.04$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

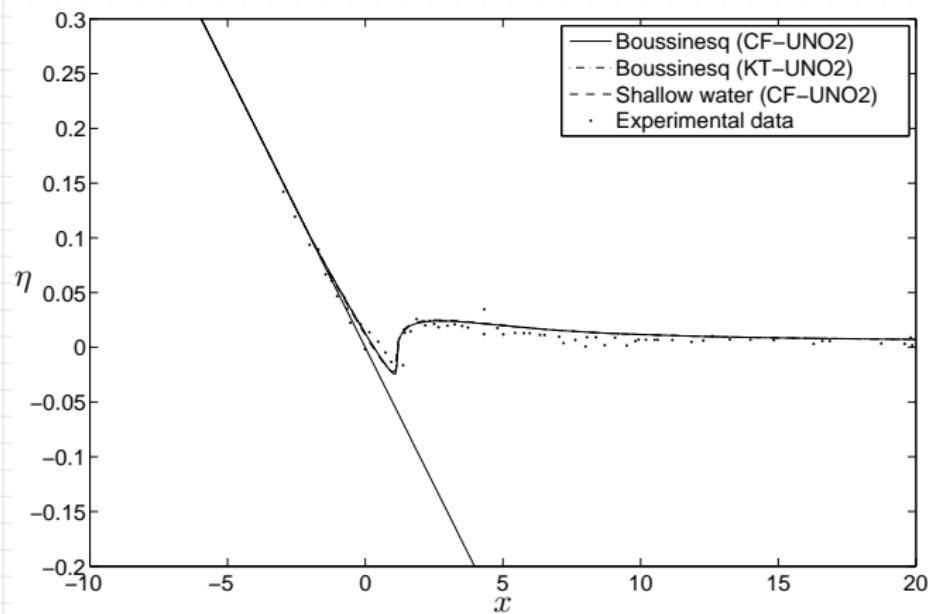


Figure: $t = 62$ s

Solitary wave runup: $A_s = 0.28$



Solitary wave runup: $A_s = 0.28$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

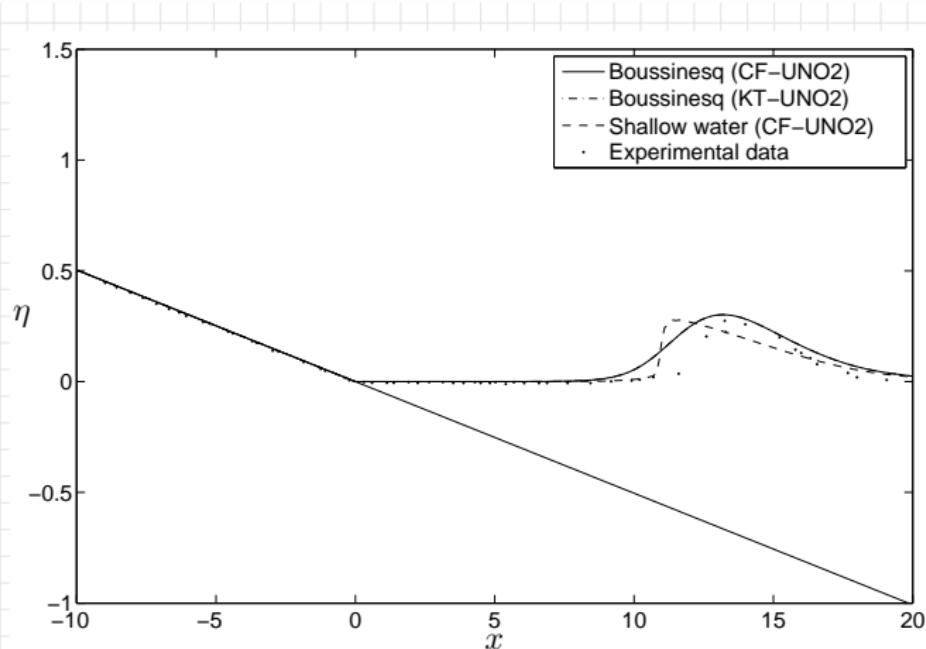


Figure: $t = 10$ s

Solitary wave runup: $A_s = 0.28$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

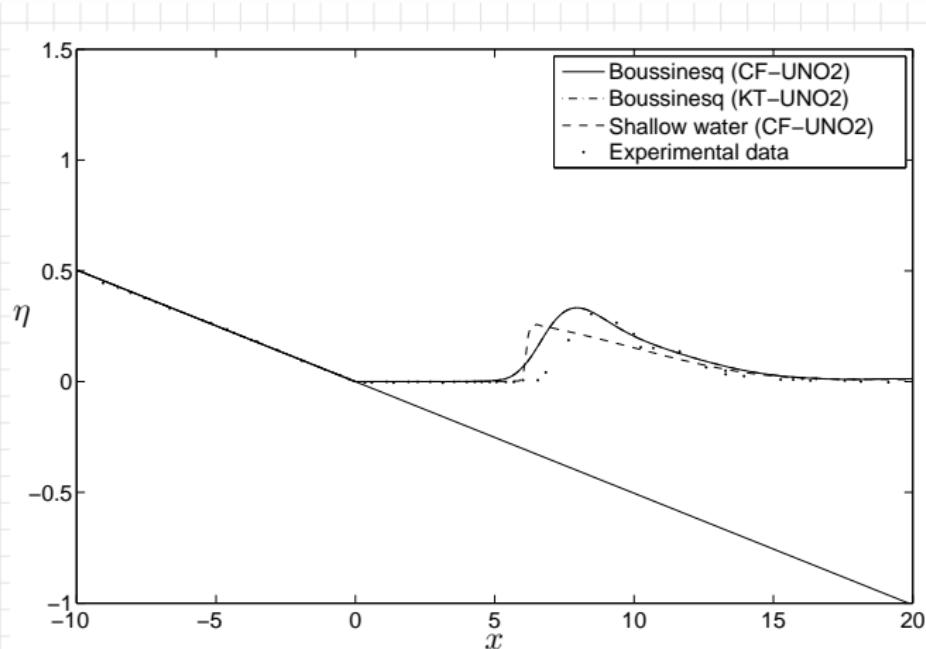


Figure: $t = 15$ s

Solitary wave runup: $A_s = 0.28$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

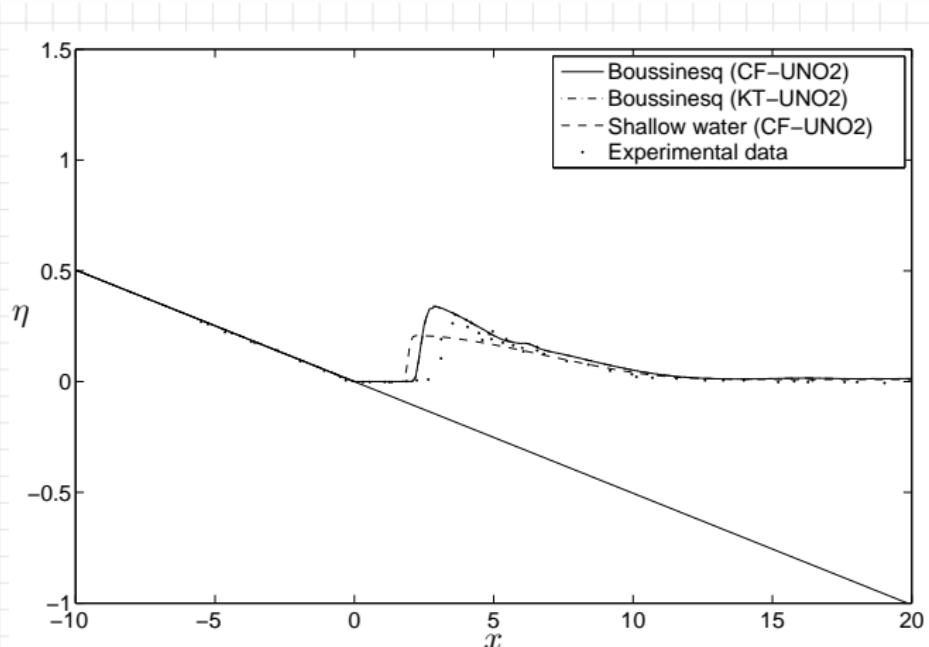


Figure: $t = 20$ s

Solitary wave runup: $A_s = 0.28$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

