

# $QH^*(G/B)$ VS $QH^*(G/P)$

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Joint with Naichung Conan Leung

Reference: math.AG 1007.1683

Fields Institute

July 13, 2010

# “Preface”

Main body:  $QH^*(G/B)$  VS  $QH^*(G/P)$ .

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Affine Schubert Calculus  $\longleftrightarrow H_*(\Omega K), H^*(\Omega K)$ .

Quantum Schubert Calculus  $\longleftrightarrow QH^*(G/P)$ .

(E.g.  $K = SU(n)$ ,  $G = SL(n, \mathbb{C})$ ,  $P = \begin{pmatrix} A_1 & * & * \\ & \ddots & * \\ & & A_k \end{pmatrix}$ .)

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Peterson's Isomorphism (Lam-Shimozono, 2007)

After taking torus-equivariant extension and localization,

$$H_*(\Omega K) \cong QH^*(G/B), \quad \frac{H_*(\Omega K)}{\text{ideal}} \cong QH^*(G/P).$$

Main body:  $QH^*(G/B)$  VS  $QH^*(G/P)$ .

# Introduction

$$F\ell_3 := \{V_1 \leq V_2 \leq \mathbb{C}^3 \mid \dim V_1 = 1, \dim V_2 = 2\}$$



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- $F\ell_3$  is a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^2$ .
- $H^*(\cdot)$  is functorial.
  - ▶  $\textcolor{red}{\iota^*} : H^*(F\ell_3) \rightarrow H^*(\mathbb{P}^1); \quad \textcolor{red}{\pi^*} : H^*(\mathbb{P}^2) \rightarrow H^*(F\ell_3).$
  - ▶  $\exists$  a filtration  $\mathcal{F}$  on  $H^*(F\ell_3)$  such that (as graded algebras)

$$Gr^{\mathcal{F}}(H^*(F\ell_3)) \cong H^*(\mathbb{P}^1) \otimes H^*(\mathbb{P}^2).$$

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**Question:** How about the quantum analog??

What is “quantum” ??

$$X = \mathbb{P}^2 = (\mathbb{C}^3 - \{\mathbf{0}\}) / \sim$$

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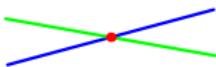
$X = \mathbb{P}^2 = \{\text{pt}\} \sqcup \mathbb{C}^1 \sqcup \mathbb{C}^2.$

- $H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$   
 $= \mathbb{Z} \cdot e \oplus \mathbb{Z} \cdot x \oplus \mathbb{Z} \cdot y$ 
  - ▶  $x \cup x = y, \quad x \cup y = 0 \implies H^*(X, \mathbb{Z}) \cong \mathbb{Z}[x]/\langle x^3 \rangle$

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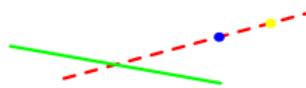
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 $x \cup y = 0 = 0 \cdot y$        $x \star y = 1 \cdot q \cdot e:$



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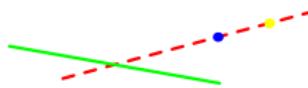
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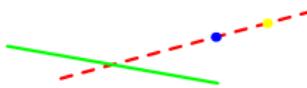
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- ▶ “deformation”:  $x \star y|_{q=0} = x \cup y$ .
- ▶ “1”: Gromov-Witten invariant;  $\exists!$  line that passes through two given points (and hits the given line).

# Aim and outline of the present talk



$$\begin{array}{ccc} G & \supseteq & P & \supseteq & B \\ (\text{semi-})\text{simple} & & \text{parabolic} & & \text{Borel} \\ \text{Lie group}/\mathbb{C} & & \text{subgroup} & & \text{subgroup} \end{array}$$

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- **Aim:** Quantum analog of the graded-algebra isomorphism  
 $Gr(H^*(G/B)) \cong H^*(P/B) \otimes H^*(G/P).$



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$\rightsquigarrow$        $P/B \rightarrow G/B \longrightarrow G/P.$   
E.g.:       $\mathbb{P}^1 \rightarrow F\ell_3 \longrightarrow \mathbb{P}^2.$

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E.g.:  $\mathbb{P}^1 \rightarrow F\ell_3 \longrightarrow \mathbb{P}^2.$

- Outline of the talk.

- Introduction (done).
- $QH^*(G/P)$  and relevant problems/developments.
  - ▶  $G/P \rightsquigarrow H^*(G/P) \rightsquigarrow QH^*(G/P).$
- On the quantum analog.

