# Combinatorics of a Tropical Integrable Model

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Affine Schubert Calculus Workshop
Fields Institute, Toronto, July 2010

# CONTENTS

# 1. Introduction

Solitons, Tropical Limits

# What are Solitons?

- Very stable solitary waves.
- They make NON-linear scatterings.
- Nevertheless they keep their original shapes after the collision.
- Usually we require such property for scattering of more than 3 solitons.

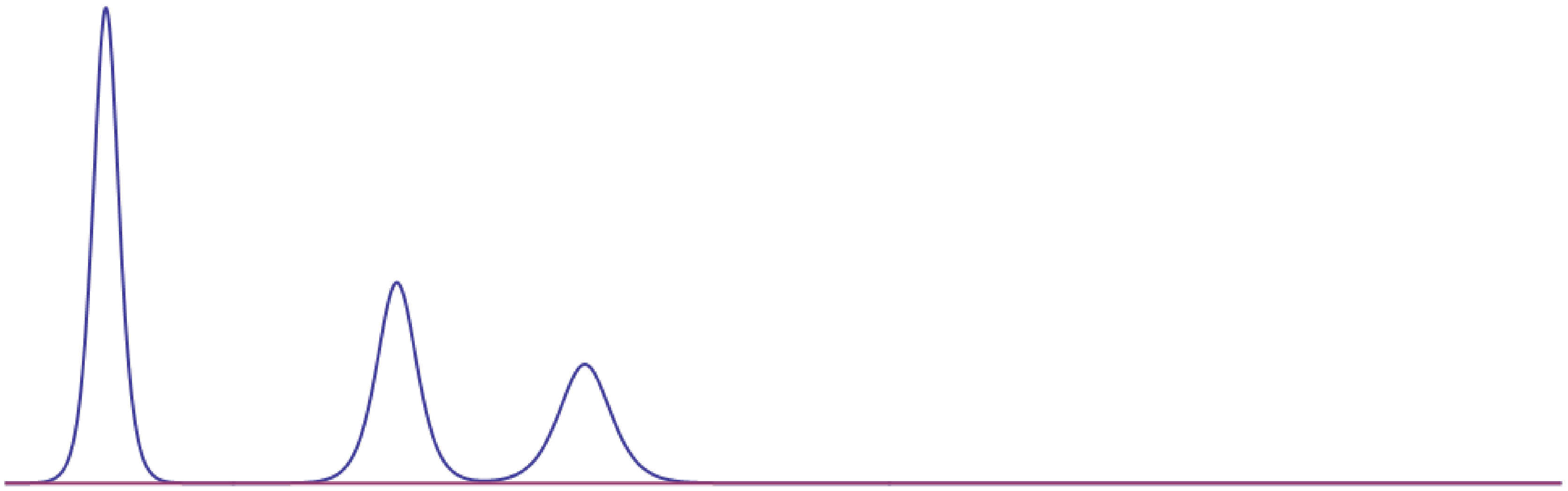
# Typical Example

- The KdV equation is a typical example of the soliton equation.

$$u_t - 6uu_x + u_{xxx} = 0$$

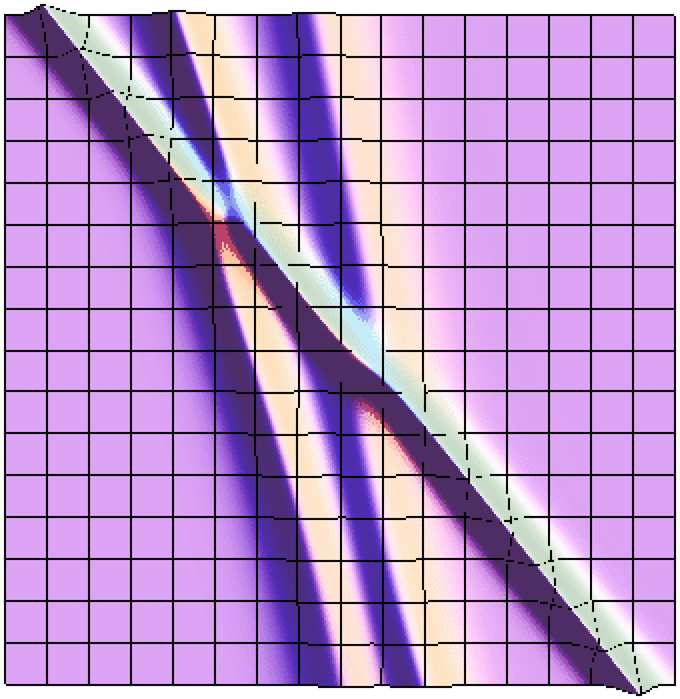
- It is a nonlinear PDE.
- N-soliton solutions obtained by Gardner-Greene-Kruskal-Miura (1967).

