## A COMBINATORIAL DESCRIPTION OF A MONOMIAL EXPANSION OF k-SCHUR FUNCTIONS

## NOTES FROM JENNIFER MORSE'S THIRD LECTURE

At the end of last time we saw that if you iterate the Pieri rule then you get a strong k-tableau.

Recall Definition: a strong k-tableau of shape  $\lambda \in \mathcal{C}^{k+1}$  and weight  $\mathfrak u$  is a sequence of cores

$$\emptyset \subset \lambda^1 \subset \lambda^2 \subset \dots \subset \lambda$$

- I)  $|\mathcal{P}^k(\lambda^x)| = |\mathcal{P}^k(\lambda^{x-1})| + 1$
- II) mark the head of (of content  $c_x$ ) in one ribbon of  $\lambda^x/\lambda^{x-1}$
- III) the contents must increase

$$c_1 < c_2 < \dots < c_{\mu_1}, c_{\mu_1+1} < c_{\mu_1+2} < \dots < c_{\mu_1+\mu_2,\dots}$$

Example: of strong k-tableaux of shape (31)

$$\emptyset \subset x \subset x \subset x$$

$$2^*$$

$$1^* \ 3 \ 3^*$$

$$0 \subset x \subset x \subset x \subset x$$

$$3^*$$

$$1^* \ 2^* \ 3$$

$$1^* \ 2^* \ 3^*$$

$$1$$

Only the last of these is a strong k-tableau of weight (3)

and

$$\frac{3}{1^* 2^* 3^*}$$

are of weight (21)

All of them have weight (111).

So this is how you construct strong k-tableaux.

With that we can gives the definition of the k-Schur function

$$h_{\mu} = \sum_{\lambda} K_{\lambda\mu}^{(k)} \mathfrak{S}_{\lambda}^{(k)}$$

where the  $K_{\lambda\mu}^{(k)}$  counts the number of strong k-tableaux of shape  $\lambda$  and weight  $\mu$  and by duality we have

$$s_{\lambda}^{(k)} = \sum K_{\lambda\mu}^{(k)} m_{\mu}$$

$$s_{(31)}^{(2)} = 1m_3 + 2m_{21} + 3m_{111}$$

Recall: we can define  $\mathfrak{S}_{\lambda}^{(k)}$  as the dual by the scalar product

$$\left\langle s_{\lambda}^{(k)}, \mathfrak{S}_{\mu}^{(k)} \right\rangle = \delta_{\lambda\mu}$$

Explicit characterization by giving the Pieri rule for  $\{s_\lambda^{(k)}\}$ 

1) one box case

$$h_1 s_{\mu}^{(k)} = \sum s_{\lambda}^{(k)}$$

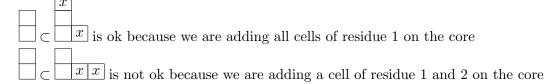
over cores  $\lambda$  where

- I)  $\mu \subseteq \lambda$
- II)  $\lambda/\mu = \text{single box ribbons}$

III)

$$|\mathcal{P}^k(\lambda)| = |\mathcal{P}^k(\mu)| + 1$$

Example: for k = 3,



Lemma:  $\lambda$  is a weak cover of  $\mu$  if an only if  $\lambda = \mu +$  all boxes of the same content mod k+1 are a k+1 residue.

$$h_1 s_{\mu}^{(k)} = \sum s_{\lambda}^{(k)}$$

if  $\lambda = \mu +$  boxes of the same residue.

Iterate to get

$$h_1 h_1 \cdots h_1 = \sum W_{\lambda 1^n}^{(k)} s_{\lambda}^{(k)}$$

$$h_1 s_{\emptyset}^{(2)} = \boxed{0}$$

$$h_1 h_1 s_{\emptyset}^{(2)} = \boxed{0 \ 1} + \boxed{2}$$

 $W_{\lambda 1^n}$  counts the number of tableaux satisfying the following defintion

Definition: a standard weak k-tableau of a shape  $\lambda \in \mathcal{C}^{k+1}$  is a sequence of cores

$$\emptyset \subset \lambda^1 \subset \lambda^2 \subset \cdots \subset \lambda^\ell = \lambda$$

such that  $\lambda^x/\lambda^{x-1}$  is skew with all values the same k+1 residue.

$$\emptyset \subset \boxed{0} \subset \boxed{0} \boxed{1} \subset \boxed{2} \boxed{2} \boxed{2}$$

where I have placed the residues (mod 3) in the partitions to see what cells are being added. This sequence of partitions corresponds to the 'standard' tableau

$$\begin{array}{c|c}
4\\3\\1&2&3\\\end{array}$$

Then take as a second example

and this is a 'standard' weak also with shape (3, 1, 1).

$$h_1^n = \sum W_{\lambda 1^n}^{(k)} s_{\lambda}^{(k)}$$

Equivalent definition of standard weak k tableaux of shape  $\lambda \in \mathcal{C}^{k+1}$ column and row increasing filling of  $\lambda$  with  $1, 2, \dots |\mathcal{P}^k(\lambda)|$  where every box has the same k+1 residue.