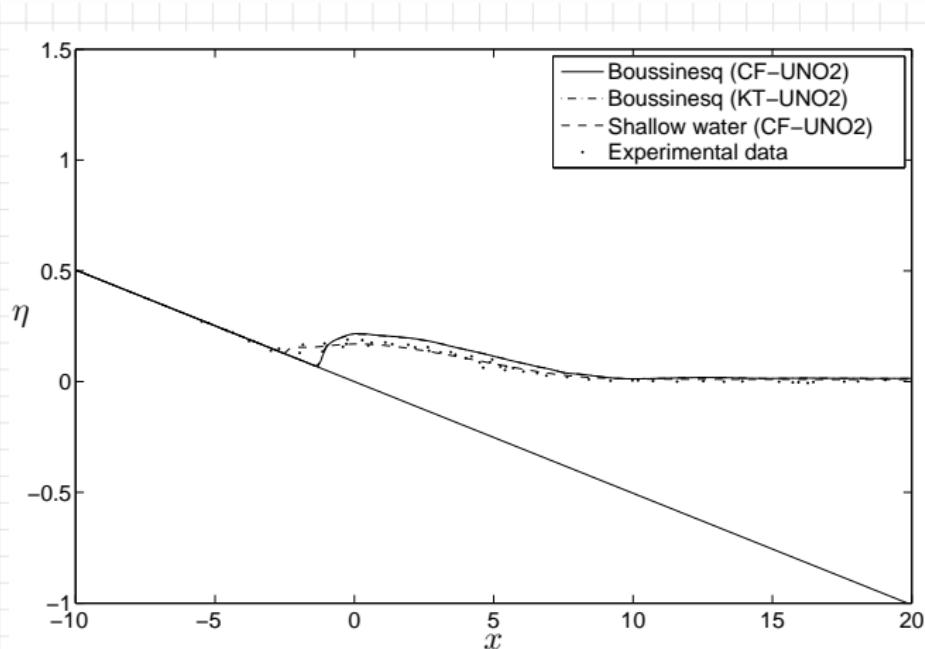


Figure: $t = 25$ s

Solitary wave runup: $A_s = 0.28$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

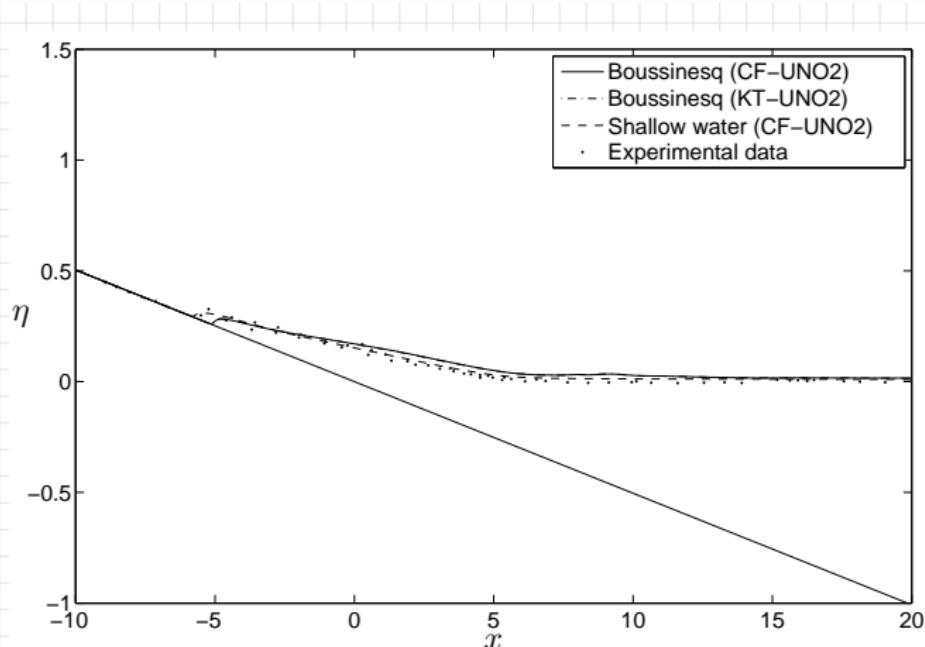


Figure: $t = 30$ s

Solitary wave runup: $A_s = 0.28$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

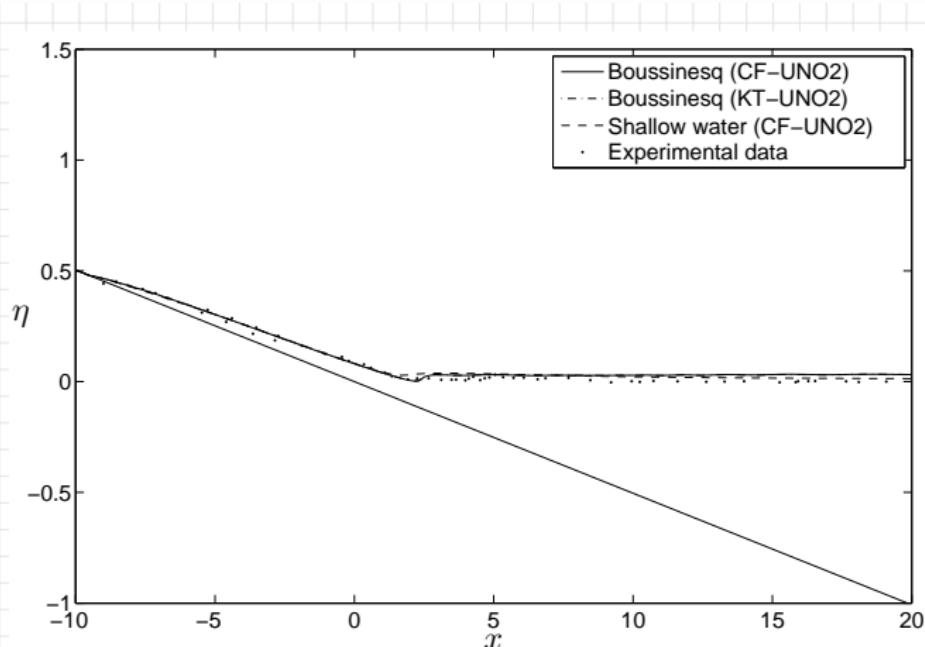


Figure: $t = 45$ s

Solitary wave runup: $A_s = 0.28$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

