PROPERTIES OF k-SCHUR FUNCTIONS, STATE OF AFFAIRS

NOTES FROM THE TALK OF LUC LAPOINTE

Abstract. First Lecture: Definitions We will present the various conjecturally equivalent characterizations of k-Schur functions $s_\lambda^{(k)}[X;t]$: the tableau atom definition, the algebraic definition using Jings operators and the definition as sums over strong tableaux with spin. We will give an overview of the properties of k-Schur functions and specify whether they are known to hold in each characterization.

Second lecture: Atomic properties. We will focus on the tableau definition of k-Schur functions. We will discuss the notions of katabolism, cyclage, Lascoux-Schutzenberger action of the symmetric group on words, etc. We will introduce the concept of copies of atoms and explain the meaning of the Pieri rule and branching coefficients (decomposition of k-Schurs into k+1-Schurs) in this context.

Third lecture: k-poset We will introduce a poset (the k-poset) on a certain type of partitions called k-shapes that allows to give an explicit expression for the branching coefficients. We will explain how the k-poset is compatible with the concept of charge and with the Lascoux-Schtzenberger action of the symmetric group on words. Finally, we will present a conjecture that relates the k-poset and tableau atoms, and give some open problems that arise from the k-poset.

Yesterday I gave two defintioins

I) One was a recursive combinatorial definition of an 'atom' (a set of tableaux)

$$A_\lambda^{(k)} = \mathcal{F}(\mathbb{A}_\lambda^{(k)}) = \mathcal{F}(\mathbb{K}_{\lambda\to k}\mathbb{B}_{\lambda_1}\mathbb{A}_{(\lambda_2,\lambda_3,\ldots,\lambda_r)})$$

II) The other was an algebraic definition in terms of symmetric functions

$$\tilde{A}_\lambda = T_{\lambda_1} B_{\lambda_1} \tilde{A}_{(\lambda_2,\lambda_3,\ldots,\lambda_r)}$$

We realized that these definitions were concrete realizations of what existed in the computer and we discovered by computer experimentation but what we really wanted was a definition which guaranteed that the k-Pieri rule holds at $t=1$, but unfortunately we are unable to prove the k-Pieri rules from these definitions.

We realized that there was a connection between these objects and the quantum cohomology but unable to prove it with these definitions.

$$s_\lambda^{(k)} = \sum_u x^{wt(u)}$$

where the sum is over all u which are strong tableaux of shape $core(\lambda)$.

Jennifer grouped these by the partition, but this definition should also be the same.

We have two proofs of this. Both are quite complicated (two 80-some-odd page papers). It is not quite obvious but that there is a bijection between words and pairs of k-tableau and dual-k tableau.

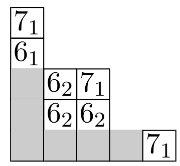
$$w \leftrightarrow (P, Q)$$

where P is a k-tableaux (weak) and Q is a dual k-tableau strong.

Sami Assaf and Sara Billey have shown something even stronger, namely that:

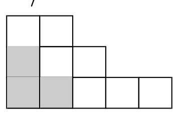
$$s_\lambda^{(k)} = \sum_u t^{spin(u)} x^{wt(u)}$$

where the sum is again over u which are strong tableaux of shape $core(\lambda)$ is Schur positive.
Example: $k = 3$



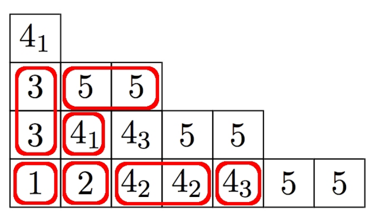
$\lambda = (5, 3, 3, 1, 1)$ is a cover of $\mu = (4, 1, 1)$.

Example/definition: The height of a ribbon is the number of rows that the ribbon occupies.

$\lambda/\mu =$  has height 3.

The spin of a tableaux is $(h - 1) \times$ (# of ribbons) + (# of ribbons above the marked one).