# Animation of the KdV Solitons



# Nonlinearity

- Nonlinearity appears in shift of phases.
- phase ≈ position of a soliton.
- Look at the previous solution from above:
- Convention: Time proceeds downwards.



# Tropical Limits

- Let us recall the tropical limits.
- For quantities

$$a = e^{A/\epsilon},\, b = e^{B/\epsilon}$$

consider the limits

$$\lim_{\epsilon \to +0} \epsilon \log(a + b) = \max(A, B)$$

$$\lim_{\epsilon \to +0} \epsilon \log(a \times b) = A + B$$

# Tropical Soliton Systems

- Tropical limit of a soliton system: Tokihiro-Takahashi-Matsukidaira-Satsuma (1996).
- According to their paper, tropical limit is called ultradiscrete limit in this context.
- Their tropical soliton system coincides with the Box-Ball System of Takahashi-Satsuma (1990).

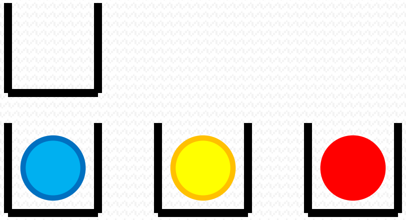
# Box-Ball System (BBS)

- During 1980's there were many attempts to find a soliton like cellular automaton.
  e.g. filter automaton of J.K. Park, K. Steiglitz & W.P. Thurston (1986)
- Takahashi-Satsuma added integrability.
- BBS is defined by a simple algorithm using balls moving on an array of boxes.
- BBS looks very simple, nevertheless there are deep mathematical structures.

# Example of the BBS

2222111133214311111111111111111
1111222211133243111111111111111
1111111122221132433111111111111
1111111111112221322433111111111
1111111111111112211322433211111
1111111111111111122111322143321

- 1　　→ empty box
- 2,3,4→ species of balls



# Mathematics Around BBS

- BBS is classical integrable system (soliton).
- BBS is also a quantum integrable system
  → Crystal bases of quantum affine algebras
  (Many authors, 1999).
- Bethe ansatz, Kostka-Foulkas polynomials,
  rigged configurations (to be explained).
- Tropical theta functions (Kuniba-S, 2006),
  Tropical geometry (Inoue-Takenawa, 2007)

# 2. Box-Ball System

Kirillov-Reshetikhin (KR) Crystals,
Combinatorial R-matrices,
Definition of the BBS

# Kirillov-Reshetikhin(KR) Crystals

- Consider only type $A_n^{(1)}$.
- Denote by $B^{r,s}$ the set of KR crystals corresponding to KR module (wt. $s\Lambda_r$).
- As the set,

  $B^{r,s}$ = {Semistandard Tableaux,
  height r, width s,
  letters from 1 to n+1}.

# KR Crystals (2)

- On the set $B^{r,s}$, define algebraic structures:
  $e_i$, $f_i$ Kashiwara operators
- Explicit algorithms for the actions of the Kashiwara operators are known.
  $i \neq 0$ signature rule
  $i = 0$ promotion op. by Shimozono (1998)
- We will not use explicit actions in this talk.

# Combinatorial R-matrices

- There is a simple algorithm for the action of Kashiwara operators on the tensor products.
- Combinatorial R-matrices reverses tensor product;
  R: $B^{r,s} \otimes B^{r',s'} \simeq B^{r',s'} \otimes B^{r,s}$
- $\simeq$ : R is a crystal isomorphism
  i.e., R commutes with all $e_i$ and $f_i$.