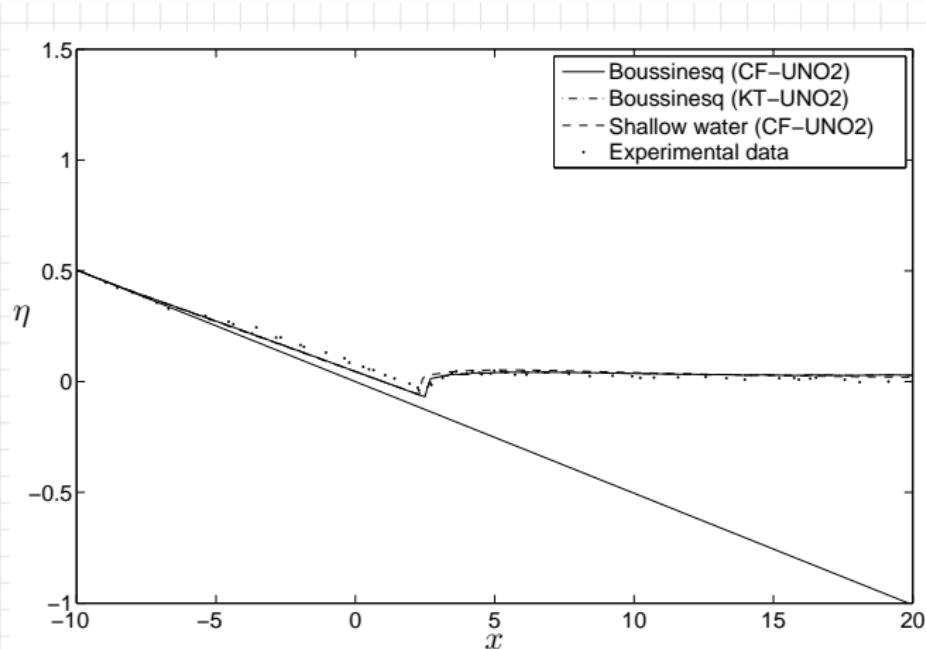


Figure: $t = 55$ s

Solitary wave runup: $A_s = 0.28$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

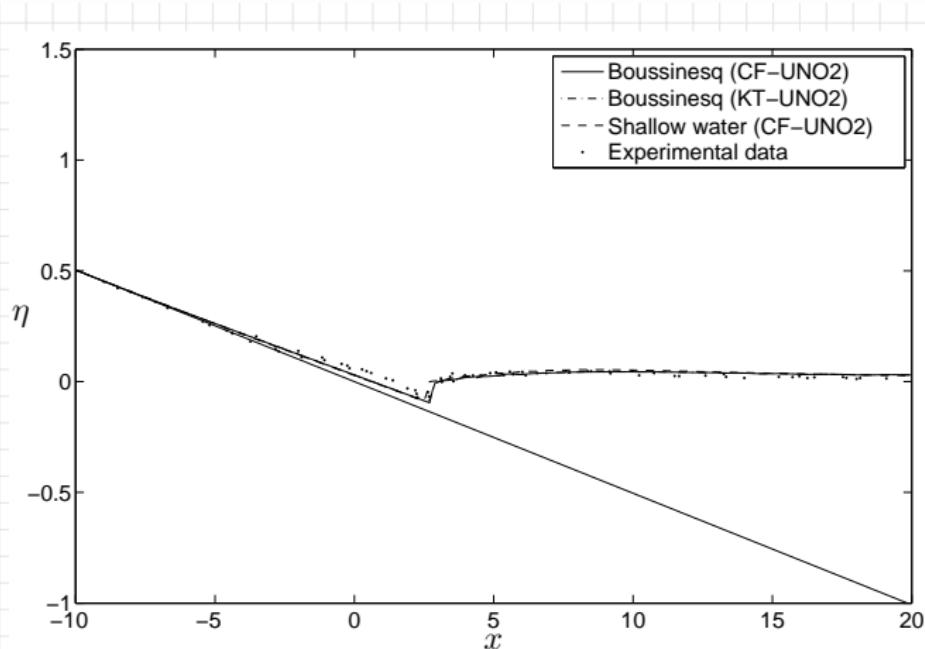


Figure: $t = 60$ s

Solitary wave runup: $A_s = 0.28$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

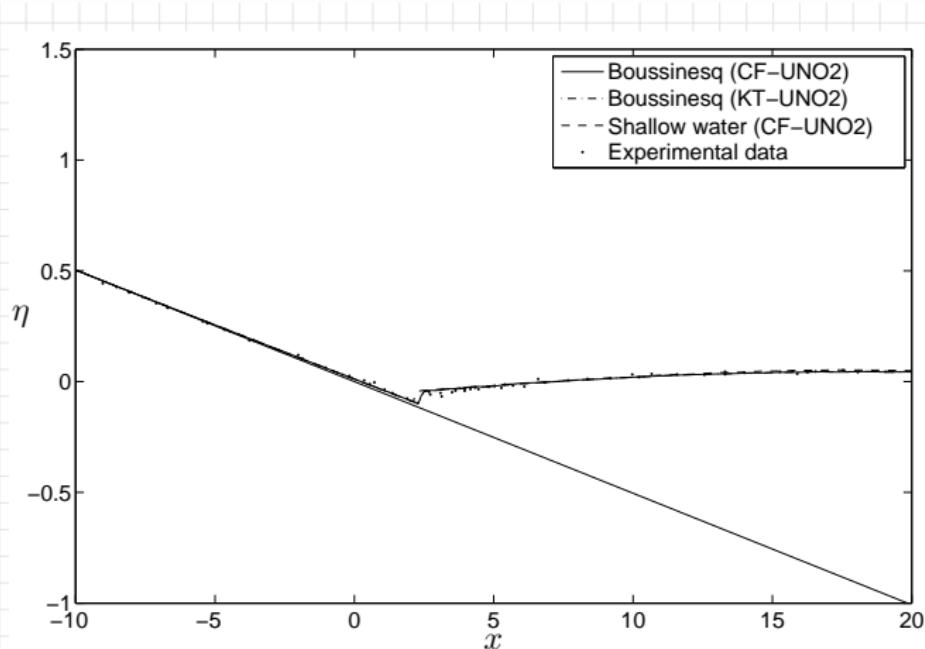


Figure: $t = 70$ s

Solitary wave runup: $A_s = 0.28$

Data from C. Synolakis (1987), $\beta = 2.88^\circ$

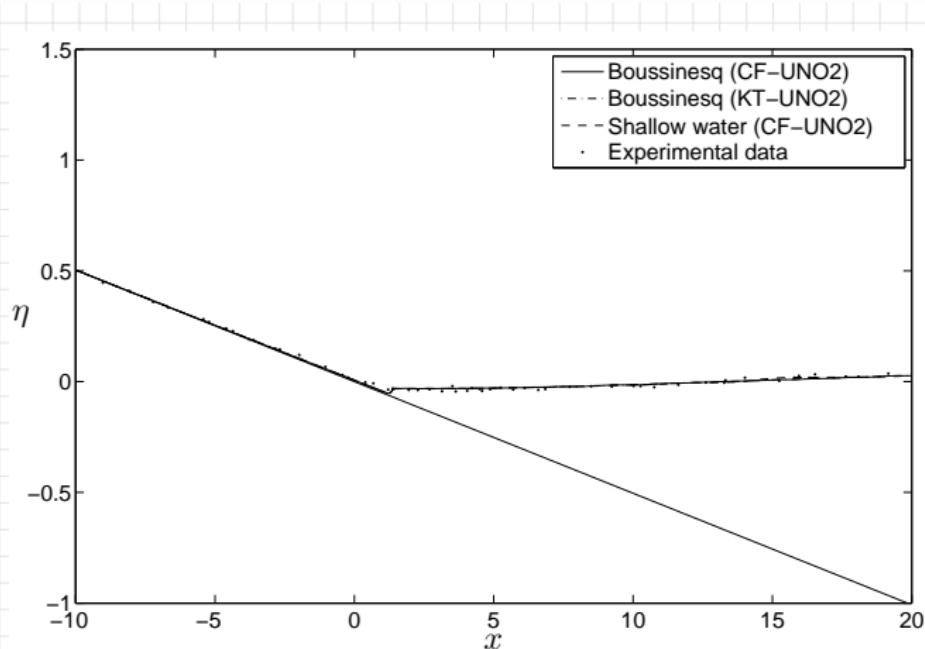


Figure: $t = 80$ s

Wave runup on random bottoms

The main motivation for this study



- Real beaches in nature are not C^∞ (smooth)
- The micro-structure is never known

