A Bijection on Core Partitions and a Parabolic Quotient of the Affine Symmetric Group

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First Description of Φ_ℓ^k

Fix $\ell \geq 2$, an integer.

Definition: ℓ-cores with first part k

We let $\mathcal{C}_\ell^k$ denote the set of ℓ-cores with first part k, and let $\mathcal{C}_\ell^{\leq k}$ denote the set of ℓ-cores with first part $\leq k$.

A map on ℓ-cores:

We define a map $\Phi_\ell^k : \mathcal{C}_\ell^k \to \mathcal{C}_{\ell-1}^{\leq k}$. To a partition $\lambda \in \mathcal{C}_\ell^k$, we just delete all rows i of λ if $h_{(i,1)}^\lambda \equiv h_{(1,1)}^\lambda \mod \ell$.

Theorem

Φ_ℓ^k is a bijection.

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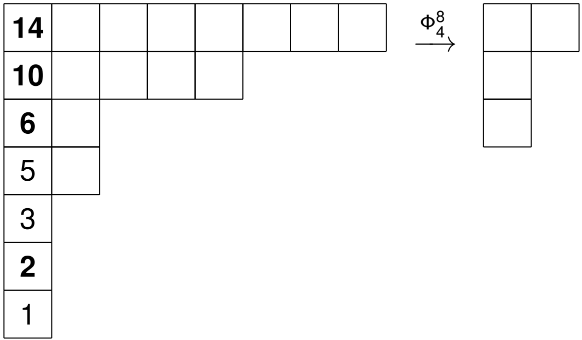
Example of Bijection: A 4-core with first part 8



From this description, it isn't obvious that Φ_4^8 maps a 4-core to a 3-core. It is also not obvious that this is a bijection. We introduce *abaci* to prove that this map is a bijection.

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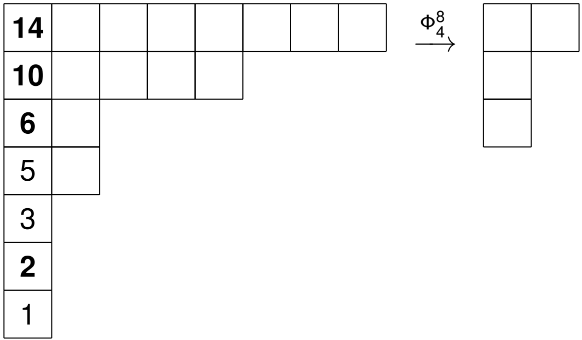

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Definition: Abacus

An **abacus diagram** is a diagram containing ℓ columns labeled $0, 1, \ldots, \ell - 1$, called **runners.** The horizontal cross-sections or rows will be called **levels** and runner *i* contains entries labeled by $r\ell + i$ on each level *r* where $-\infty < r < \infty$.

Definition: Beads and Gaps

Entries in the abacus diagram may be circled; such circled elements are called **beads.** Entries which are not circled will be called **gaps**.

Abaci corresponding to partitions

An abacus for λ will be any abacus diagram such that the i^{th} largest bead has λ_i gaps in smaller positions.

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Example of an Abacus: $\ell = 4$ and $\lambda = (10, 10, 4, 2, 2)$

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
-8 & -7 & -6 & -5 \\
-4 & -3 & -2 & -1 \\
0 & 1 & 2 & 3 \\
4 & 5 & 6 & 7 \\
8 & 9 & 10 & 11 \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

Description of Bijection on Abaci

Lemma: ℓ-cores on an abacus

A partition is an ℓ-core if and only if for every runner has no bead below a gap.

Proposition: Φ_ℓ^k on abaci

Given an abacus for λ, find the largest bead. Delete the entire runner containing it.

Why abaci?

In this setting, it is very easy to see that Φ_ℓ^k is invertible.

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$\xrightarrow{\Phi_4^8}$

⋮	⋮	⋮	⋮
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-3	-2	$\times$	-1
0	1	$\times$	2
3	4	$\times$	5
6	7	$\times$	8
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To show that Φ is inverible: We just need to insert a column of beads in such a way that the first part becomes 8.

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Connections to Lapointe Morse bijection

Definition

We let $\mathcal{P}_\ell^k$ denote the set of all ℓ bounded partitions with length k, and let $\mathcal{P}_\ell^{\leq k}$ denote the ℓ bounded partitions with length less than or equal to k.