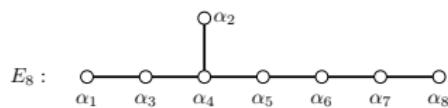
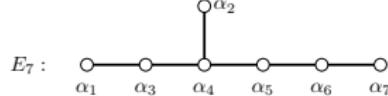
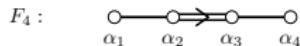
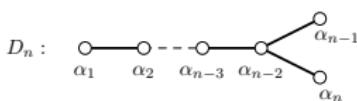
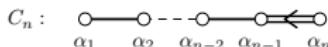
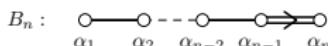
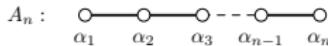
Flag varieties  $G/P$ 's  $\longleftrightarrow (\Delta, \Delta_P)$ 's

- $G$ : a simply-connected complex simple Lie group (of rank  $n$ ).
- $P$ : a parabolic subgroup of  $G$ , i.e. a Lie subgroup of  $G$  such that  $G/P$  is (a smooth) projective (variety).

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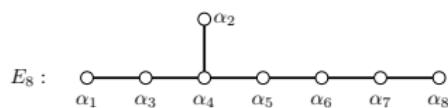
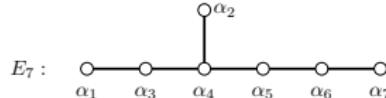
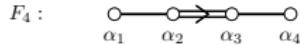
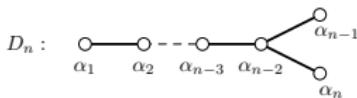
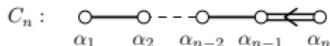
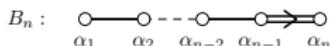
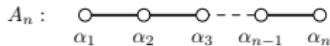


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- ▶  $\Delta := \{\alpha_1, \dots, \alpha_n\}$  is a base.

- $P$ : a parabolic subgroup of  $G$ , i.e. a Lie subgroup of  $G$  such that  $G/P$  is (a smooth) projective (variety).
  - ▶  $P$  corresponds to a subset  $\Delta_P$  of  $\Delta$ .

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$F\ell_3 = G/B = K/T$  is a (complete) flag variety of type  $A_2$ .

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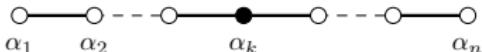
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$F\ell_3 = G/B = K/T$  is a (complete) flag variety of type  $A_2$ .

- Flag varieties  $G/P$  of  $A_n$ -type are all of the form:

$G/P = \{V_{a_1} \leqslant \cdots \leqslant V_{a_k} \leqslant \mathbb{C}^{n+1} \mid \dim V_{a_j} = a_j, j = 1, \dots, k\}.$

- ▶  $G/P = Gr(k, n+1)$ , then  $\Delta_P = \Delta \setminus \{\alpha_k\}$ .



# (Quick review) Weyl group $W \triangleq \langle s_1, \dots, s_n \rangle$

- Cartan matrices  $(A_{ij})_{n \times n}$ ;  $A_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$ .

$$A_n : \quad \begin{array}{ccccccccc} \circ & - & \circ & - & \cdots & - & \circ & - & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_{n-1} & & \alpha_n \end{array}$$

$$B_n : \quad \begin{array}{ccccccccc} \circ & - & \circ & - & \cdots & - & \circ & \nearrow & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_{n-2} & & \alpha_{n-1} & & \alpha_n \end{array}$$

$$C_n : \quad \begin{array}{ccccccccc} \circ & - & \circ & - & \cdots & - & \circ & \leftarrow & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_{n-2} & & \alpha_{n-1} & & \alpha_n \end{array}$$

$$D_n : \quad \begin{array}{ccccccccc} \circ & - & \circ & - & \cdots & - & \circ & \nearrow & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_{n-3} & & \alpha_{n-2} & & \alpha_{n-1} \\ & & & & & & & \searrow & \\ & & & & & & & & \circ \\ & & & & & & & & \alpha_n \end{array}$$

$\rightsquigarrow$

$$F_4 : \quad \begin{array}{ccccccccc} \circ & - & \circ & \nearrow & \circ & - & \circ & - & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \end{array}$$

$$E_6 : \quad \begin{array}{ccccccccc} & & & & \circ & \alpha_2 \\ & & & & | & & & & \\ \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ \alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 \end{array}$$

$$E_7 : \quad \begin{array}{ccccccccc} & & & & \circ & \alpha_2 \\ & & & & | & & & & \\ \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ \alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 \\ & & & & & & & & \alpha_7 \end{array}$$

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$$G_2 : \quad \begin{array}{ccccc} & \circ & \nearrow & \circ & \\ & | & & | & \\ \circ & & \alpha_1 & & \alpha_2 \end{array}$$

- $A_{ii} \triangleq 2$ .
- For  $i \neq j$ ,  $A_{ij} \triangleq \begin{cases} -1, & \text{if } \exists \text{ arrow from } \alpha_j \text{ to } \alpha_i \\ -\#\overrightarrow{\alpha_i \alpha_j}, & \text{otherwise} \end{cases}$ .
- E.g. type  $G_2$ :  $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

