A COMBINATORIAL DESCRIPTION OF A MONOMIAL EXPANSION OF k-SCHUR FUNCTIONS

NOTES FROM JENNIFER MORSE'S THIRD LECTURE

At the end of last time we saw that if you iterate the Pieri rule then you get a strong k-tableau.

Recall Definition: a strong k-tableau of shape $\lambda \in \mathcal{C}^{k+1}$ and weight $\mathfrak{u}$ is a sequence of cores

$$\emptyset \subset \lambda^1 \subset \lambda^2 \subset \cdots \subset \lambda$$

I) $|\mathcal{P}^k(\lambda^x)| = |\mathcal{P}^k(\lambda^{x-1})| + 1$
II) mark the head of (of content c_x) in one ribbon of λ^x/λ^{x-1}
III) the contents must increase

$$c_1 < c_2 < \ldots < c_{\mu_1}, c_{\mu_1+1} < c_{\mu_1+2} < \ldots < c_{\mu_1+\mu_2,\ldots}$$

Example: of strong k-tableaux
of shape (31)

$$\emptyset \subset \begin{array}{|c|}\hline x \\ \hline\end{array} \subset \begin{array}{|c|}\hline x \\ \hline \\ \hline\end{array} \subset \begin{array}{|c|c|c|}\cline{1-1} & \multicolumn{2}{c}{} \\ \hline & & x \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\cline{1-1} 2^* & \multicolumn{2}{c}{} \\ \hline 1^* & 3 & 3^* \\ \hline\end{array}$$

$$\emptyset \subset \begin{array}{|c|}\hline x \\ \hline\end{array} \subset \begin{array}{|c|c|}\hline & x \\ \hline\end{array} \subset \begin{array}{|c|c|c|}\cline{1-1} & \multicolumn{2}{c}{} \\ \hline \cdot & & x \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\cline{1-1} 3^* & \multicolumn{2}{c}{} \\ \hline 1^* & 2^* & 3 \\ \hline\end{array}$$

$$\emptyset \subset \begin{array}{|c|}\hline x \\ \hline\end{array} \subset \begin{array}{|c|c|}\hline & x \\ \hline\end{array} \subset \begin{array}{|c|c|c|}\cline{1-1} & \multicolumn{2}{c}{} \\ \hline \cdot & & x \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\cline{1-1} 3 & \multicolumn{2}{c}{} \\ \hline 1^* & 2^* & 3^* \\ \hline\end{array}$$

Only the last of these is a strong k-tableau of weight (3)

$$\begin{array}{|c|cc}\hline 3^* \\ \hline 1^* & \multicolumn{1}{|c|}{2^*} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

and

$$\begin{array}{|c|cc}\hline 3 \\ \hline 1^* & \multicolumn{1}{|c|}{2^*} & \multicolumn{1}{c|}{3^*} \\ \hline \end{array}$$

are of weight (21)

All of them have weight (111).

So this is how you construct strong k-tableaux.

With that we can gives the definition of the k-Schur function

$$h_\mu = \sum_\lambda K^{(k)}_{\lambda\mu} \mathfrak{S}^{(k)}_\lambda$$

where the $K^{(k)}_{\lambda\mu}$ counts the number of strong k-tableaux of shape λ and weight μ and by duality we have

$$s^{(k)}_\lambda = \sum K^{(k)}_{\lambda\mu} m_\mu$$

$$s^{(2)}_{(31)} = 1m_3 + 2m_{21} + 3m_{111}$$

Recall: we can define $\mathfrak{S}^{(k)}_\lambda$ as the dual by the scalar product

$$\left\langle s^{(k)}_\lambda, \mathfrak{S}^{(k)}_\mu \right\rangle = \delta_{\lambda\mu}$$

Explicit characterization by giving the Pieri rule for $\{s^{(k)}_\lambda\}$

1) one box case

$$h_1 s^{(k)}_\mu = \sum s^{(k)}_\lambda$$

over cores λ where

I) $\mu \subseteq \lambda$

II) $\lambda/\mu =$ single box ribbons

III)

$$|\mathcal{P}^k(\lambda)| = |\mathcal{P}^k(\mu)| + 1$$

Example: for $k = 3$,

$\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \subset \begin{array}{|c|c}\hline x \\ \hline \\ \hline & \multicolumn{1}{|c|}{x} \\ \hline \end{array}$ is ok because we are adding all cells of residue 1 on the core

$\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \subset \begin{array}{|c|cc}\hline \\ \hline & \multicolumn{1}{|c|}{x} & \multicolumn{1}{c|}{x} \\ \hline \end{array}$ is not ok because we are adding a cell of residue 1 and 2 on the core

Lemma: λ is a weak cover of μ if an only if $\lambda = \mu +$ all boxes of the same content mod $k+1$ are a $k+1$ residue.

$$\begin{array}{|c|}\hline 2\\\hline 0\\\hline\end{array} \subset \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{}\\\cline{1-1} 2 & \multicolumn{1}{c}{}\\\hline 0 & 1\\\hline\end{array} \subset \begin{array}{|c|}\hline 0\\\hline 1\\\hline 20\\\hline 01\\\hline\end{array} \subset \begin{array}{|c|}\hline 2\\\hline 0\\\hline 12\\\hline 20\\\hline 012\\\hline\end{array}$$

$$h_1 s_\mu^{(k)} = \sum s_\lambda^{(k)}$$

if $\lambda = \mu +$ boxes of the same residue.