Spin of a strong tableau = sum of the spins of each cover.
Example: Here is an example of a strong tableau using the definition of a marked cover that was given by in Jennifer's talk yesterday. Here we have a 3 core.



The circled groups of labels are the marked ribbons.

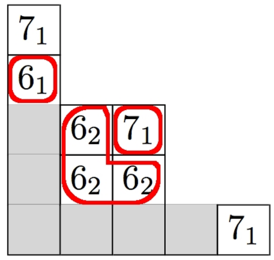
This tableau has weight (11131) because there are three groups of cells labeled 4 (note that they are labeled 4_1, 4_2 and 4_3) and one each for the labels 1,2,3,5.

If we want to calculate the spins we have $0 + 0 + 1 + 1 + 0 + 1 + 0 = 3$.

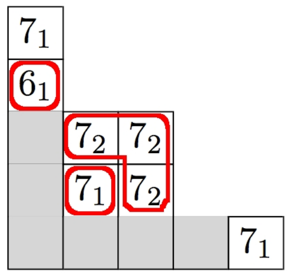
We claim that $s_\lambda^{(k)}$ is a symmetric function and that it is Schur positive. This was proved by Assaf and Billey using dual equivalence graphs.

Want: a spin preserving involution which switches weights $(\alpha_1, \alpha_2, \dots, \alpha_n)$ to $(\alpha_1, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_n)$.

The following tableau has spin 2 and two 6 labels and one 7 label



but note that in this one the 7's are not subscripted (I think he left it off as shorthand). And we want this to be switched with



because this has two 7 labels and one 6 label and also has spin two.

One thing that you might be a bit confused about is "why the names 'strong' and 'weak'?" "Strong" comes from the strong bruhat order of $\tilde{S}_{k+1}$ and "weak" from the weak order.

What we have is a bijection from Grassmannian permutations $\leftrightarrow$ $(k+1)$-cores. On the left hand side of this bijection you have a strong covers in the Bruhat order, on the right you have (containment of cores and degree increases by one).

The following is a list of conjectured properties that are held by k-Schur functions as well as some comments about what has been proven and what has not been proven for each of the three definitions. $A_\lambda^{(k)}$, $\tilde{A}_\lambda^{(k)}$, $s_\lambda^{(k)}$ are three defintions and they are conjectured to be equivalent.

Properties:

(1) k-Schur reduces to the Schur function when k is large.
    This is proven for $A_\lambda^{(k)}$, $\tilde{A}_\lambda^{(k)}$, $s_\lambda^{(k)}$
(2) Schur positivity
    $A_\lambda^{(k)}$ this is automatic by definition
    $\tilde{A}_\lambda^{(k)}$ this seems like it is hopeless to prove unless you make connection between these definitions (or if you prove Schur positivity of the k-split basis).
    $s_\lambda^{(k)}$ yes, by Assaf and Billey

(3) satisfies the Pieri rule (at t=1)
  $A_\lambda^{(k)}$ we do not have a proof of this
  $\tilde{A}_\lambda^{(k)}$ we do not have a proof of this
  $s_\lambda^{(k)}$ yes, by definition

(4) k-conjugation (at $t = 1$) Neat property: If λ is a core then $\lambda^{(k)}$ is its conjugate. then $\omega(s_\lambda) = s_{\lambda'}$ then $\omega(s_\lambda^{(k)}) = s_{\lambda^{(k)}}^{(k)})$.
  $A_\lambda^{(k)}$ and $\tilde{A}_\lambda^{(k)}$ we do not have a proof of this
  $s_\lambda^{(k)}$ we do have a proof of this.

(5) for t generic k-conjugation Neat property: If λ is a core then $\lambda^{(k)}$ is its conjugate. then $\omega(s_\lambda) = s_{\lambda'}$ then $\omega(s_\lambda^{(k)}[X;t]) = t^* s_{\lambda^{(k)}}^{(k)}[X;1/t]$.
  $A_\lambda^{(k)}$ and $\tilde{A}_\lambda^{(k)}$ we do not have a proof of this
  $s_\lambda^{(k)}[X;t]$ we do not have a proof of this yet, but we are hopeful that we have a way of attacking this problem.