# (Quick review) Weyl group $W \triangleq \langle s_1, \dots, s_n \rangle$

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$$C_n : \quad \begin{array}{ccccccccc} & \circ & - & \circ & - & \cdots & - & \circ & \swarrow & \circ \\ & \alpha_1 & & \alpha_2 & & \alpha_{n-2} & & \alpha_{n-1} & & \alpha_n \end{array}$$

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$$G_2 : \quad \begin{array}{ccccccccc} & & & & \circ & \nearrow & \circ & \alpha_2 \\ & & & & \alpha_1 & & & \end{array}$$

- $s_i : \mathfrak{h} \rightarrow \mathfrak{h}$ ;  $s_i(\lambda) \triangleq \lambda - \langle \alpha_i, \lambda \rangle \alpha_i^\vee$ .

►  $\mathfrak{h}^* := \bigoplus_{i=1}^n \mathbb{C}\alpha_i$ ;  $(\mathfrak{h}^*)^* = \mathfrak{h} = \bigoplus_{i=1}^n \mathbb{C}\alpha_i^\vee$ .

► reflection along the hyperplane  $H_{\alpha_i} := \{\lambda \in \mathfrak{h} \mid \langle \alpha_i, \lambda \rangle = 0\}$ .

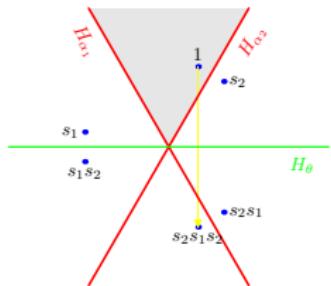
►  $s_i : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ ;  $s_i(\beta) \triangleq \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i$

## (Quick review) Weyl group $W \triangleq \langle s_1, \dots, s_n \rangle$

- Cartan matrices  $(A_{ij})_{n \times n}$ ;  $A_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$ .
- $s_i : \mathfrak{h} \rightarrow \mathfrak{h}$ ;  $s_i(\lambda) \triangleq \lambda - \langle \alpha_i, \lambda \rangle \alpha_i^\vee$ .
  - ▶  $\mathfrak{h}^* := \bigoplus_{i=1}^n \mathbb{C}\alpha_i$ ;  $(\mathfrak{h}^*)^* = \mathfrak{h} = \bigoplus_{i=1}^n \mathbb{C}\alpha_i^\vee$ .
  - ▶ reflection along the hyperplane  $H_{\alpha_i} := \{\lambda \in \mathfrak{h} \mid \langle \alpha_i, \lambda \rangle = 0\}$ .
  - ▶  $s_i : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ ;  $s_i(\beta) \triangleq \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i$
- Length function  $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ .
  - ▶  $\ell(\text{id}) \triangleq 0$ .
  - ▶ For  $w \neq \text{id}$ ,  $\ell(w) \triangleq \min\{k \mid w = s_{i_1} \cdots s_{i_k}\}$ .

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- E.g. type  $A_2$ :  $G = SL(3, \mathbb{C})$ ,  $\Delta = \{\alpha_1, \alpha_2\}$ .  $\theta := \alpha_1 + \alpha_2$ .

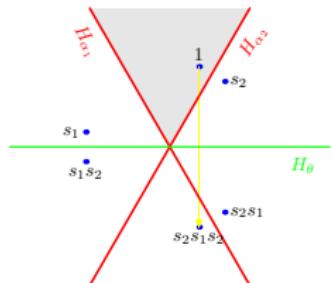


$$W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_2s_1s_2\}.$$

In particular,  $W \cong S_3$  (via  $s_1 \mapsto (12)$ ,  $s_2 \mapsto (23)$ ),  $|W| = 6$ .

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Note that

$$H^*(F\ell_3) \cong H^*(\mathbb{P}^1) \otimes H^*(\mathbb{P}^2) \quad \text{as vector spaces}$$

$$\implies \dim H^*(F\ell_3) = \dim H^*(\mathbb{P}^1) \times \dim H^*(\mathbb{P}^2) = 2 \times 3 = 6 = |W|$$

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$W_P \triangleq \langle s_i : \alpha_i \in \Delta_P \rangle; W^P \triangleq \{w \in W | \ell(w) \leq \ell(v), \forall v \in wW_P\}.$

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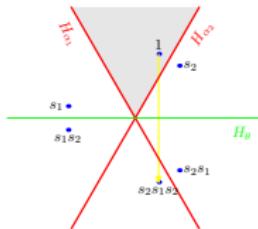
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$W^P = \{1, s_2, s_1s_2\}$ . Note that  $G/P = \mathbb{P}^2$  and  $G/B = F\ell_3$ .  
 $\dim H^*(G/P, \mathbb{Z}) = 3 = |W^P|$ ,  $\dim H^*(G/B, \mathbb{Z}) = 6 = |W^B|$ .

