Another basis and pattern for irreducible modules

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Simple complex Lie algebra

- $\mathfrak{g}$ simple complex Lie algebra
- $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$
- P, P^+ (dominant) weights, R, R^+ (positive) roots
- $\mathfrak{n}^+ = \langle e_\alpha \mid \alpha \in R^+ \rangle, \mathfrak{n}^- = \langle f_\alpha \mid \alpha \in R^+ \rangle, \mathfrak{h} = \langle \alpha^\vee \mid \alpha \in R^+ \rangle$

For $\lambda \in P^+$, $V(\lambda)$ denotes the irreducible highest weight module of highest weight λ generated by a nonzero highest weight vector v_λ.

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Current algebra

The *current algebra* is defined to be

$$\mathfrak{g} \otimes \mathbb{C}[t]$$

with Lie bracket

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}.$$

$$\mathfrak{g} = \mathfrak{g} \otimes 1 \subset \mathfrak{g} \otimes \mathbb{C}[t].$$

Given $a \in \mathbb{C}$, then one can define a $\mathfrak{g} \otimes \mathbb{C}[t]$-action on $V(\lambda)$ by

$$x \otimes t.v = ax.v \text{ for } v \in V.$$

Clearly, the $\mathfrak{g}$-action is not changed!

Fusion product

For $\lambda, \mu \in P^+$ and $a \neq b \in \mathbb{C}$ the tensor product $V_a(\lambda) \otimes V_b(\mu)$ is cyclic generated by $v_\lambda \otimes v_\mu$. The natural filtration on $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])$, given by the degree of t, induces a filtration on $V_a(\lambda) \otimes V_b(\mu)$.
The associated graded module is called the *fusion product* (due to B.Feigin/Loktev), denoted by $V_a(\lambda) * V_b(\mu)$. Clearly

$$V_a(\lambda) \otimes V_b(\mu) \cong_{\mathfrak{g}} V_a(\lambda) * V_b(\mu).$$

Since the module is cyclic one has

$$V_a(\lambda) * V_b(\mu) \cong \mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])/I_{a,b}(\lambda, \mu)$$

for some ideal $I_{a,b}(\lambda, \mu)$.
How does this ideal look like? And why might this be interesting?

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How does this ideal look like? And why might this be interesting?

Conjecture

The ideal $I_{a,b}(\lambda, \mu)$ is generated by the set

- $\mathfrak{n}^+ \otimes \mathbb{C}[t]$
- $h \otimes t^r - 0^r(\lambda + \mu)(h)$
- $(f_\alpha \otimes 1)^{(\lambda+\mu)(\alpha^\vee)+1}$
- $(f_\alpha \otimes t)^{min\{\lambda(\alpha^\vee),\mu(\alpha^\vee)\}+1}$.
- $\mathfrak{g} \otimes t^2\mathbb{C}[t]$

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- The ideal contains these elements, so $V_a(\lambda) * V_b(\mu)$ is a quotient of $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])/(\text{elements})$.
- The ideal would be independent of a, b.
- The subalgebra $\mathbf{U}(\mathfrak{g} \otimes t\mathbb{C}[t])$ acts as a commutative algebra.

Consider the subspace $\mathbf{U}(\mathfrak{g} \otimes t\mathbb{C}[t]).1$, it decomposes into $\mathfrak{h}$ weight spaces, and the weight multiplicity is conjectured to be $c^{\tau}_{\lambda,\mu}$, the multiplicity of $V(\mu)$ in the tensor product $V(\lambda) \otimes V(\mu)$.

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PBW filtration

There is a natural filtration on $\mathbf{U}(\mathfrak{n}^-)$, called the "degree filtration"

$$U(\mathfrak{n}^-)_s := \{x_1 \cdots x_l \mid x_j \in \mathfrak{n}_-; j \leq s\}$$

with

$$gr\mathbf{U}(\mathfrak{n}^-) \cong S(\mathfrak{n}^-).$$

$V(\lambda) = \mathbf{U}(\mathfrak{n}^-).v_\lambda$, so there is an induced filtration, called "PBW filtration", on $V(\lambda)$ given by

$$V(\lambda)_s = \mathbf{U}(\mathfrak{n}^-)_s v_\lambda$$

We are mainly interested in the associated graded space $grV(\lambda)$.
$grV(\lambda)$ is a $S(\mathfrak{n}^-)$-module by construction and since $V(\lambda)$ is cyclic, we have for some ideal $I(\lambda) \subset S(\mathfrak{n}^-)$