# Affinization

- For crystal B, define its affinization. As the set,
  $$\mathrm{Aff}(B) := \{b[d] \mid b \in B,\ d \in \mathbb{Z}\}$$
- There are actions of the Kashiwara op.
- Combinatorial R-matrices for affine crystals:
  $$R : b[d] \otimes b'[d'] \simeq \tilde{b}'[d' - H] \otimes \tilde{b}[d + H]$$
  where $H = H(b \otimes b')$ is the energy function.
- There is a recursive defining axiom for the energy functions.
- The Yang-Baxter relation holds for affine crystals.

# Algorithm for R-matrices (1)

- Row word of a tableau t: row(t)

  Example:

  t =

| | | |
|---|---|---|
| a | b | c |
| d | e | f |
| h | i | j |

  row(t) = hij def abc

- Row insertion:

  $$t \longleftarrow ab = ((t \longleftarrow a) \longleftarrow b)$$

# Algorithm for R-matrices (2)

- Theorem [Shimozono 1998]

  (c.f. [Schilling-Warnaar 1998] )

  $$R : b \otimes b' \simeq \tilde{b}' \otimes \tilde{b}$$

  if and only if

  $$(b' \leftarrow row(b)) = (\tilde{b} \leftarrow row(\tilde{b}')).$$

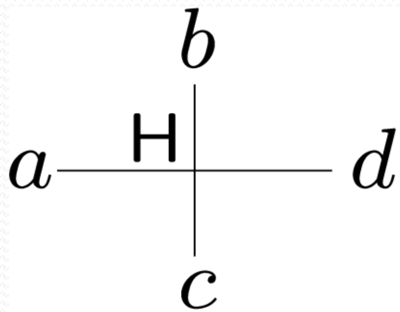
  $H = H(b \otimes b')$ is the number of boxes of $(b' \leftarrow row(b))$ outside of the concatenation of the shapes of b and b'.

# Example

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 3 & 6 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & 4 \\ \hline 4 & 5 \\ \hline \end{array} \simeq \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 3 & 3 & 4 \\ \hline 4 & 5 & 6 \\ \hline \end{array},$$

$$H\left(\begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 3 & 6 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & 4 \\ \hline 4 & 5 \\ \hline \end{array}\right) = 3.$$

- Vertex diagram.

  For $a \otimes b \simeq c \otimes d$, write 

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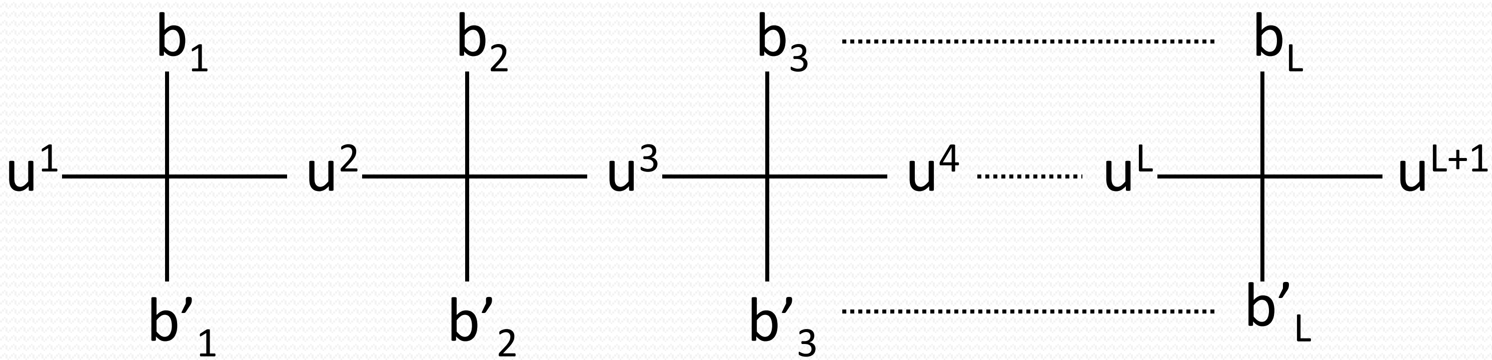
# Definition of the BBS