$$H^*(G/P, \mathbb{Z}). \quad (\Delta_P \subset \Delta)$$

Bruhat decomposition:  $G/P = \bigsqcup_{w \in W^P} B\dot{w}P/P.$



- $H_*(G/P, \mathbb{Z}) = \bigoplus_{w \in W^P} \mathbb{Z}\sigma_w; H^*(G/P, \mathbb{Z}) = \bigoplus_{w \in W^P} \mathbb{Z}\sigma^w.$ 
  - ▶  $\sigma_w = [\overline{B\dot{w}P/P}] \in H_{2\ell(w)}(G/P, \mathbb{Z}).$
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- $H_2(G/P, \mathbb{Z}) \cong Q^\vee/Q_P^\vee$ , where  $Q_P^\vee := \bigoplus_{\alpha \in \Delta_P} \mathbb{Z}\alpha^\vee$ .
  - ▶  $H_2(G/B, \mathbb{Z}) \cong Q^\vee$ .

## Problems/Answers on $H^*(G/P)$

- ① A (nice) presentation of the cohomology ring  $H^*(G/P)$ .
- ② A “combinatorial” formula/algorithm for  $N_{u,v}^w$ 's,

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“trivial”: functorial properties, graded-algebra isomorphisms  
 $\implies$  presentation of  $H^*(G/P)$

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# Kontsevich's Moduli space of stable maps

↔ BIG/SMALL quantum cohomology.

- Let  $\lambda \in H_2(X, \mathbb{Z})$ . Let  $\overline{\mathcal{M}}_{0,3}(X, \lambda)$  denote the moduli space of stable maps of degree  $\lambda$  of 3-pointed genus 0 curves into  $X$ :

$$\overline{\mathcal{M}}_{0,3}(X, \lambda) = \{(f : C \rightarrow X; p_1, p_2, p_3) \mid f_*([C]) = \lambda, \text{ stab.}, \dots\} / \sim .$$

- $C$  is a projective, connected, reduced, (at worst) nodal curve.
- $g_a(C) = 0$ .
- $p_1, p_2, p_3$ : distinct, nonsingular, marked points in  $C$ .
- “stab.”:  $|Aut(f : C \rightarrow X; p_1, p_2, p_3)| < +\infty$ .
- The moduli space  $\overline{\mathcal{M}}_{0,3}(G/P, \lambda)$  is well-behaved. It is an irreducible, rational, normal, projective variety with (at worst) finite quotient singularities.
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For any  $u, v \in W^P$ , the (small) **quantum product** is defined by

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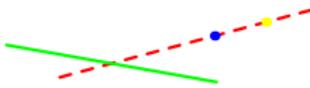
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  - ▶  $(\sigma^w)^\sharp$ 's lie in  $H^*(G/P)$  such that  $\int_{[G/P]} (\sigma^w)^\sharp \cup \sigma^{w'} = \delta_{w,w'}$ .
  - ▶ Finite sum.  $N_{u,v}^{w, \lambda_P} = 0$  unless  $q_{\lambda_P} \in \mathbb{Q}[\mathbf{q}]$ .

# $QH^*(G/P)$ VS $H^*(G/P)$

- Similarity.
  - ▶  $QH^*(G/P)$  is commutative and associative, and has unit 1.
  - ▶  $N_{u,v}^{w,\lambda_P}$ 's have geometrical meanings. For instance for  $G/P = \mathbb{P}^2$ , we have  $\textcolor{red}{x} \star \textcolor{blue}{y} = \textcolor{red}{1} \cdot q \cdot \textcolor{blue}{e}$ .
- Thus  $N_{\textcolor{red}{x},\textcolor{blue}{y}}^{e,q} = \textcolor{red}{1}$ ; that means
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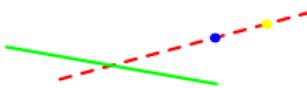


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- Difference: there is “NO” functorial property for  $QH^*(\cdot)$ .  
In general,  $f : X \rightarrow Y$  does not induce  
 $f^* : QH^*(Y) \rightarrow QH^*(X)$ .

# Problems on $QH^*(G/P)$

- ① A (nice) presentation of the quantum cohomology ring:

$$QH^*(G/P) = \frac{\mathbb{Q}[x_1, \dots, x_M, q_1, \dots, q_r]}{(\text{relations})}.$$

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# Developments on $QH^*(G/P)$

- ① A presentation of the quantum cohomology ring  $QH^*(G/P)$ . Known for (1)  $P = B$ ; (2)  $G = SL(n+1, \mathbb{C})$ ; (3) (almost all)  $P$  is maximal.
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  - Lam-Shimozono (preprint, 2007; Acta. Math. 2010):  
 $QH^*(G/B) \cong H_*(\Omega K)$  after taking torus-equivariant extension and localization (D. Peterson's isomorphism).