Iterate to get

$$h_1 h_1 \cdots h_1 = \sum W_{\lambda 1^n}^{(k)} s_\lambda^{(k)}$$

$$h_1 s_\emptyset^{(2)} = \begin{array}{|c|}\hline 0\\\hline\end{array}$$

$$h_1 h_1 s_\emptyset^{(2)} = \begin{array}{|c|c|}\hline 0 & 1\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 0\\\hline\end{array}$$

$$h_1 h_1 h_1 s_\emptyset^{(2)} = \begin{array}{|c|c|c|}\cline{1-1} 2 & \multicolumn{2}{c}{}\\\hline 0 & 1 & 2\\\hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{}\\\cline{1-1} 2 & \multicolumn{1}{c}{}\\\hline 0 & 1\\\hline\end{array}$$

$W_{\lambda 1^n}$ counts the number of tableaux satisfying the following defintioin

Definition: a standard weak k-tableau of a shape $\lambda \in \mathcal{C}^{k+1}$ is a sequence of cores

$$\emptyset \subset \lambda^1 \subset \lambda^2 \subset \cdots \subset \lambda^\ell = \lambda$$

such that $\lambda^x / \lambda^{x-1}$ is skew with all values the same $k+1$ residue.

$$\emptyset \subset \begin{array}{|c|}\hline 0\\\hline\end{array} \subset \begin{array}{|c|c|}\hline 0 & 1\\\hline\end{array} \subset \begin{array}{|c|c|c|}\cline{1-1} 2 & \multicolumn{2}{c}{}\\\hline 0 & 1 & 2\\\hline\end{array} \subset \begin{array}{|c|c|c|}\cline{1-1} 1 & \multicolumn{2}{c}{}\\\cline{1-1} 2 & \multicolumn{2}{c}{}\\\hline 0 & 1 & 2\\\hline\end{array}$$

where I have placed the residues (mod 3) in the partitions to see what cells are being added. This sequence of partitions corresponds to the 'standard' tableau

$$\begin{array}{|c|c|c|}\cline{1-1} 4 & \multicolumn{2}{c}{}\\\cline{1-1} 3 & \multicolumn{2}{c}{}\\\hline 1 & 2 & 3\\\hline\end{array}$$

Then take as a second example

$$\emptyset \subset \boxed{0} \subset \begin{array}{|c|}\hline 2 \\ \hline 0 \\ \hline\end{array} \subset \begin{array}{|c|c}\cline{1-1} 1 & \\ \cline{1-1} 2 & \\ \hline 0 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \subset \begin{array}{|c|cc}\cline{1-1} 1 & & \\ \cline{1-1} 2 & & \\ \hline 0 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline\end{array}$$

$$\begin{array}{|c|cc}\cline{1-1} 3 & & \\ \cline{1-1} 2 & & \\ \hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline\end{array}$$

and this is a 'standard' weak also with shape $(3, 1, 1)$.

$$h_1^n = \sum W_{\lambda 1^n}^{(k)} s_\lambda^{(k)}$$

Equivalent defnition of standard weak k tableaux of shape $\lambda \in \mathcal{C}^{k+1}$

column and row increasing filling of λ with $1, 2, \ldots |\mathcal{P}^k(\lambda)|$ where every box has the same $k+1$ residue.

Want more generally $h_\mu = \sum W_{\lambda\mu}^{(k)} s_\lambda^{(k)}$.

The general Pieri rule.

$$h_r s_\mu^{(k)} = \sum s_\lambda^{(k)}$$

over $\lambda+$ a weak r strip.

Defintition λ/μ $(\lambda, \mu \in \mathcal{C}^{k+1})$ is a weak r strip if

1) λ/μ is a horizontal strip with r distinct residues

2) $|\mathcal{P}^k(\lambda)| = |\mathcal{P}^k(\mu)| + r$

Example:

$$h_2 s_{(1)}^{(2)} = s_{(31)}^{(2)}$$

Definition: (weak) k-tableau of shape λ and weight μ is a column strict filling with μ_1 residues labelleing ones, μ_2 residues labeleding twos, etc.