Bijection

For $\lambda \in \mathcal{P}_\ell^k$, define $\Psi_\ell^k(\lambda)$ to be the partition obtained by removing the first column of λ. Then Ψ_ℓ^k is a bijection between $\mathcal{P}_\ell^k$ and $\mathcal{P}_{\ell-1}^{\leq k}$.

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Viewed in terms of Lapointe-Morse bijection

Theorem

The following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}_\ell^k \arrow[r, "\Phi_\ell^k"] \arrow[d, "t"] & \mathcal{C}_{\ell-1}^{\leq k} \arrow[d, "t"] \\
\mathcal{C}_\ell^{len=k} \arrow[d, "\rho_\ell"] & \mathcal{C}_{\ell-1}^{len\leq k} \arrow[d, "\rho_{\ell-1}"] \\
\mathcal{P}_{\ell-1}^k \arrow[r, "\Psi_{\ell-1}^k"] & \mathcal{P}_{\ell-2}^{\leq k}
\end{tikzcd}$$

All arrows here are bijections!

An equivalent description of Lapointe-Morse bijection

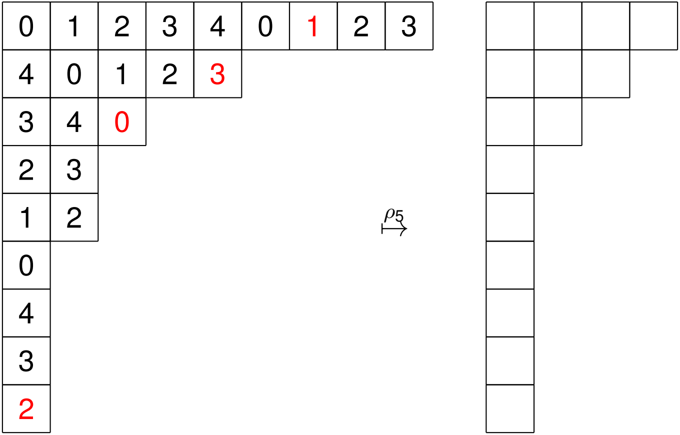
Out of the proof that the diagram commutes, we derived the following corollary.

Corollary

The Lapointe-Morse bijection can be described on ℓ-cores by keeping only the first column of each residue in a Young diagram.

Example

Let $\ell = 5$ and $\lambda = (9, 5, 3, 2, 2, 1, 1, 1, 1)$. We retain the columns $1, 3, 5, 7$ resulting in $\rho_5(\lambda) = (4, 3, 2, 1, 1, 1, 1, 1, 1)$.

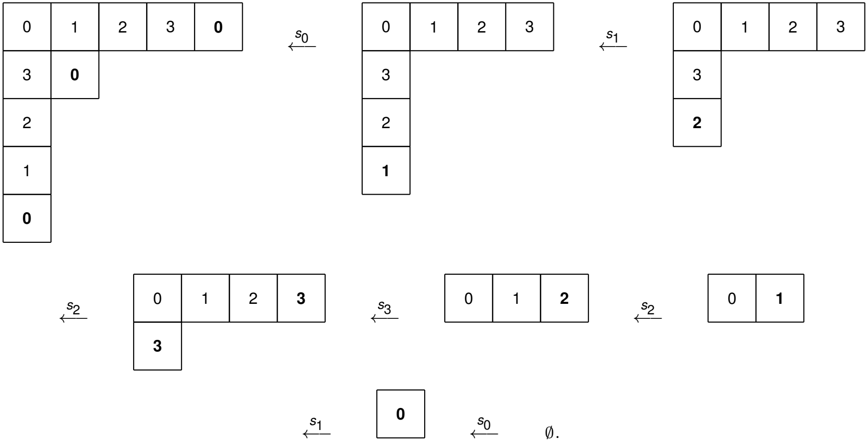


Association of cores to reduced words

Recall that ℓ-cores are also in bijection with the reduced words in $\widetilde{S_\ell}$ whose reduced words all end in s_0 (called affine grassmannian permutations).

Equivalently, these words are minimal length coset representatives of $\widetilde{S}_\ell/S_\ell$.

Example



This shows that the 4-core $(5, 2, 1, 1, 1)$ is identified with the affine grassmannian permutation $s_0 s_1 s_2 s_3 s_2 s_1 s_0$.

Bijection on Words

We can now interpret our bijection as a map

$$\Phi_\ell : \widehat{S}_\ell / S_\ell \to \widehat{S}_{\ell-1} / S_{\ell-1}$$

Proposition:

For a word w, $len(\Phi_\ell(w)) = len(w) - (w\emptyset)_1$.

For a description of Φ_ℓ, see our paper.