Possible solutions:

- Addition of ad-hoc friction terms (Manning, Chézy, ...)
- Model the bottom roughness by a *random* perturbation

Rough sloping beach construction

Discrete formulation

Space discretization:

$$\mathcal{T} = \{x_i\}_{i \in \mathbb{Z}} \in \mathbb{R}, \quad \mathcal{C}_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], \quad \Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}.$$

Rough beach = sloping bottom + random noise

$$d_i = -d_0 + x_i \tan(\delta) + \xi_i, \quad \xi_i \sim \mathcal{N}(0, \sigma^2)$$

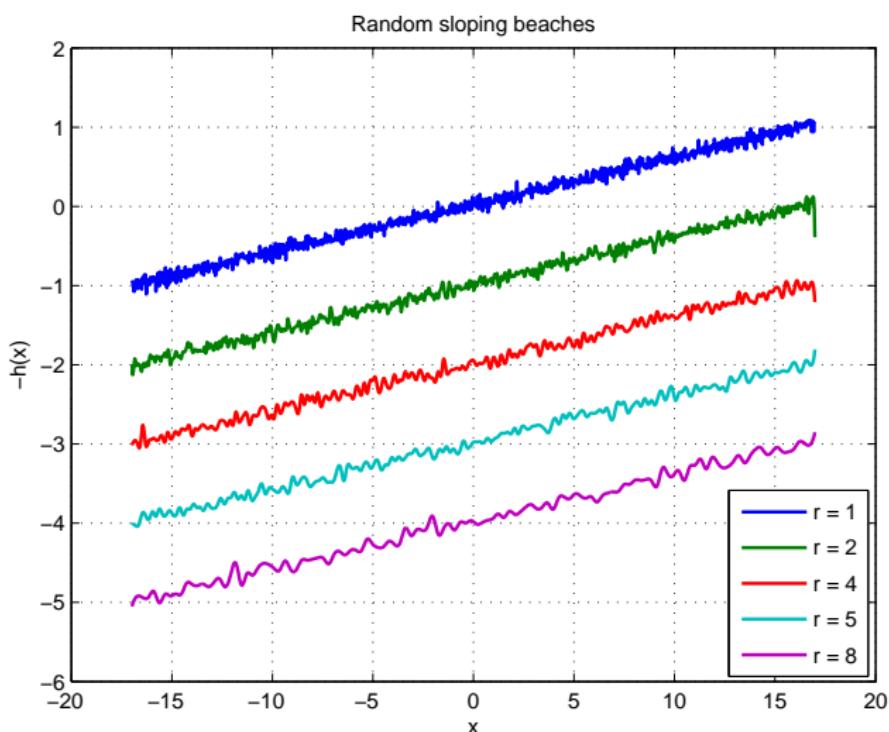
More regular noise:

- Regularity parameter $r \in \mathbb{Z}_+$

$$\mathcal{T}_r = \{x_{ri}\}_{i \in \mathbb{Z}} \subseteq \mathcal{T}, \quad \xi_i = \mathcal{P}(\xi_{ri})$$

