Another basis and pattern for irreducible modules

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- P, P^+ (dominant) weights, R, R^+ (positive) roots

$$\bullet \ \mathfrak{n}^+ = \langle e_\alpha \mid \alpha \in R^+ \rangle, \, \mathfrak{n}^- = \langle f_\alpha \mid \alpha \in R^+ \rangle, \, \mathfrak{h} = \langle \alpha^\vee \mid \alpha \in R^+ \rangle$$

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Current algebra

The current algebra is defined to be

$$\mathfrak{g}\otimes\mathbb{C}[t]$$

with Lie bracket

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}.$$

$$\mathfrak{g} = \mathfrak{g} \otimes \mathfrak{1} \subset \mathfrak{g} \otimes \mathbb{C}[t].$$

Given $a \in \mathbb{C}$, then one can define a $\mathfrak{g} \otimes \mathbb{C}[t]$ -action on $V(\lambda)$ by

$$x \otimes t.v = ax.v$$
 for $v \in V$.

Clearly, the g-action is not changed!



Fusion product

For $\lambda, \mu \in P^+$ and $a \neq b \in \mathbb{C}$ the tensor product $V_a(\lambda) \otimes V_b(\mu)$ is cyclic generated by $v_\lambda \otimes v_\mu$. The natural filtration on $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])$, given by the degree of t, induces a filtration on $V_a(\lambda) \otimes V_b(\mu)$.

The associated graded module is called the *fusion product* (due to B.Feigin/Loktev), denoted by $V_a(\lambda) * V_b(\mu)$. Clearly

$$V_a(\lambda) \otimes V_b(\mu) \cong_{\mathfrak{g}} V_a(\lambda) * V_b(\mu).$$

Since the module is cyclic one has

$$V_a(\lambda) * V_b(\mu) \cong \mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])/I_{a,b}(\lambda,\mu)$$

for some ideal $I_{a,b}(\lambda,\mu)$.

How does this ideal look like? And why might this be interesting?

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How does this ideal look like? And why might this be interesting?

- ullet $\mathfrak{n}^+\otimes \mathbb{C}[t]$
- $h \otimes t^r 0^r (\lambda + \mu)(h)$
- $(f_{\alpha} \otimes 1)^{(\lambda+\mu)(\alpha^{\vee})+1}$
- $(f_{\alpha} \otimes t)^{\min\{\lambda(\alpha^{\vee}),\mu(\alpha^{\vee})\}+1}$.
- $\mathfrak{g} \otimes t^2 \mathbb{C}[t]$

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- The ideal contains these elements, so $V_a(\lambda) * V_b(\mu)$ is a quotient of $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])/(\text{elements})$.
- The ideal would be independent of *a*, *b*.
- The subalgebra $\mathbf{U}(\mathfrak{g} \otimes t\mathbb{C}[t])$ acts as a commutative algebra.

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There is a natural filtration on $\mathbf{U}(\mathfrak{n}^-)$, called the "degree filtration"

$$U(\mathfrak{n}^-)_s := \{x_1 \cdots x_l \mid x_j \in \mathfrak{n}_-; j \leq s\}$$

with

$$gr\mathbf{U}(\mathfrak{n}^-)\cong \mathcal{S}(\mathfrak{n}^-).$$

 $V(\lambda) = \mathbf{U}(\mathfrak{n}^-).v_{\lambda}$, so there is an induced filtration, called "PBW filtration", on $V(\lambda)$ given by

$$V(\lambda)_{\mathcal{S}} = \mathbf{U}(\mathfrak{n}^-)_{\mathcal{S}} V_{\lambda}$$

We are mainly interested in the associated graded space $grV(\lambda)$.

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- $f_{\alpha}^{\lambda(\alpha^{\vee})+1} \in I(\lambda)$
- $U(n^+)$ is acting on $grV(\lambda)$, since $\mathfrak{n}^+V(\lambda)_s\subset V(\lambda)_s$, so

$$\mathbf{U}(\mathfrak{n}^+) \circ \operatorname{span}\{f_{\alpha}^{\lambda(\alpha^{\vee})+1}\} \subset I(\lambda)$$

From now on let \mathfrak{g} be of type A_n

Theorem (FFL)

We have for $\lambda \in P^{-}$

$$I(\lambda) = S(\mathfrak{n}^-) \left(\mathbf{U}(\mathfrak{n}^+) \circ span\{f_{\alpha}^{\lambda(\alpha^{\vee})+1}\} \right)$$

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Motivation Another story Pattern and basis Conclusion

How to prove this?

Find a generating set for $S(\mathfrak{n}^-)/I(\lambda)$ and show it parametrizes a linear independent set in $V(\lambda)$.

As a byproduct we obtain a new class of pattern and basis for irreducible A_n -modules. This basis was conjectured by Vinberg (2005).

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Dyck path

Denote by α_i the simple roots for A_n and set

$$\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j,$$

all positive roots for A_n have this form.