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 $N_{u,v}^{w,\lambda_P} = N_{u,v}^{w',\lambda_B}$ .  $\dashrightarrow QH^*(G/B) \text{ VS } QH^*(G/P) \dashleftarrow$
- “Quantum to classical” principle for (almost all) Grassmannians (Buch-Kresch-Tamvakis, Chaput-Manivel-Perrin).

## Grading map on $H^*(G/B)$

- E.g. type  $A_2$  and  $\Delta_P = \{\alpha_1\}$ .  $P/B \rightarrow G/B \longrightarrow G/P$  with  
 $H^*(G/P) = H^*(\mathbb{P}^2) = \mathbb{Q}\{1, x, x^2\} = \mathbb{Q}\{1, \sigma^{s_2}, \sigma^{s_1 s_2}\}$ ,  
 $H^*(P/B) = H^*(\mathbb{P}^1) = \mathbb{Q}\{1, z\} = \mathbb{Q}\{1, \sigma^{s_1}\}$  and  $G/B = F\ell_3$ .

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  - ▶ As vector spaces:  $H^*(G/B) \cong H^*(P/B) \otimes H^*(G/P)$ .  
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  - Grading map  $gr : W \rightarrow \mathbb{Z}^2$ ;  $gr(w) = (i, j)$ .

	P/B				G/P			
4								
3								
2								
1	$z$	$zx$	$zx^2$					
0	1	$x$	$x^2$					
-1	0	0	0					
-2	0	0	0	0	0	0	0	
$i \backslash j$	0	1	2	3	4	5	6	

## $\mathbb{Z}^2$ -filtration on $H^*(G/B)$

- $\mathbb{Z}^2$ -filtration  $\mathcal{F} = \{F_{\mathbf{a}}\}_{\mathbf{a} \in \mathbb{Z}^2}$ , where  $F_{\mathbf{a}} = \bigoplus_{gr(w) \leq \mathbf{a}} \mathbb{Q}\sigma^w$ .

- ▶ Lexicographical order:  $\mathbf{a} = (a_1, a_2) < (b_1, b_2) = \mathbf{b}$  if and only if either (i)  $a_1 < b_1$  or (ii)  $a_1 = b_1$  and  $a_2 < b_2$ .
- ▶  $H^*(G/B)$  is a  $\mathbb{Z}^2$ -filtered algebra. That is,  $F_{\mathbf{a}}F_{\mathbf{b}} \subseteq F_{\mathbf{a}+\mathbf{b}}$ .
- ▶ Associated graded algebra  $Gr^{\mathcal{F}} = \bigoplus Gr_{\mathbf{a}}$ .  
 $Gr_{\mathbf{a}} = F_{\mathbf{a}} / \bigcup_{\mathbf{b} < \mathbf{a}} F_{\mathbf{b}} = 0 \iff \mathbf{a} \text{ lies in the gray blocks.}$

								$P/B$
								$G/P$
								$G/P$
4								
3								
2								
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## $\mathbb{Z}^2$ -filtration on $H^*(G/B)$

- $H^*(G/B)$  is a  $\mathbb{Z}^2$ -filtered algebra with filtration  $\mathcal{F}$
- $\mathcal{F}$  is compatible with  $P/B \xhookrightarrow{i} G/B \xrightarrow{\pi} G/P$ : there exist algebra isomorphisms  $\bar{i}^*$  and  $\bar{\pi}^*$  such that

$$\begin{array}{ccc}
 H^*(G/B) & & A \\
 proj \downarrow & \searrow i^* & \downarrow proj \\
 H^*(G/B)/I & \xrightarrow[\cong]{\bar{i}^*} & H^*(P/B), \\
 & & H^*(G/P) \xrightarrow[\cong]{\bar{\pi}^*} A/J,
 \end{array}$$

- $A := \pi^*(H^*(G/P))$  is a subalgebra of  $H^*(G/B)$  with  $J = 0 \triangleleft A$
- $I \triangleleft H^*(G/B)$  is spanned by  $\{x, x^2, zx, zx^2\}$ .

		P/B				G/P			
		1	2	3	4	5	6	7	8
0	1	x	$xz$	$xz^2$					
	0	1	$x$	$x^2$					
-1	0	0	0						
-2	0	0	0	0	0	0	0	0	
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- $A$  is a subalgebra of  $H^*(G/B)$  with  $J = 0 \triangleleft A$
- $I \triangleleft H^*(G/B)$ .
- ▶ As  $\mathbb{Z}^2$ -graded algebras,  
 $Gr^{\mathcal{F}}(H^*(G/B)) \cong H^*(P/B) \otimes H^*(G/B).$ 
  - ▶  $H^*(G/P) \cong \frac{\mathbb{Q}[\mathfrak{h}^*]^{w_P}}{\mathbb{Q}[\mathfrak{h}^*]_{>0}^{w_P}}$