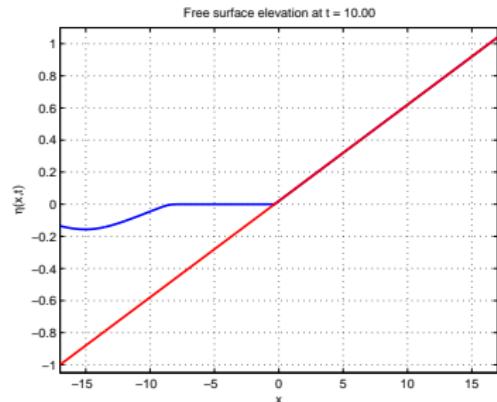
Rough sloping beach construction

Discrete formulation: $\sigma = 0.05$

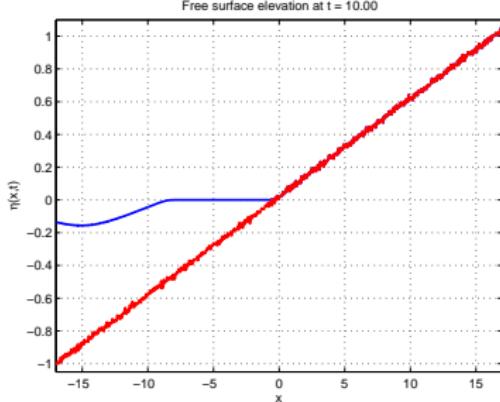


Smooth and rough cases

One numerical result with NSWE: $\sigma = 0.01$, $r = 1$



(a) Smooth bottom

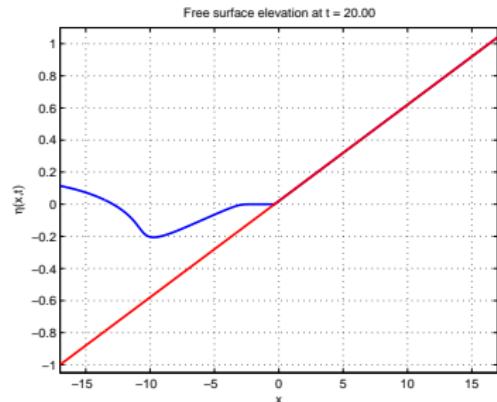


(b) Random bottom

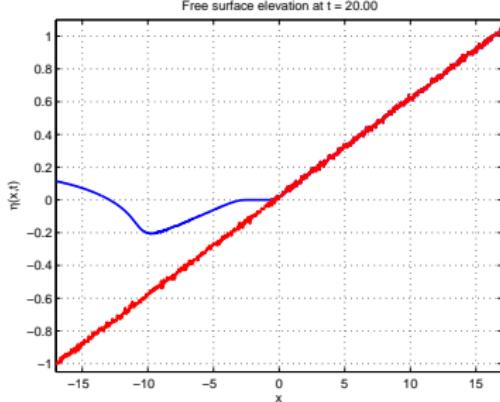
Figure: $T = 10$ s

Smooth and rough cases

One numerical result with NSWE: $\sigma = 0.01$, $r = 1$



(a) Smooth bottom

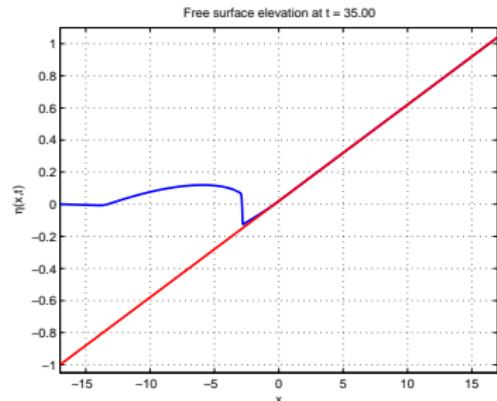


(b) Random bottom

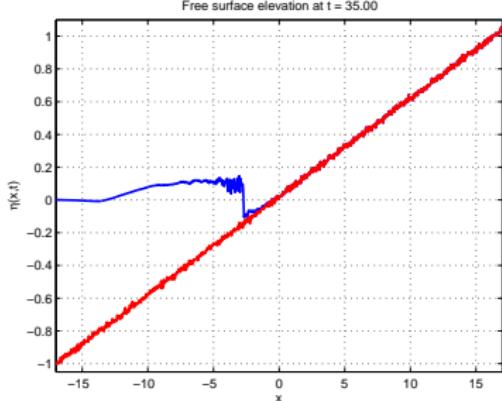
Figure: $T = 20$ s

Smooth and rough cases

One numerical result with NSWE: $\sigma = 0.01$, $r = 1$



(a) Smooth bottom

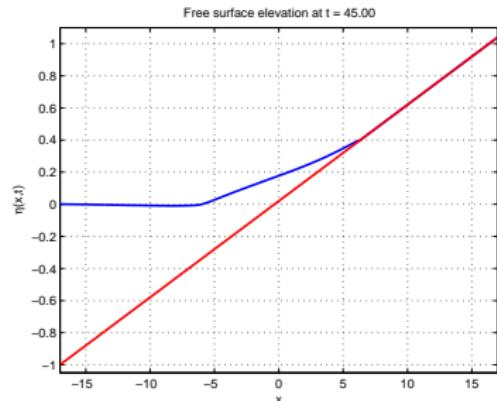


(b) Random bottom

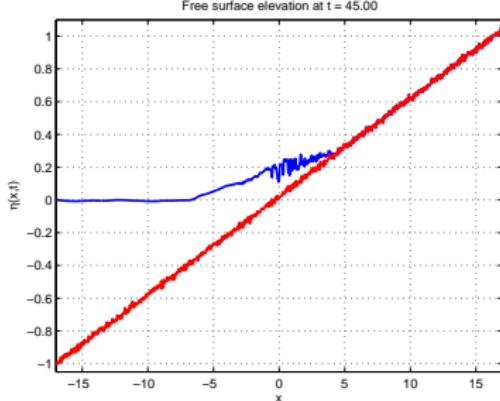
Figure: $T = 35$ s

Smooth and rough cases

One numerical result with NSWE: $\sigma = 0.01$, $r = 1$



(a) Smooth bottom

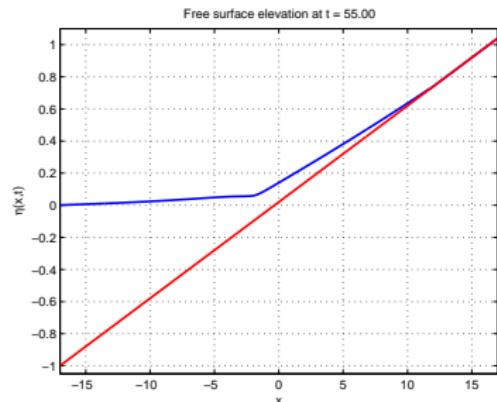


(b) Random bottom

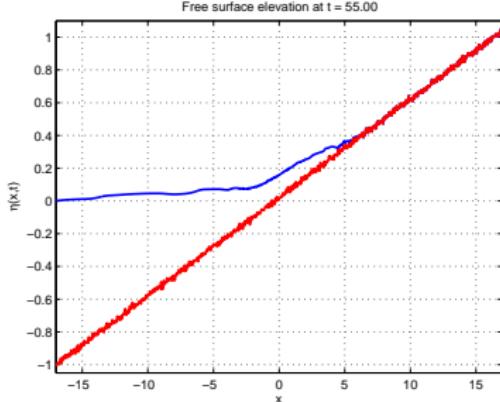
Figure: $T = 45$ s

Smooth and rough cases

One numerical result with NSWE: $\sigma = 0.01$, $r = 1$



(a) Smooth bottom

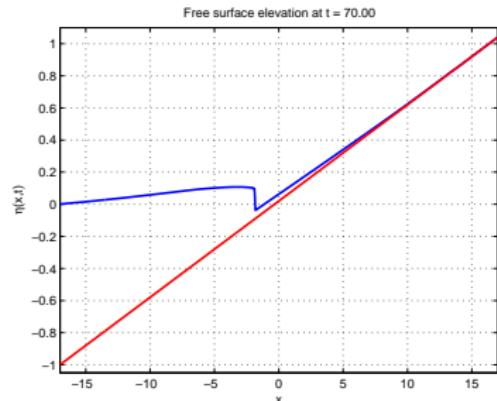


(b) Random bottom

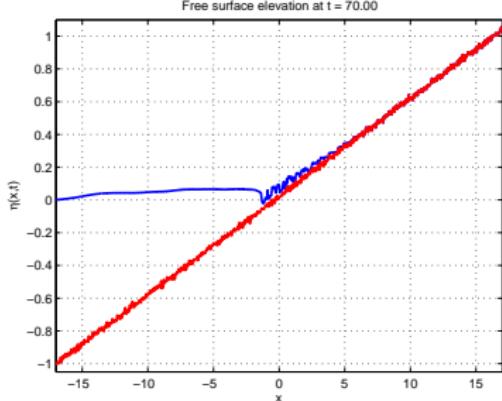
Figure: $T = 55$ s

Smooth and rough cases

One numerical result with NSWE: $\sigma = 0.01$, $r = 1$



(a) Smooth bottom

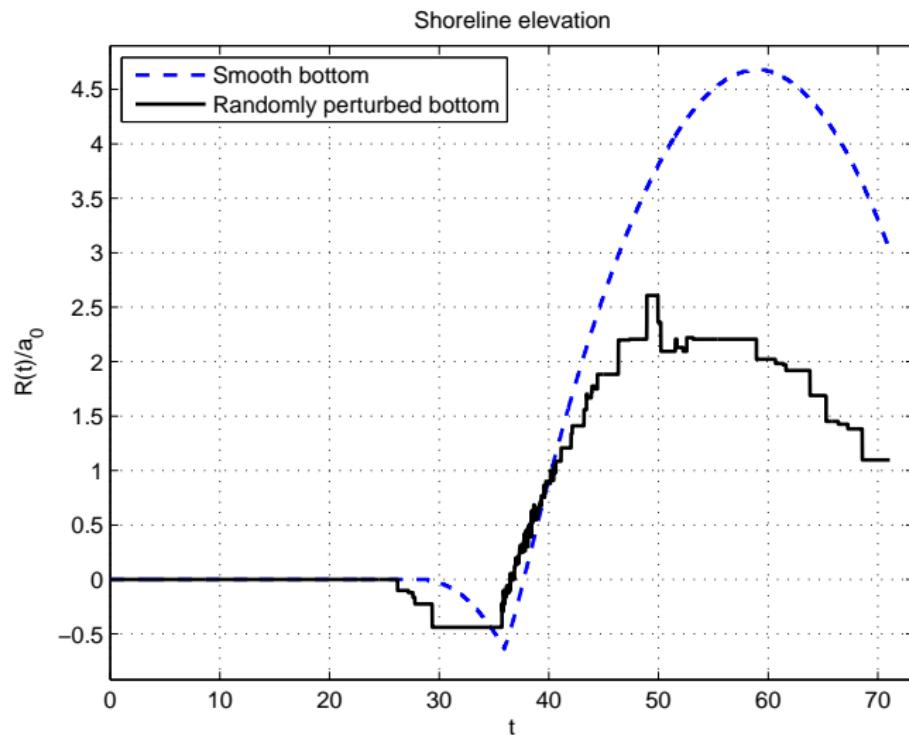


(b) Random bottom

Figure: $T = 70$ s

Smooth and rough cases

One numerical result with NSWE: $\sigma = 0.01, r = 1$



Wave runup on random beaches

Runup height reduction quantification

Question:

How to quantify the maximum runup value reduction by bottom irregularities?