The Root Lattice

Pick an orthonormal basis $\epsilon_1, \dots, \epsilon_\ell$ of $\mathbb{R}^\ell$. Let $R = \{\epsilon_i - \epsilon_j : i \neq j\}$. Let Λ_R be the lattice of R and let V be the subspace of $\mathbb{R}^n$ spanned by R.

There is a natural action of S_ℓ on $\mathbb{R}^\ell$. S_ℓ just acts by permuting the indices. This actually gives an action of S_ℓ on R and hence on Λ_R and V.

The action of S_ℓ on V can be extended to an action of $\widetilde{S_\ell}$: the generator s_0 acts by reflection of an affine hyperplane.

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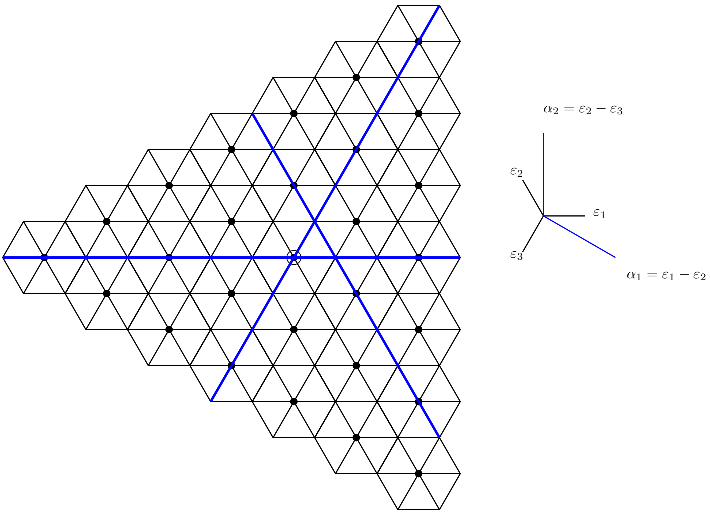
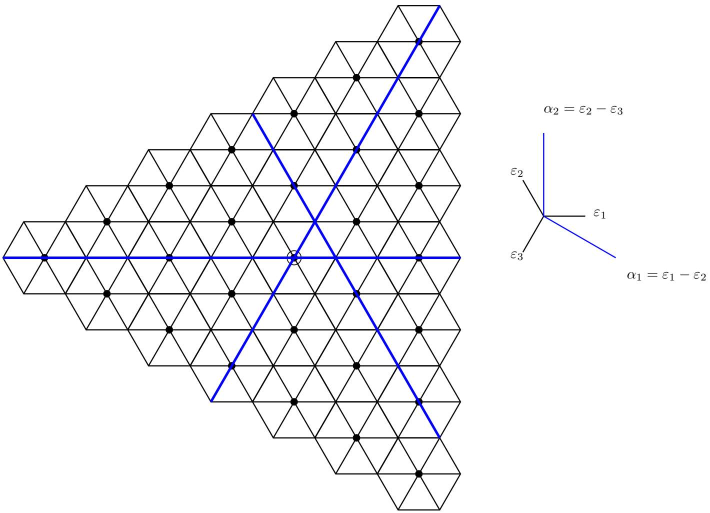


The action of $\widetilde{S}_\ell$ on the chambers is transitive and faithful.
Hence we can associate each element of $\widetilde{S}_\ell$ to a chamber.