$$grV(\lambda) \simeq S(\mathfrak{n}^-)/I(\lambda)$$

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Main theorem

- $f_\alpha^{\lambda(\alpha^\vee)+1} \in I(\lambda)$
- $\mathbf{U}(n^+)$ is acting on $grV(\lambda)$, since $\mathfrak{n}^+ V(\lambda)_s \subset V(\lambda)_s$, so

$$\mathbf{U}(\mathfrak{n}^+) \circ \text{ span}\{f_\alpha^{\lambda(\alpha^\vee)+1}\} \subset I(\lambda).$$

From now on let $\mathfrak{g}$ be of type A_n.

Theorem (FFL)

We have for $\lambda \in P^+$

$$I(\lambda) = S(\mathfrak{n}^-)\left(\mathbf{U}(\mathfrak{n}^+) \circ \ span\{f_\alpha^{\lambda(\alpha^\vee)+1}\}\right)$$

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How to prove this?

Find a generating set for $S(\mathfrak{n}^-)/I(\lambda)$ and show it parametrizes a linear independent set in $V(\lambda)$.

As a byproduct we obtain a new class of pattern and basis for irreducible A_n-modules. This basis was conjectured by Vinberg (2005).

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Dyck path

Denote by α_i the simple roots for A_n and set

$$\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j,$$

all positive roots for A_n have this form.

A *Dyck path* $\mathbf{p}$ of length k is a sequence of roots $\beta(0), \ldots, \beta(k)$ satisfying the following rules

- If $k = 0$, then $\beta(0) = \alpha_i$ for some i, so assume $k > 0$:
- $\beta(0), \beta(k)$ are simple roots, say $\beta(0) = \alpha_i, \beta(k) = \alpha_j$, with $i < j$.
- If $\beta(\ell) = \alpha_{i,j}$, then $\beta(\ell) = \alpha_{i,j+1}$ or $\beta(\ell) = \alpha_{i+1,j}$

Example

$\mathbf{p} = (\alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4, \alpha_4 + \alpha_5, \alpha_5)$

Denote by $\mathbb{D}$ the set of all Dyck paths.

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Dyck path

Denote by α_i the simple roots for A_n and set

$$\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j,$$

all positive roots for A_n have this form.
A *Dyck path* **p** of length k is a sequence of roots $\beta(0), \ldots, \beta(k)$ satisfying the following rules

- If $k = 0$, then $\beta(0) = \alpha_i$ for some i, so assume $k > 0$:
- $\beta(0), \beta(k)$ are simple roots, say $\beta(0) = \alpha_i, \beta(k) = \alpha_j$, with $i < j$.
- If $\beta(\ell) = \alpha_{i,j}$, then $\beta(\ell) = \alpha_{i,j+1}$ or $\beta(\ell) = \alpha_{i+1,j}$

Example

$\mathbf{p} = (\alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4, \alpha_4 + \alpha_5, \alpha_5)$

Denote by $\mathbb{D}$ the set of all Dyck paths.

To have a picture:

$$\begin{pmatrix}
\alpha_1 & & & & & & & \\
\alpha_{12} & \alpha_2 & & & & & & \\
 & \downarrow & & & & & & \\
\alpha_{123} & \alpha_{23} & & \alpha_3 & & & & \\
 & \downarrow & & & & & & \\
\alpha_{1234} & \alpha_{234} & \rightarrow & \alpha_{34} & \rightarrow & \alpha_4 & & \\
 & & & & & \downarrow & & \\
\alpha_{12345} & \alpha_{2345} & & \alpha_{345} & & \alpha_{45} & \rightarrow & \alpha_5
\end{pmatrix}$$

Polytope

Let $\lambda = \sum m_i\omega_i \in P^+$, define $P(\lambda) \subset \mathbb{R}_{\geq 0}^{\sharp\text{roots}}$ by

$$P(\lambda) := \left\{ (s_\alpha)_{\alpha>0} \mid \begin{array}{l} \forall \mathbf{p} \in \mathbb{D}: \text{ If } \beta(0) = \alpha_i, \beta(k) = \alpha_j, \text{ then} \\ s_{\beta(0)} + \cdots + s_{\beta(k)} \leq m_i + \cdots + m_j \end{array} \right\}.$$