- Highest weight element $u^1 \in B^{r,s}$
- Initial state: $b = b_1 \otimes \cdots \otimes b_L \in \bigotimes B^{r_i,s_i}$.
- Definition of the BBS time evolution op. $T^{r,s}$;




$$T^{r,s}(b) := b'_1 \otimes \cdots \otimes b'_L$$

# 3. Rigged Configurations

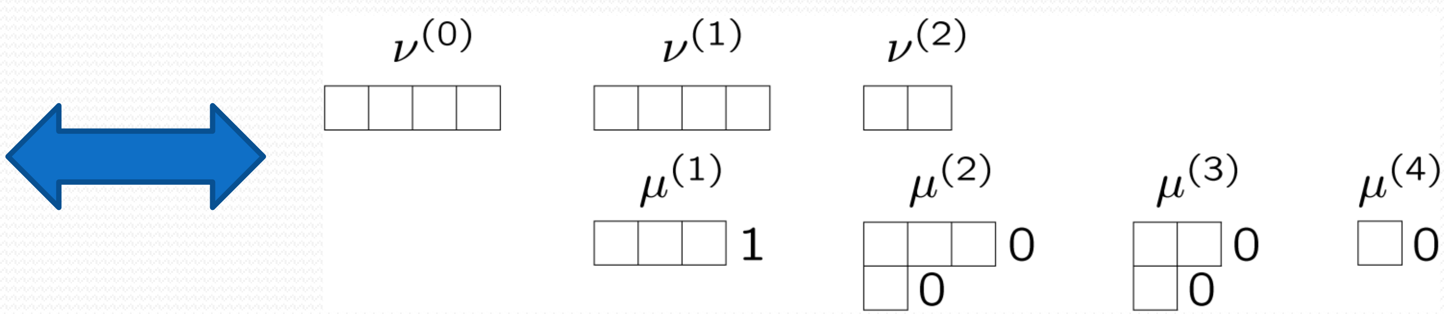
Rigged Configurations

Inverse Scattering Formalism of the BBS

# Rigged Configuration Bijection

- is a bijection between two sets:
  - Highest weight elements of tensor products of KR crystals (paths)
  - Rigged Configurations (RC)

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline 3&4\\\hline\end{array} \otimes \begin{array}{|c|c|c|c|}\hline 1&1&2&4\\\hline 2&2&3&5\\\hline\end{array}$$



# RC Bijection (2)

- Originally, RC bijection arises through a combinatorial analysis of the Bethe ansatz.
- Give proofs of the “Fermionic formula” of the Kostka-Foulkas polynomials.
- Type A
  - Kerov-Kirillov-Reshetikhin 1986, …………,
  - Kirillov-Schilling-Shimozono 1999
- Types D, etc.
  - Okado-Schilling-Shimozono 2002, …………

# RC and BBS are related.

- A Basic Observation by Kuniba-Okado-S-Takagi-Yamada (2006)
- RC bijection
  = inverse scattering formalism of the BBS

  i.e., it gives a complete set of
  - Conserved quantities &
  - Linearization parameters

  for nonlinear dynamics of the BBS.

# Statement

- Theorem [KOSTY 2006]

  Let the RC corresponding to p be

  $$(\mu, r) = \{(\mu^{(1)}, r^{(1)}), \cdots, (\mu^{(n)}, r^{(n)})\}.$$

  Then the RC corresponding to $T^{a,l}(p)$ be

  $$\{(\mu^{(1)}, r^{(1)}), \cdots, (\mu^{(a)}, r'^{(a)}) \cdots, (\mu^{(n)}, r^{(n)})\}$$

  where the rigging $r'^{(a)}$ shows linear evolution

  $$r_i'^{(a)} = r_i^{(a)} + \min(l, \mu_i^{(a)}).$$

# 4. Tropical Tau Functions

Energy Statistics,

Tropical Tau Functions

# Energy Statistics (1)

- Definition.

  Crystal formulation by Nakayashiki-Yamada.

  C.f. Lascoux-Schützenberger's charge.