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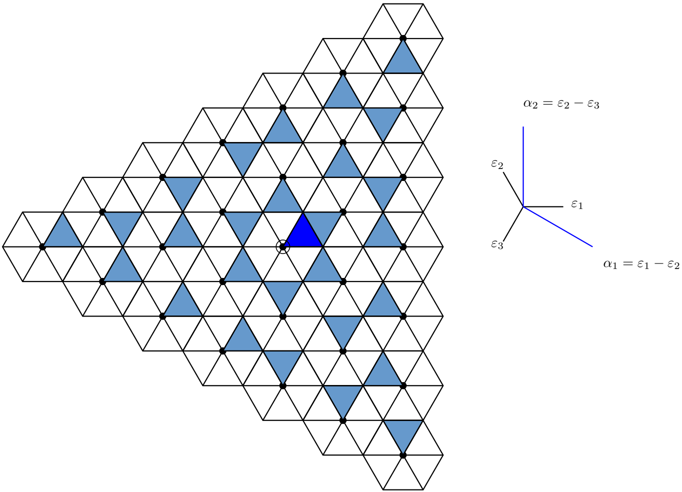
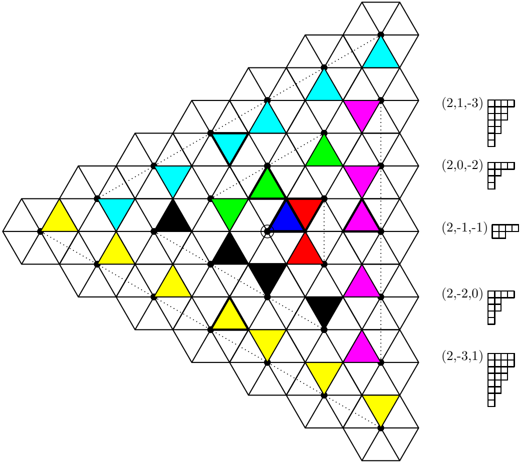
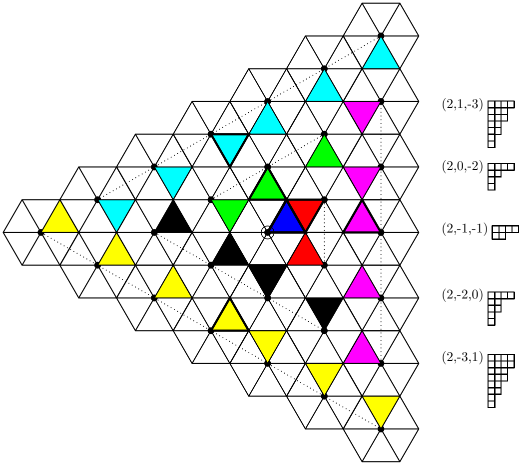


Cosets of $\widetilde{S}_\ell / S_\ell$ are in bijection with translations from the root lattice. Hence ℓ-cores are also in bijection with translations by the root lattice. All triangles which are 3 cores are blue.



Proposition:

The ℓ-cores with first part k all lie on an affine hyperplane.



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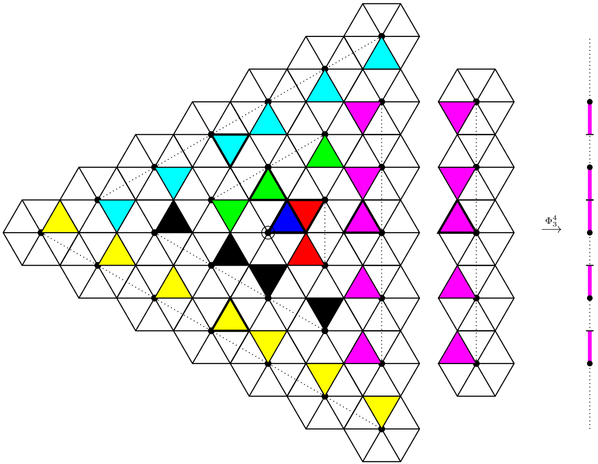

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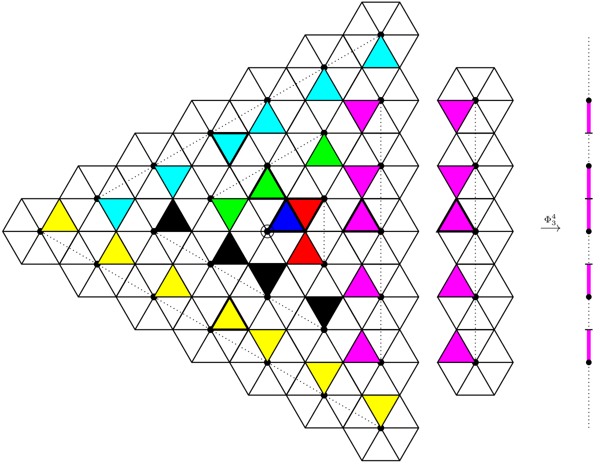
Projecting the minimal length cosets corresponding to ℓ-cores with first part k onto this hyperplane is a description of Φ_ℓ^k.



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Applications of bijection

Corollary of bijection

The number of ℓ-cores with first part k is $\binom{k+\ell-2}{k}$.

Problem

Count the set $S_\mu(n) :=$

$$\{\lambda \vdash n : core_\ell(\lambda) = \mu \text{ and } S^\lambda \text{ is an irreducible representation of } H_n(q), \text{ for } q \text{ an } \ell^{th} \text{ root of unity.}\}.$$

Solution

The number of partitions of $\frac{n-|core_\ell(\mu)|}{\ell}$ of length at most r, where r is minimal such that $\mu_r - \mu_{r+1} \neq \ell - 1$.

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