(6) That these elements form a basis for $\Lambda_t^{(k)}$
  $A_\lambda^{(k)}$ we do not have a proof of this
  $\tilde{A}_\lambda^{(k)}$ there is a a proof of this
  $s_\lambda^{(k)}$ we do not have a proof of this and we don't even know that these elements are in the correct space.

(7) k-rectangles
  A rectangular partition whose maximal hook length is k ($\ell^{(k-\ell+1)}$)
  At $t = 1$,
$$s_\lambda^{(k)} = s_{R_1} s_{R_2} \cdots s_{R_\ell} s_\mu^{(k)}$$
where $R_1, \dots, R_\ell$ are k-rectangles. μ is k-irreducible.
$$s^{(4)}_{(44333222222111111)} = s_4 s_4 s_{33} s_{222} s_{222} s_{1111} s^{(4)}_{311}$$
where the (311).

What is really interesting is that there are $k!$ irreducible elements (as there are $k!$ k-bounded partitions with no k-rectangles). What I believe is that when we really understand k-Schur functions we will see that these irreducible elements will be at the heart of the matter.

When $k = \infty$ all partitions index irreducible.

When $t = 1$ the rectangle property for $A_\lambda^{(k)}$ we do not have a proof of this

$\tilde{A}_\lambda^{(k)}$ we do have a proof of this (it was one of the hardest things we have done) and it holds for all t.

$s_\lambda^{(k)}$ we have a proof of this.

(8) In general (when t is not necessarily 1) we observe the the analogous property
$$s_\lambda^{(k)}[X;t] = B_{R_1} B_{R_2} \cdots B_{R_\ell} s_\mu^{(k)}[X;t]$$

where the B_R are the vertex operators of Shimozono and Zabrocki.
in the cases $A_\lambda^{(k)}$, $s_\lambda^{(k)}$ we do not have a proof of this
$\tilde{A}_\lambda^{(k)}$ we do have a proof of this

(9) k-Littlewood-Richardson rule when $t = 1$

$$s_\mu^{(k)} s_\nu^{(k)} = \sum c_{\mu\nu}^{\lambda(k)} s_\lambda^{(k)}$$

We observe that when we multiply these elements multiply positive
$A_\lambda^{(k)}$, $\tilde{A}_\lambda^{(k)}$ we do not know how this holds
but for $s_\lambda^{(k)}$ through the work of Thomas Lam we know that it is true.
One of the main open problems we hope to arrive at a combinatorial rule for calculating the $c_{\mu\nu}^{\lambda(k)}$.

(10) Connection with the geometry of affine Grassmannian
Connection with quantum cohomoogy of flags
$A_\lambda^{(k)}$, $\tilde{A}_\lambda^{(k)}$ we do not know how this holds
$s_\lambda^{(k)}$ there is a proof of this

(11) Affine insertion at $t = 1$ (this is about to be published in Memoirs of AMS) we know that $s_\lambda^{(k)}$ satisfies a k-branching and k-poset.
Most important fact is that k-Schur is $(k+1)$-Schur positive
$A_\lambda^{(k)}$, $\tilde{A}_\lambda^{(k)}$ we do not know how this holds

Remark: (ask about this) at $t = 1$ the coproduct is positive
or equivalently $\mathfrak{S}_\lambda^{(k)} \mathfrak{S}_\mu^{(k)}$ is Schur positive.

The main conjecture is that these three bases are equivalent.

$$A_\lambda^{(k)} =?= \tilde{A}_\lambda^{(k)} =?= s_\lambda^{(k)}$$

Our computer algorithms are all implemented with the first definition. We have checked that the first and the last are the same up to degree 20 (not checked so high for generic t or between the first and the second).