Claim: $h_\mu = \sum W_{\lambda\mu}^{(k)} s_\lambda^{(k)}$ where $W_{\lambda\mu}^{(k)}$ = the number of weak k tableaux of shape λ and weight μ.

Definition:

$$\mathfrak{S}_\lambda^{(k)} = \sum W_{\lambda\mu}^{(k)} m_\mu$$

Example:

$$\mathfrak{S}^{(2)}_{(31)} = \begin{array}{|c|c|c|}\cline{1-1} 3_2 \\ \hline 1_0 & 2_1 & 3_2 \\ \hline\end{array} + \begin{array}{|c|c|c|}\cline{1-1} 2_2 \\ \hline 1_0 & 1_1 & 2_2 \\ \hline\end{array}$$

$$\mathfrak{S}^{(2)}_{(31)} = m_{111} + m_{21}$$

So I discussed with Luc and decided what this has to do with the Affine Stanley symmetric functions. We decided that a good thing to finish with is the explicit connection between these two families.

Recall that $w =$ affine grassmannian permutation in $\tilde{S}_{k+1}$.

Definition:

$$\tilde{F}_w = \sum_{w=v_1v_2\cdots v_\ell} x_1^{|v_1|}x_2^{|v_2|}\cdots x_\ell^{|v_\ell|}$$

where v_i is a cyclically decreasing word.

$$= \sum_{\mu\ partition} a^{(k)}_{w\mu} m_\mu$$

where $a^{(k)}_{w\mu}$ is equal to the number of μ factorizations for w
i.e. $w = v_1v_2\cdots v_\ell$ for cyclically decreasing v_i and $|v_i| = \mu_i$.
How does this connect to k-Schurs?
We convert affine grassmannian permutations to cores.

$$s_0s_1s_2s_1s_0 \to core$$

by successively adding corners of residue i.

$$\begin{array}{|c|}\hline 0 \\ \hline\end{array} \subset \begin{array}{|c|c|}\hline 0 & 1 \\ \hline\end{array} \subset \begin{array}{|c|c|c|}\cline{1-1} 2 \\ \hline 0 & 1 & 2 \\ \hline\end{array} \subset \begin{array}{|c|c|c|}\cline{1-1} 1 \\ \cline{1-1} 2 \\ \hline 0 & 1 & 2 \\ \hline\end{array} \subset \begin{array}{|c|c|c|c|}\cline{1-1} 1 \\ \cline{1-2} 2 & 0 \\ \hline 0 & 1 & 2 & 0 \\ \hline\end{array} \subset \begin{array}{|c|c|c|c|}\cline{1-1} 0 \\ \cline{1-1} 1 \\ \cline{1-2} 2 & 0 \\ \hline 0 & 1 & 2 & 0 \\ \hline\end{array}$$

check

Remember that since every reduce word ends in 0.

It turns out that no matter what reduced word I pick, I will always construct the same core.

This defines a bijection between cores and affine permutations.

Claim:

$$\tilde{F}^{(k)}_{w_\lambda} = \mathfrak{S}^{(k)}_\lambda = \sum W^{(k)}_{\lambda\mu} m_\mu$$

Suffices to show that that there is a bijection between μ-factorizations for w_λ and weak k-tableaux of shape $\lambda \in \mathcal{C}^{k+1}$ weight μ.

This was one of the big breakthroughs for us figuring out that there was a correspondence between the standard k tableaux of shape λ and the reduced words for w_λ.

From words $s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_2} s_{i_1}$ for permutation $w \in \tilde{S}_{???}$

fill boxes successively putting 1 in all boxes of residue i_1 putting 2 in a all boxes of residue i_2, etc. In the example below I've subscripted the labels with the residues (mod 3).

Example: From the word $s_0 s_1 s_2 s_0 s_1 s_2 s_1 s_0$ we construct

$$\begin{array}{|c|c|c|c|c|}
\cline{1-1} 8_0 \\
\cline{1-1} 7_1 \\
\cline{1-2} 6_2 & 8_0 \\
\cline{1-2} 5_0 & 7_1 \\
\cline{1-3} 4_1 & 6_2 & 8_0 \\
\cline{1-3} 3_2 & 5_0 & 7_1 \\
\hline 1_0 & 2_1 & 3_2 & 5_0 & 7_1 \\
\hline
\end{array}$$

which is a 3-core.

Semistandard case

(0)(1)(2)(10)

$$\begin{array}{|c|c|c|c|}
\cline{1-1} 4_0 \\
\cline{1-1} 3_1 \\
\cline{1-2} 2_2 & 4_0 \\
\hline 1_0 & 1_1 & 2_2 & 4_0 \\
\hline
\end{array}$$

This is a tableau of weight $(2, 1, 1, 1)$