A *Dyck path* **p** of length k is a sequence of roots $\beta(0), \ldots, \beta(k)$ satisfying the following rules

- If k = 0, then $\beta(0) = \alpha_i$ for some i, so assume k > 0:
- $\beta(0), \beta(K)$ are simple roots, say $\beta(0) = \alpha_i, \beta(K) = \alpha_j$, with i < j.
- If $\beta(\ell) = \alpha_{i,i}$, then $\beta(\ell) = \alpha_{i,i+1}$ or $\beta(\ell) = \alpha_{i+1,i}$

Example

 $\mathbf{p} = (\alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4, \alpha_4 + \alpha_5, \alpha_5)$

Denote by \mathbb{D} the set of all Dyck paths.



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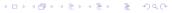
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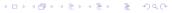
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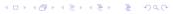
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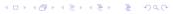
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To have a picture:

$$\begin{pmatrix} \alpha_{1} \\ \alpha_{12} & \alpha_{2} \\ \downarrow \\ \alpha_{123} & \alpha_{23} & \alpha_{3} \\ \downarrow \\ \alpha_{1234} & \alpha_{234} & \rightarrow & \alpha_{34} & \rightarrow & \alpha_{4} \\ \downarrow \\ \alpha_{12345} & \alpha_{2345} & \alpha_{345} & \alpha_{45} & \rightarrow & \alpha_{5} \end{pmatrix}$$

Polytope

Let
$$\lambda = \sum m_i \omega_i \in P^+$$
, define $P(\lambda) \subset \mathbb{R}_{\geq 0}^{\sharp \mathsf{roots}}$ by

$$P(\lambda) := \left\{ (s_{\alpha})_{\alpha > 0} \mid \begin{array}{l} \forall \mathbf{p} \in \mathbb{D} : \text{ If } \beta(0) = \alpha_{i}, \beta(k) = \alpha_{j}, \text{ then} \\ s_{\beta(0)} + \dots + s_{\beta(k)} \leq m_{i} + \dots + m_{j} \end{array} \right\}.$$

Let $S(\lambda)$ be the set of integer points in $P(\lambda)$

$$S(\lambda) = P(\lambda) \cap \mathbb{Z}_{>0}^{\sharp \mathsf{roots}}.$$

Pattern

For \mathfrak{g} of type A_2 , there are only three Dyck paths

- the two of length 1 corresponding to the simple roots
- the path which involves all positive roots.

For $\lambda = m_1 \omega_1 + m_2 \omega_2$ the associated polytope is

$$P(\lambda) = \left\{ \begin{pmatrix} s_1 \\ s_{12} & s_2 \end{pmatrix} \mid \begin{array}{l} 0 \le s_1 \le m_1, 0 \le s_2 \le m_2, \\ s_1 + s_2 + s_{12} \le m_1 + m_2 \end{array} \right\},$$



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For a tuple $\mathbf{s} = (s_{\alpha})_{\alpha}$ we define

$$f^{\mathbf{s}} = \prod_{lpha} f_{lpha}^{oldsymbol{s}_{lpha}} \in \mathbf{S}(\mathfrak{n}^-)$$

Theorem (FFL)

The set $\{\prod f^{s_{\beta}}_{\beta}v_{\lambda}\mid \mathbf{s}\in S(\lambda)\}$ is a basis for $grV(\lambda).$

If we fix an order in every tuple s, then we obtain

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Motivation Another story Pattern and basis Conclusion

This concept also works for type C_n , the definition of a Dyck path has to be adjusted.

A Dyck path ends either in a simple root or in the highest root of a C_r subalgebra of C_n (coming from a C_r subdiagram).

The polytope, pattern and basis is defined in the same way.

There are a lot of other patterns for irreducible modules already known, for example Gelfand-Tsetlin pattern $GT(\lambda)$. In this pattern there are \sharp roots-variables

$$r_{i,j} \mid 1 \leq j \leq n, j \leq i \leq n,$$

and two inequalities for every variable

$$r_{i-1,j-1} \geq r_{i,j} \geq r_{i,j-1},$$

where $r_{i,0} := m_n + \ldots + m_{i+1}$, for $i = 0, \ldots, n$.

Compare to $S(\lambda)$

- Same polytope for A₂
- For n > 2, $S(\lambda)$ has more inequalities, much more. Every Dyck path gives an inequality and the number of Dyck
 - paths is huge.
- So what is the advantage??



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- We obtain generators and relation for $grV(\lambda)$.
- Obvious generalization to arbitrary types (exists for GT-pattern as well, more complicated).
- We obtain an graded character formula.
- With the Minkowski sum we have $S(\lambda) + S(\mu) = S(\lambda + \mu)$. Is $P(\lambda) + P(\mu) = P(\lambda + \mu)$??
- We obtain pattern for certain Demazure modules.



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The ideal $I_{a,b}(\lambda,\mu)$ is generated by the set

$$\mathfrak{n}^+\otimes \mathbb{C}[t], n\otimes t^*=0^\circ(\lambda+\mu)(n), \mathfrak{g}\otimes t^*\mathbb{C}[t]$$

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