Parameters:

- σ : measure of noise amplitude since $\mathbb{P}\{|\xi_i| < 1.96\sigma\} = 0.95$
- r : measure of noise regularity

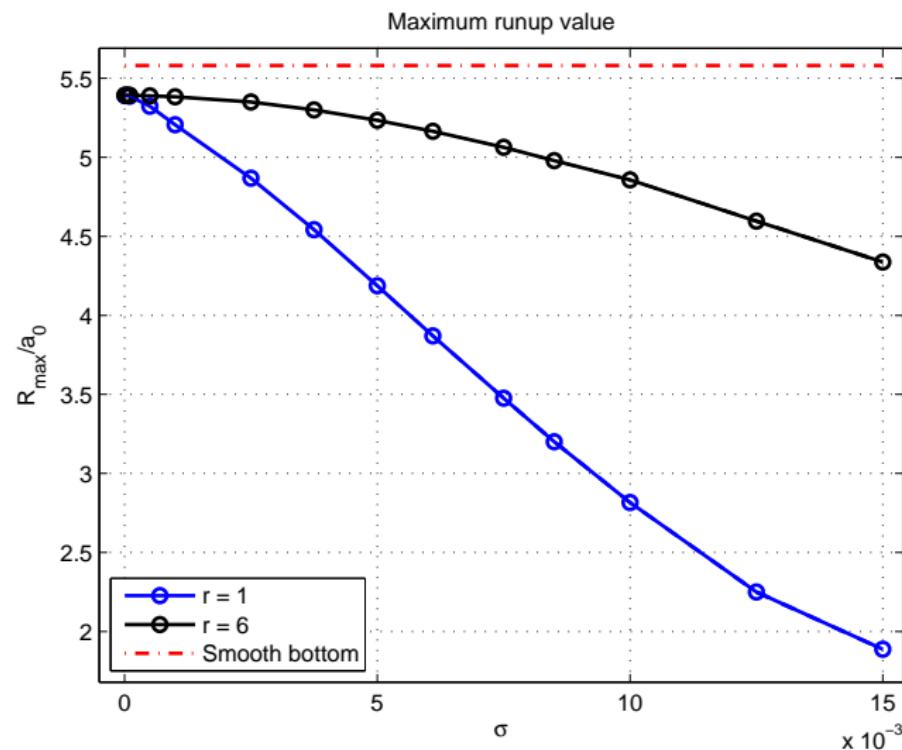
Dimension of the problem:

$$\{\xi_i\} \in \mathbb{R}^m, \quad m = \frac{N}{r}, \quad N = |\mathcal{T}| = |\{x_i\}|$$

- No polynomial chaos expansion if $m > 2$
- No Quasi-Monte-Carlo if $m > 200$
- Classical Monte-Carlo simulation

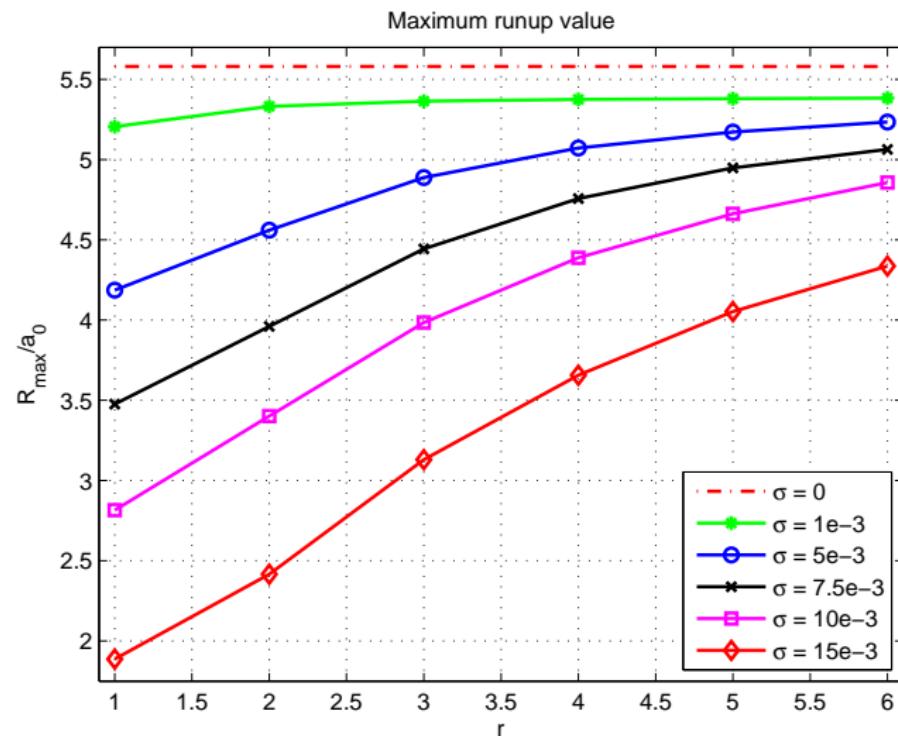
The effect of the noise magnitude

For the fixed regularity parameter



The effect of the noise regularity

For the fixed noise magnitude



Performance of the bottom friction

Manning friction law

- Monte-Carlo is expensive
- Everybody uses various ad-hoc *friction* terms

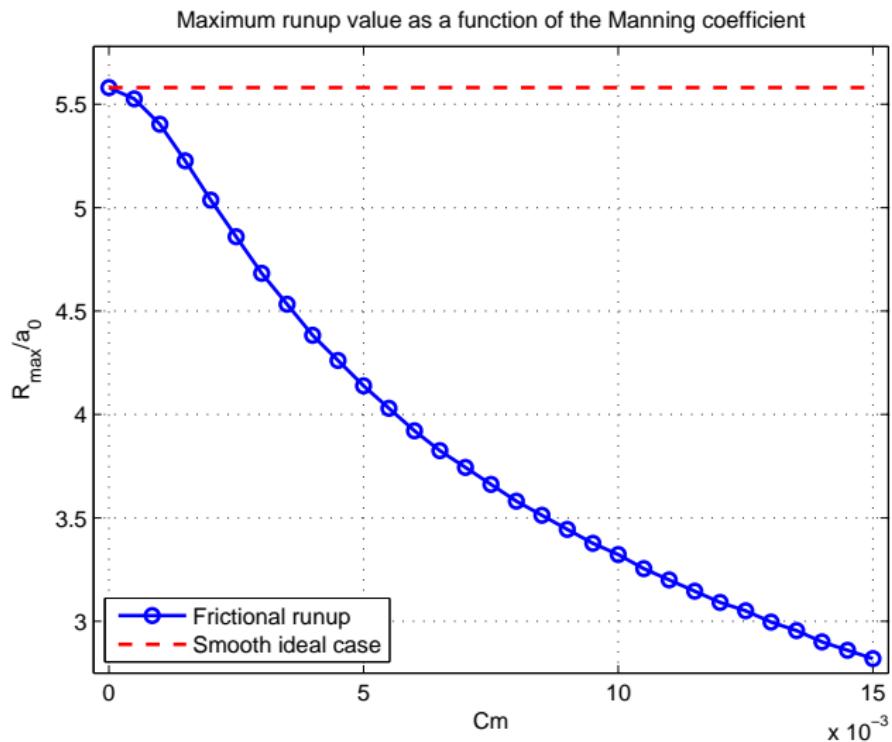
NSWE with the Manning term:

$$\begin{aligned} H_t + (Hu)_x &= 0, \\ (Hu)_t + \left(Hu^2 + \frac{g}{2}H^2 \right)_x &= gHd_x - gC_m^2 \frac{u|u|}{H^{\frac{1}{3}}} \end{aligned}$$

- Coefficient C_m is a measure of the bottom roughness
- How does R_{\max} depend on C_m ?