## Step 1 for quantum analog: known information

- Quantum Chevalley formula for  $G/B$ :  $u \in W, 1 \leq i \leq n$ ,  
 $\sigma^u \star \sigma^{s_i} = \sum_{\gamma \in \Gamma_1} \langle \chi_i, \gamma^\vee \rangle \sigma^{us_\gamma} + \sum_{\gamma \in \Gamma_2} \langle \chi_i, \gamma^\vee \rangle q_{\gamma^\vee} \sigma^{us_\gamma}$ , where  
 $\Gamma_1 := \{\gamma \in R^+ \mid \ell(us_\gamma) = \ell(u) + 1\}$  and  
 $\Gamma_2 := \{\gamma \in R^+ \mid \ell(us_\gamma) = \ell(u) + 1 - \langle 2\rho, \gamma^\vee \rangle\}.$ 
  - ▶  $\{\chi_1, \dots, \chi_n\} \subset \mathfrak{h}^*$ : fundamental weights.  $\langle \chi_i, \alpha_j^\vee \rangle = \delta_{i,j}$ .
  - ▶  $\rho := \sum_{i=1}^n \chi_i$ .  $R^+ = W \cdot \Delta \cap (\bigoplus_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i)$ .  $w(\alpha_j)^\vee = w(\alpha_j^\vee)$ .
  - ▶ Quantum Chevalley formula **determines** the ring structure of  $QH^*(G/B)$  completely.
- Peterson-Woodward comparison formula: For every  $u, v, w \in W^P$ , we have

$$N_{u,v}^{w,\lambda_P} = N_{u,v}^{\tilde{w},\lambda_B},$$

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## Step 2: typical example and expected grading map

E.g. type  $A_2$  with  $\Delta_P = \alpha_1$ :  $G/B = F\ell_3$ ,  $G/P = \mathbb{P}^2$ .

- Quantum products for  $F\ell_3$ :

$$\sigma^{s_1} \star \sigma^{s_1} = \sigma^{s_2 s_1} + q_1, \quad \sigma^{s_1} \star \sigma^{s_1 s_2} = \sigma^{s_1 s_2 s_1}, \quad \sigma^{s_1} \star \sigma^{s_2 s_1} = q_1 \sigma^{s_2},$$

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$$\sigma^{s_1 s_2} \star \sigma^{s_1 s_2 s_1} = q_1 q_2 \sigma^{s_2}, \quad \sigma^{s_1 s_2} \star \sigma^{s_1 s_2} = q_2 \sigma^{s_2 s_1}, \quad \sigma^{s_1} \star \sigma^{s_1 s_2 s_1} = q_1 \sigma^{s_1 s_2} + q_1 q_2,$$

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- $QH^*(\mathbb{P}^2) = \frac{\mathbb{Q}[x, t]}{\langle x^3 - t \rangle}$ .  $x \star_P x^2 = t$ .

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Recall that  $P/B = \mathbb{P}^1$ ,

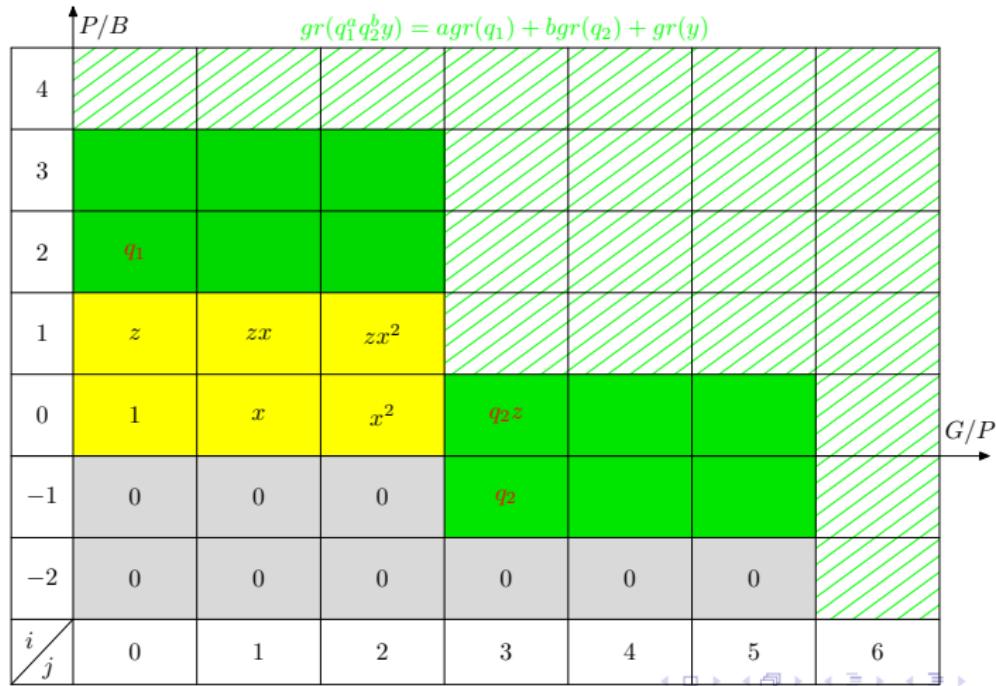
$$H^*(\mathbb{P}^2) = \mathbb{Q}\{1, x, x^2\}, \quad H^*(\mathbb{P}^1) = \mathbb{Q}\{1, z\},$$