Want more generally  $h_{\mu} = \sum W_{\lambda\mu}^{(k)} s_{\lambda}^{(k)}$ . The general Pieri rule.

$$h_r s_{\mu}^{(k)} = \sum s_{\lambda}^{(k)}$$

over  $\lambda$ + a weak r strip.

Defintition  $\lambda/\mu$   $(\lambda, \mu \in \mathcal{C}^{k+1})$  is a weak r strip if 1)  $\lambda/\mu$  is a horizontal strip with r distinct residues

- 2)  $|\mathcal{P}^k(\lambda)| = |\mathcal{P}^k(\mu)| + r$

Example:

$$h_2 s_{(1)}^{(2)} = s_{(31)}^{(2)}$$

Definition: (weak) k-tableau of shape  $\lambda$  and weight  $\mu$  is a column strict filling with  $\mu_1$ residues labelleing ones,  $\mu_2$  residues labelleding twos, etc.

Claim:  $h_{\mu} = \sum W_{\lambda\mu}^{(k)} s_{\lambda}^{(k)}$  where  $W_{\lambda\mu}^{(k)}$  = the number of weak k tableaux of shape  $\lambda$  and weight  $\mu$ .

Definition:

$$\mathfrak{S}_{\lambda}^{(k)} = \sum W_{\lambda\mu}^{(k)} m_{\mu}$$

Example:

$$\mathfrak{S}_{(31)}^{(2)} = \frac{\boxed{3_2}}{\boxed{1_0}\boxed{2_1}\boxed{3_2}} + \frac{\boxed{2_2}}{\boxed{1_0}\boxed{1_1}\boxed{2_2}}$$
$$\mathfrak{S}_{(31)}^{(2)} = m_{111} + m_{21}$$

So I discussed with Luc and decided what this has to do with the Affine Stanley symmetric functions. We decided that a good thing to finish with is the explicit connection between these two families.

Recall that  $w = \text{affine grassmannian permutation in } \tilde{S}_{k+1}$ .

Definition:

$$\tilde{F}_w = \sum_{w=v_1v_2\cdots v_\ell} x_1^{|v_1|} x_2^{|v_2|} \cdots x_\ell^{|v_\ell|}$$

where  $v_i$  is a cyclically decreasing word.

$$= \sum_{\mu \ partition} a_{w\mu}^{(k)} m_{\mu}$$

where  $a_{w\mu}^{(k)}$  is equal to the number of  $\mu$  factorizations for w i.e.  $w = v_1 v_2 \cdots v_\ell$  for cyclically decreasing  $v_i$  and  $|v_i| = \mu_i$ . How does this connect to k-Schurs?

We convert affine grassmannian permutations to cores.

$$s_0s_1s_2s_1s_0 \rightarrow core$$

by successively adding corners of residue i.

check

Remember that since every reduce word ends in 0.

It turns out that no matter what reduced word I pick, I will always construct the same core.

This defines a bijection between cores and affine permutations.

Claim:

$$\tilde{F}_{w_{\lambda}}^{(k)} = \mathfrak{S}_{\lambda}^{(k)} = \sum W_{\lambda\mu}^{(k)} m_{\mu}$$

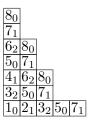
Suffices to show that that there is a bijection between  $\mu$ -factorizations for  $w_{\lambda}$  and weak k-tableaux of shape  $\lambda \in \mathcal{C}^{k+1}$  weight  $\mu$ .

This was one of the big breakthroughs for us figuring out that there was a correspondence between the standard k tableaux of shape  $\lambda$  and the reduced words for  $w_{\lambda}$ .

From words  $s_{i_{\ell}}s_{i_{\ell-1}}\cdots s_{i_2}s_{i_1}$  for permutation  $w\in \tilde{S}_{???}$ 

fill boxes successively putting 1 in all boxes of residue  $i_1$  putting 2 in a all boxes of residue  $i_2$ , etc. In the example below I've subscripted the labels with the residues (mod 3).

Example: From the word  $s_0s_1s_2s_0s_1s_2s_1s_0$  we construct



which is a 3-core.

Semistandard case

(0)(1)(2)(10)

This is a tableau of weight (2, 1, 1, 1)