  For (not necessarily highst weight elements)

  $$p = b_1 \otimes \cdots \otimes b_L \in \bigotimes_i B^{r_i, s_i},$$

  define its energy statistic E(p) by

  $$E(p) = \sum_{i<j} H(b_i \otimes b_j^{(i+1)}).$$

# Energy Statistics (2)

- Meaning of $b_j^{(i)}$

$$\cdots \otimes b_i \otimes b_{i+1} \otimes \cdots \otimes b_{j-1} \otimes b_j \otimes \cdots \simeq$$

$$\cdots \otimes b_i \otimes b_{i+1} \otimes \cdots \otimes b_j^{(j-1)} \otimes b'_{j-1} \otimes \cdots \simeq$$

$$\cdots\cdots\cdots \simeq$$

$$\cdots \otimes b_i \otimes b_j^{(i+1)} \otimes \cdots \otimes b'_{j-2} \otimes b'_{j-1} \otimes \cdots$$

- Recall $E(p) = \sum_{i<j} H(b_i \otimes b_j^{(i+1)})$

# Tau Functions

- Generalization of the energy statistics.
- For the highest element $u^{r,s} \in B^{r,s}$ define

$$\tau^{r,s}(p) := E(u^{r,s} \otimes p).$$

- For the highest elements $p \in \bigotimes_i B^{r_i,s_i}$ we have

$$\tau^{r,s}(p) = E(p).$$

# Tropical Tau Functions (1)

- Tropical tau functions are defined for the case of crystals for symmetric tensor reps:

$$p \in \bigotimes_i B^{1,\lambda_i}.$$

- Notation: rigged configuration corresponding to p:

$$(\mu, r) = \{(\mu^{(1)}, r^{(1)}), \cdots, (\mu^{(n)}, r^{(n)})\}$$

$\mu^{(i)} = (\mu_j^{(i)})$ : i-th Young diagram (from left)

$r_j^{(i)}$ : the rigging corresp. to j-th row

# Tropical Tau Functions (2)

- Charge Function (of the Fermionic Formula)

$$c(\lambda, \mu, r) = \frac{1}{2} \sum_{a,b=1}^{n} C_{ab} \min(\mu^{(a)}, \mu^{(b)}) - \min(\lambda, \mu^{(1)}) + \sum_{a=1}^{n} |r^{(a)}|$$

where $\min(\lambda, \mu) = \sum_{i,j} \min(\lambda_i, \mu_j)$, $|r| = \sum_i r_i$ and $C_{a,b}$ is the type $A_n$ (finite) Cartan matrix.

# Tropical Tau Functions (3)

- Definition.
  For $1 \leq i \leq n$,
  $$\tau_i(\lambda) = -\min_{(\nu,s)\subset(\mu,r)} \{c(\lambda,\nu,s) + |\nu^{(i)}|\}.$$
  Here, we consider (μ,r) as the set whose elements are the pairs $(\mu_i^{(a)}, r_i^{(a)})$.
  We take minimum over all possible subsets of (μ,r) in this sense.

# Tropical Plücker Relation

- Notations: (1) $\tau_{k,i} := \tau_i(\lambda_1, \ldots, \lambda_k)$.
  (2) Use $\bar{\tau}$ for tau functions for $T^{1,\infty}$(RC).
- Theorem [Kuniba-S-Yamada 2006]
  Tropical Plücker (or UD Hirota bilinear) relation
  $$\bar{\tau}_{k,i-1} + \tau_{k-1,i}$$
  $$= \max(\bar{\tau}_{k,i} + \tau_{k-1.i-1}, \bar{\tau}_{k-1,i-1} + \tau_{k,i} - \lambda_k).$$
- Related with the Weyl group description of the time evolution operators $T^{1,\infty}$.

# Piecewise Linear Formula

- Tropical tau functions give a piecewise linear (PL) formula for energy statistics and their generalizations as follows:
- Theorem [Kuniba-S-Yamada 2006 & S 2006] Suppose that p corresponds to RC. Then

$$\tau^{1,\infty}(p) = \tau_{n+1}(\mathrm{RC}).$$

# Tropical RC Bijection

- Tropical tau functions give an explicit piecewise linear formula for the RC bijection.
- For the path $p = p_1 \otimes \cdots \otimes p_L \in \bigotimes_i B^{r_i,s_i}$ denote the number of i in the tableaux representation of $p_k$ by $x_{k,i}$.
- Theorem [Kuniba-S-Yamada 2006 & S 2006] For the image of the map RC→p, we have
$$x_{k,i} = \tau_{k,i} - \tau_{k-1,i} - \tau_{k,i-1} + \tau_{k-1,i-1}.$$

# Initial value problem for BBS

- Combining
  - Inverse scattering formalism for the BBS
  - PL formula for the RC→path map

  we obtain a solution for the initial value problem for the BBS.