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## Step 2: typical example and expected grading map

$$G/B = \mathcal{F}\ell_3, \quad G/P = \mathbb{P}^2, \quad P/B = \mathbb{P}^1.$$

- $QH^*(G/B)$  is naturally labelled by a subset of  $W \times Q^\vee$ .  
Can define a grading map  $gr : W \times Q^\vee \rightarrow \mathbb{Z}^2$



## Step 3: statements $\rightarrow$ modifications $\rightarrow$ proofs

- Can define a filtration  $\mathcal{F} = \{F_{\mathbf{a}}\}_{\mathbf{a} \in \mathbb{Z}^2}$  on  $QH^*(G/B)$  similarly.

- ▶  $\mathcal{F}$  is a deformation of the classical one.
- ▶  $\mathcal{F} = \{F_{\mathbf{a}}\}_{\mathbf{a} \in \mathbb{Z}^2}$  define a  $\mathbb{Z}^2$ -filtered algebra structure on  $QH^*(G/B)$ . That is,  $F_{\mathbf{a}} * F_{\mathbf{b}} \subset F_{\mathbf{a} + \mathbf{b}}$ .

**Guess** that  $QH^*(G/B)$  is a  $\mathbb{Z}^2$ -filtered algebra, keeping similar properties to the classical ones'.

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- Check one more example of type  $A_3$ . Need to modify the “statement”.  
**Correction:** consider a  $\mathbb{Z}^{r+1}$ -filtration instead, where  $r = |\Delta_P|$ .

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- **Proofs** by using **induction** together with quantum Chevalley formula and Peterson-Woodward comparison formula.

## PW-lifting map $\psi_{\Delta, \Delta_P}$

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- 

$$\begin{aligned}\psi_{\Delta, \Delta_P} : W^P \times Q^\vee / Q_P^\vee &\hookrightarrow W \times Q^\vee; \\ (w, \lambda_P) &\mapsto (w\omega_P\omega_{P'}, \lambda_B).\end{aligned}$$

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- ▶ E.g. type  $A_2$ , for  $\lambda_P = \alpha_2^\vee + Q_P^\vee$  where  $\Delta_P = \{\alpha_1\}$ , we have  $\psi_{\Delta, \Delta_P}(1, \lambda_P) = (s_1, \alpha_2^\vee)$  (note  $\omega_P = s_1$  and  $\omega_{P'} = \text{id}$ ).

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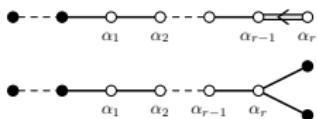
- $$\begin{aligned}\psi_{\Delta, \Delta_P} : W^P \times Q^\vee / Q_P^\vee &\hookrightarrow W \times Q^\vee; \\ (w, \lambda_P) &\mapsto (w\omega_P\omega_{P'}, \lambda_B).\end{aligned}$$

- ▶ E.g. type  $A_2$ , for  $\lambda_P = \alpha_2^\vee + Q_P^\vee$  where  $\Delta_P = \{\alpha_1\}$ , we have  $\psi_{\Delta, \Delta_P}(1, \lambda_P) = (s_1, \alpha_2^\vee)$  (note  $\omega_P = s_1$  and  $\omega_{P'} = \text{id}$ ).
- Peterson-Woodward comparison formula does not seem to tell us information on the relationships between the ring structure of  $QH^*(G/P)$  and  $QH^*(G/B)$ .

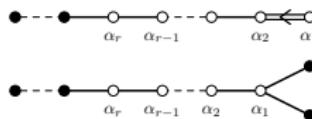
# Definition of grading map

- ① Give a “canonical order” on  $\Delta_P$ :

USE



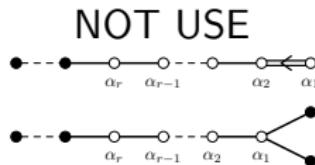
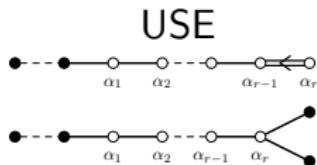
NOT USE



Set  $\Delta_{P_j} := \{\alpha_1, \alpha_2, \dots, \alpha_j\}$  for  $1 \leq j \leq r$ .  $P_{r+1} := G$ .