Let $S(\lambda)$ be the set of integer points in $P(\lambda)$

$$S(\lambda) = P(\lambda) \cap \mathbb{Z}_{\geq 0}^{\sharp\text{roots}}.$$

Pattern

$$
\left(\begin{array}{cccccccc}
\mathbf{m_1} & & & & & & & \\
s_1 & & & & & & & \\
 & \mathbf{m_2} & & & & & & \\
s_{12} & s_2 & & & & & & \\
 & \downarrow & & \mathbf{m_3} & & & & \\
s_{123} & s_{23} & & s_3 & & & & \\
 & \downarrow & & & & \mathbf{m_4} & & \\
s_{1234} & s_{234} & \rightarrow & s_{34} & \rightarrow & s_4 & & \\
 & & & & & \downarrow & & \mathbf{m_5} \\
s_{12345} & s_{2345} & & s_{345} & & s_{45} & \rightarrow & s_5
\end{array}\right)
$$

Example

For $\mathfrak{g}$ of type A_2, there are only three Dyck paths

- the two of length 1 corresponding to the simple roots
- the path which involves all positive roots.

For $\lambda = m_1\omega_1 + m_2\omega_2$ the associated polytope is

$$P(\lambda) = \left\{ \begin{pmatrix} s_1 & \\ s_{12} & s_2 \end{pmatrix} \mid \begin{array}{l} 0 \le s_1 \le m_1, 0 \le s_2 \le m_2, \\ s_1 + s_2 + s_{12} \le m_1 + m_2 \end{array} \right\},$$

This is just a transformation of the Gelfand-Tsetlin pattern for A_2 and highest weight λ.

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Basis

For a tuple $\mathbf{s} = (s_\alpha)_\alpha$ we define

$$f^{\mathbf{s}} = \prod_\alpha f_\alpha^{s_\alpha} \in \mathbf{S}(\mathfrak{n}^-)$$

Theorem (FFL)

The set $\{\prod f_\beta^{s_\beta} v_\lambda \mid \mathbf{s} \in S(\lambda)\}$ *is a basis for* $grV(\lambda)$.

If we fix an order in every tuple $\mathbf{s}$, then we obtain

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This concept also works for type C_n, the definition of a Dyck path has to be adjusted.
A Dyck path ends either in a simple root or in the highest root of a C_r subalgebra of C_n (coming from a C_r subdiagram).
The polytope, pattern and basis is defined in the same way.

GT pattern

There are a lot of other patterns for irreducible modules already known, for example Gelfand-Tsetlin pattern $GT(\lambda)$. In this pattern there are $\sharp$roots-variables

$$r_{i,j} \mid 1 \leq j \leq n, j \leq i \leq n,$$

and two inequalities for every variable

$$r_{i-1,j-1} \geq r_{i,j} \geq r_{i,j-1},$$

where $r_{i,0} := m_n + \ldots + m_{i+1}$, for $i = 0, \ldots, n$.

Compare to $S(\lambda)$:

- Same polytope for A_2.
- For $n > 2$, $S(\lambda)$ has more inequalities, much more. Every Dyck path gives an inequality and the number of Dyck paths is huge.
- So what is the advantage??

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Advantages

- We obtain generators and relation for $grV(\lambda)$.
- Obvious generalization to arbitrary types (exists for GT-pattern as well, more complicated).
- We obtain an graded character formula.
- With the Minkowski sum we have $S(\lambda) + S(\mu) = S(\lambda + \mu)$. Is $P(\lambda) + P(\mu) = P(\lambda + \mu)$??
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$$V(\lambda)_a * V(\lambda)_b \cong \mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])/I_{a,b}(\lambda,\mu)$$

Conjecture

The ideal $I_{a,b}(\lambda,\mu)$ is generated by the set

$$\mathfrak{n}^+ \otimes \mathbb{C}[t], h \otimes t^r - 0^r(\lambda+\mu)(h), \mathfrak{g} \otimes t^2\mathbb{C}[t]$$

$$(f_\alpha \otimes 1)^{(\lambda+\mu)(\alpha^\vee)+1}, (f_\alpha \otimes t)^{min\{\lambda(\alpha^\vee),\mu(\alpha^\vee)\}+1}.$$

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