# 5. Macdonald Polynomials

Macdonald polynomials & tau functions

# Macdonald Polynomials

- We consider the following version of the Macdonald polynomials: $\tilde{H}_\mu(x;q,t) = \sum_\lambda \tilde{K}_{\lambda,\mu}(q,t)s_\lambda(x)$, where $\tilde{K}_{\lambda,\mu}(q,t)$ is the Kostka-Macdonald polynomials and $s_\lambda(x)$ is the Schur Polynomial.
- We consider the Haglund-Haiman-Loehr (2004) formula for the Macdonald polynomials.

# Tau functions with partition

- Consider $(B^{1,1})^{\otimes L}$ type path

$$p = p_1 p_2 \cdots p_L.$$

- Let $\mu = (\mu_1 \cdots \mu_m)$ be a composition. Denote by $p_{[1]}$ the first $\mu_1$ letters of p, by $p_{[2]}$ the next $\mu_2$ letters of p, and so on.
- Definition

$$\tau_\mu^{r,s}(p) = \sum_i \tau^{r,s}(p_{[i]}).$$

# Conjecture

- Conjecture [Kirillov-S 2009]

$$q^{-\sum_{i>r}\alpha_i} \sum_{p\in\mathcal{P}(\alpha)} q^{\tau_\mu^{r,1}(p)} = \sum_{\eta\vdash|\mu|} K_{\eta,\alpha}\tilde{K}_{\eta,\mu}(q,1).$$

  $\mathcal{P}(\alpha)$ : set of all paths of weight α.

- $r = \infty$; t=1 specialization of a simplified version of the Haglund-Haiman-Loehr formula. $\tau^{\infty,1}$ coincides with Haglund's maj statistics.
  $r = 1$ ; tau functions for the BBS.

# Example

Table of all paths of $L = 6$, wt$= (4, 1, 1)$

| | | | | | |
|---|---|---|---|---|---|
| **111123** | **111132** | **111213** | **111231** | **111312** | **111321** |
| **112113** | **112131** | **112311** | **113112** | **113121** | **113211** |
| **121113** | **121131** | **121311** | **123111** | **131112** | **131121** |
| **131211** | **132111** | **211113** | **211131** | **211311** | **213111** |
| **231111** | **311112** | **311121** | **311211** | **312111** | **321111** |

# Example

Table of $\tau^{3,1}_{(4,2)}(p)$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 0 | 2 | 1 | 2 | 1 |
| 3 | 2 | 3 | 3 | 2 | 2 |
| 4 | 3 | 4 | 5 | 4 | 3 |
| 4 | 3 | 1 | 0 | 1 | 2 |
| 3 | 1 | 0 | 1 | 2 | 0 |

$$\sum q^{\tau} = (q^5 + 4q^4 + 7q^3 + 7q^2 + 7q + 4)$$

# Example

Table of $\tau^{2,1}_{(4,2)}(p)$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 2 | 3 | 2 | 1 |
| 3 | 4 | 3 | 3 | 2 | 2 |
| 4 | 5 | 4 | 5 | 4 | 3 |
| 4 | 3 | 1 | 2 | 1 | 2 |
| 3 | 5 | 4 | 5 | 6 | 4 |

blue$\rightarrow +2$, red$\rightarrow +4$

$$\sum q^{\tau} = q(q^5 + 4q^4 + 7q^3 + 7q^2 + 7q + 4)$$

# Example

Table of $\tau^{1,1}_{(4,2)}(p)$

| | | | | | |
|---|---|---|---|---|---|
| 3 | 2 | 2 | 3 | 2 | 3 |
| 3 | 4 | 3 | 3 | 4 | 2 |
| 4 | 5 | 4 | 5 | 4 | 5 |
| 4 | 3 | 5 | 6 | 5 | 6 |
| 7 | 5 | 6 | 5 | 6 | 4 |

blue$\rightarrow$+2, red$\rightarrow$+4

$$\sum q^{\tau} = q^2(q^5 + 4q^4 + 7q^3 + 7q^2 + 7q + 4)$$

# Thank you!