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- ② Considering  $\{P_j/B \rightarrow P_{j+1}/B \longrightarrow P_{j+1}/P_j\}_{j=1}^r$

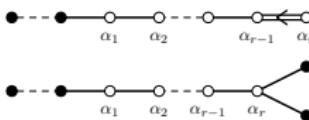
Obtain a  $\mathbb{Z}^{r+1}$ -filtration on  $H^*(G/B)$ , by defining (for  $w \in W$ )

$$gr(w) = \sum_{j=1}^r |Inv(w) \cap (R_{P_j}^+ \setminus R_{P_{j-1}}^+)| \mathbf{e}_j \in \mathbb{Z}^{r+1},$$

where  $Inv(w) = \{\beta \in R^+ \mid w(\beta) \notin R^+\}$  and  $R_{P_0}^+ := \emptyset$ .

## Definition of grading map

- ① Give a “canonical order” on  $\Delta_P$ :  $\Delta_{P_j}$ ’s.
- ② Define a grading map  $gr : W \rightarrow \mathbb{Z}^{r+1}$ .
- ③ Extend the grading map to  $gr : W \times Q^\vee \rightarrow \mathbb{Z}^{r+1}$ .  
Remain to define  $gr(q_j)$ ’s recursively (by using Peterson-Woodward comparison formula).



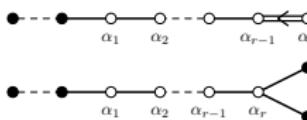
For instance, if  $\Delta_P$  is of  $A$ -type, then we define

- $gr(q_j) = (1-j)\mathbf{e}_{j-1} + (1+j)\mathbf{e}_j$ , for  $1 \leq j \leq r$ .

- Obtain a  $\mathbb{Z}^{r+1}$ -filtration  $\mathcal{F}$  via the grading map  $gr$ .

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where  $\omega_P \omega_{P'}$  and  $a_i$ ’s satisfy

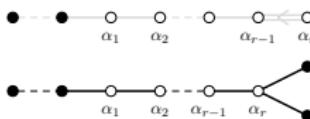
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- $QH^*(G/B)$  has an  $\mathbb{Z}^{r+1}$ -filtered algebra structure with filtration  $\mathcal{F}$ .
- There is a canonical algebra isomorphism

$$QH^*(G/B)/\mathcal{I} \xrightarrow{\sim} QH^*(P/B)$$

for an ideal  $\mathcal{I}$  (which is explicitly defined) of  $QH^*(G/B)$ .

		$gr(q_1^a q_2^b y) = agr(q_1) + bgr(q_2) + gr(y)$									
		$P/B$								$G/P$	
		4									
		3									
		2	$q_1$								
		1	$z$	$zx$	$zx^2$						
		0	1	$x$	$x^2$	$q_2 z$					
		-1	0	0	0	$q_2$					
		-2	0	0	0	0	0	0			
$i$	$j$	0	1	2	3	4	5	6			

# Theorem (Leung-Li, math.AG 1007.1683)

Suppose  $\text{Dyn}(\Delta_P)$  is a disjoint union of  $A$ -type ones  $\Delta^{(k)}$ 's.

- There exists a subalgebra  $\mathcal{A}$  of  $QH^*(G/B)$  together with an ideal  $\mathcal{J}$  of  $\mathcal{A}$ , such that  $QH^*(G/P)$  is canonically isomorphic to  $\mathcal{A}/\mathcal{J}$  as algebras.
- As graded algebras, (after localization)  $Gr^{\mathcal{F}}(QH^*(G/B))$  is isomorphic to  $(\bigotimes_k \bigotimes_{i_k=1}^{r_k} QH^*(\mathbb{P}^{i_k})) \otimes QH^*(G/P)$ , where  $r_k := |\Delta^{(k)}|$ .

		$gr(q_1^a q_2^b y) = agr(q_1) + bgr(q_2) + gr(y)$							
		P/B							
		4	3	2	1	0	-1	-2	$i/j$
4									
3									
2	q <sub>1</sub>								
1	z	$zx$	$zx^2$						
0	1	x	$x^2$	q <sub>2</sub> z					
-1	0	0	0	q <sub>2</sub> z					
-2	0	0	0	0	0	0			
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Furthermore for each remaining type, more than half of the homogeneous varieties  $G/P$ ’s also satisfy this.
- (In preparation.) Could expect that for the remaining cases,  $QH^*(G/P)$  is isomorphic to a graded subalgebra  $\mathcal{A}'$  of  $Gr_{(r+1)}^{\mathcal{F}}(QH^*(G/B)) := \bigoplus_{k \geq 0} Gr_{k\mathbf{e}_{r+1}}^{\mathcal{F}}$ . (It is always a proper subalgebra except for one special case.)

## One more question

How about the analog on the side of based loop groups?

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$$?? \rightarrow \Omega K \rightarrow ??$$

↷ a similar filtration on  $H_*(\Omega K)$ ?

Thank you!!...