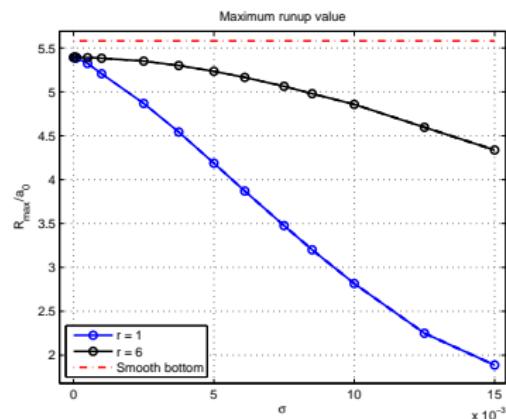
Performance of the bottom friction

Manning friction law

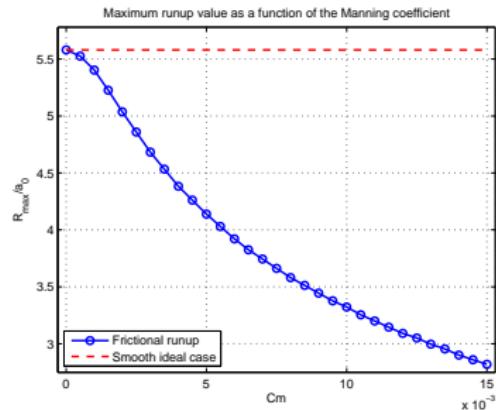


Performance of the bottom friction

Manning friction law



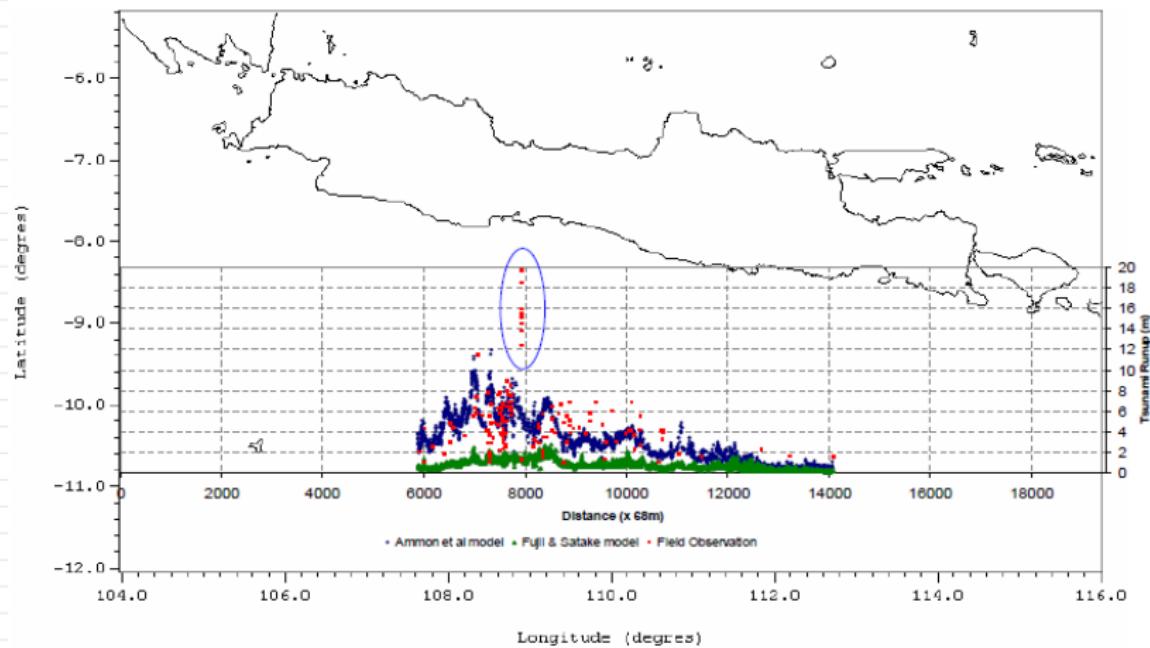
(a) MC simulation



(b) Manning friction

July 17, 2006 Java Tsunami

By courtesy of Widjo Kongko (FI-LUH, Hannover)

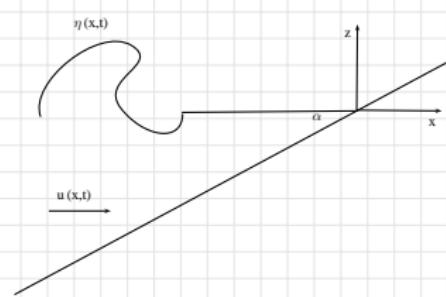


How to explain extreme runup values?

Simple academic test-case

Monochromatic wave runup

- Left boundary condition:
 $H_0(t) = d_0 + a_0 \sin(\omega t)$
- Incoming periodic monochromatic wave



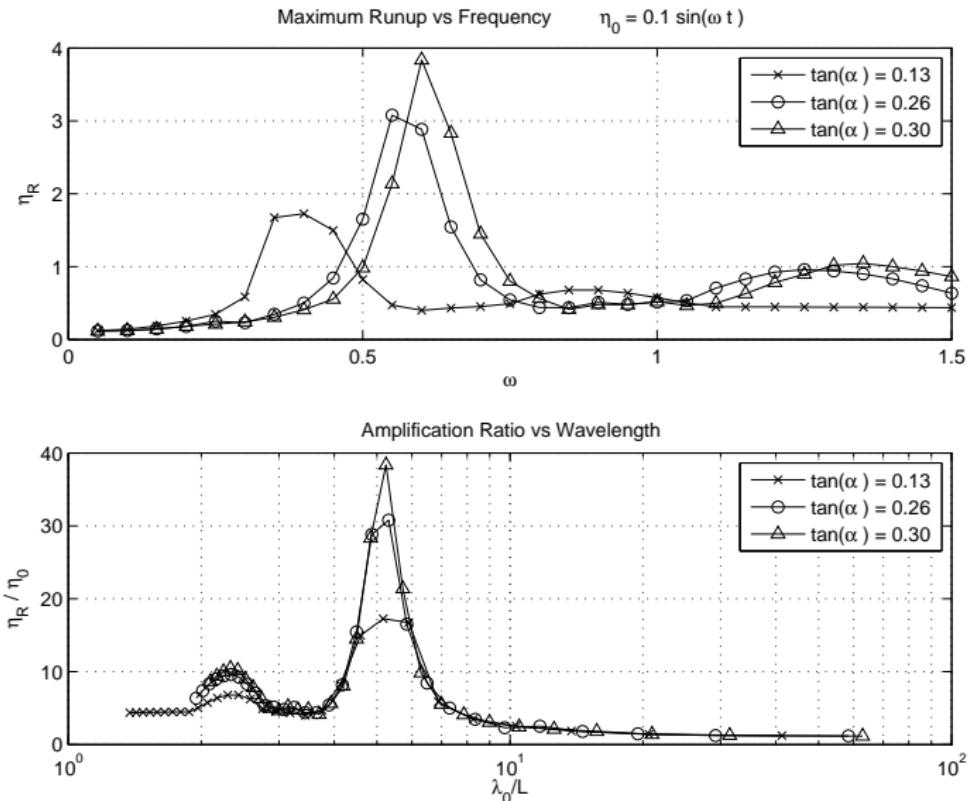
Reference:

I. Didenkulova, E. Pelinovsky. Oceanology, **48**, 2008

- Linear theory was shown to predict correctly at least the maximal runup
- $R_{\max} \sim \sqrt{\omega}$
- We compute numerically the R_{\max} for various values of ω

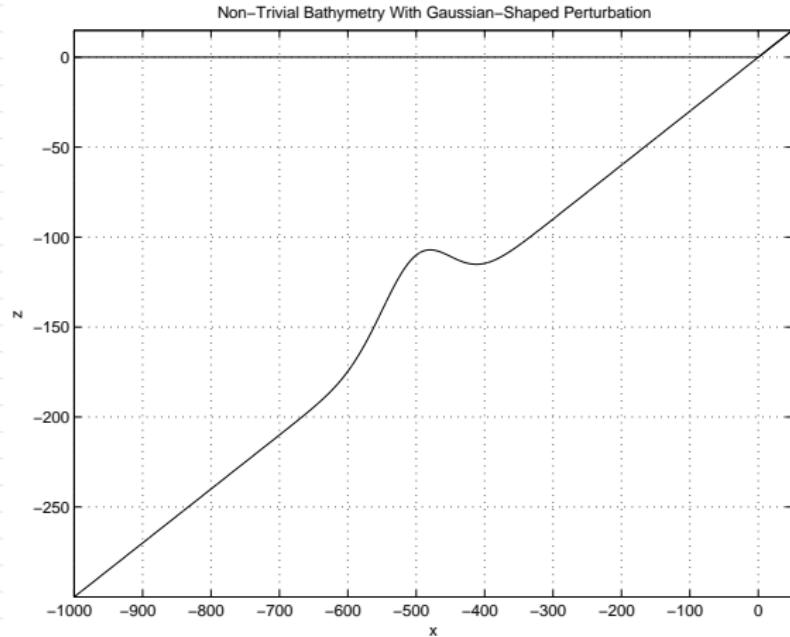
Runup amplification

Constant sloping beach: some analytical considerations by K. Kajiura (1976)



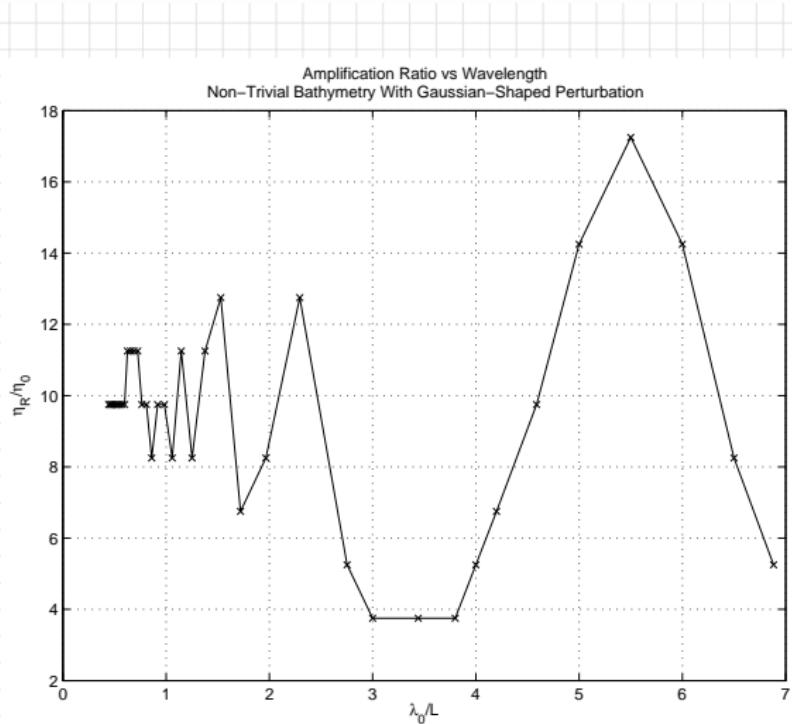
Non-trivial bathymetry - I

Gaussian bump on a sloping beach



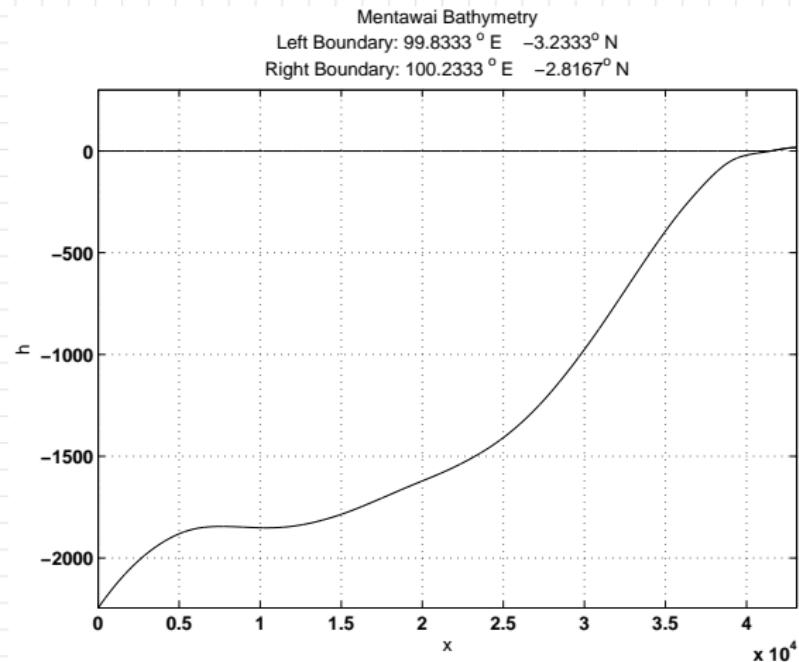
Non-trivial bathymetry - I

Gaussian bump on a sloping beach



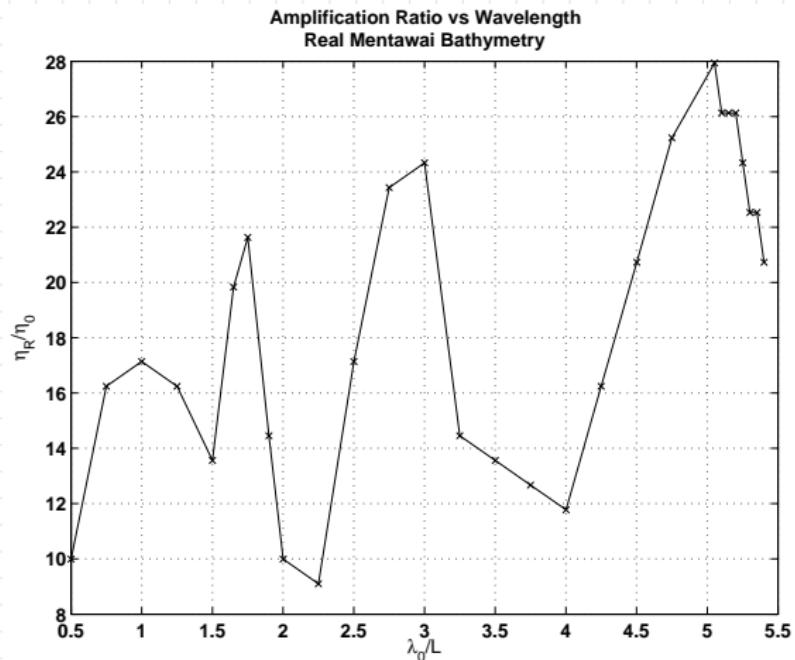
Non-trivial bathymetry - II

Mentawai Island bathymetry slice



Non-trivial bathymetry - II

Mentawai Island bathymetry slice



Conclusions

- Several schemes to discretize dispersive wave equations have been proposed and tested
- FVs are sufficiently accurate for solitons dynamics
- Dispersive effects are beneficial for the description of breaking waves

- Quantification of the wave runup on random beaches
- Resonant amplification of the maximum runup value on non-flat sloping bottoms



Thank you for your attention!

<http://www.lama.univ-savoie.fr/~dutykh/>



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