

A Murnaghan-Nakayama Rule For k -Schur Functions

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Outline

History

The Murnaghan-Nakayama Rule

The affine Murnaghan-Nakayama rule

Non-commutative symmetric functions

Sketch of non-commutative proof

The dual formulation

Early history - Representation theory

Theorem (Frobenius, 1900)

The map from functions on S_n to symmetric functions given by

$$f \mapsto \frac{1}{n!} \sum_{w \in S_n} f(w) p_{\lambda(w)}$$

sends

$$(\text{ trace function on } \lambda\text{-irrep of } S_n) \mapsto s_\lambda$$

Ferdinand
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Corollary

$$s_\lambda = \sum_{\mu} \frac{1}{z_\mu} \chi_\lambda(\mu) p_\mu \qquad p_\mu = \sum_{\lambda} \chi_\lambda(\mu) s_\lambda$$

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Early History - Combinatorics

Theorem (Littlewood-Richardson, 1934)

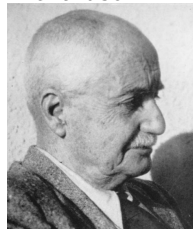
$$p_r s_\mu = \sum_{\lambda} (-1)^{\text{ht}(\lambda/\mu)} s_\lambda$$

where the summation is over all λ such that λ/μ is a border strip of size r .

Dudley Littlewood



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Corollary

Iteration gives

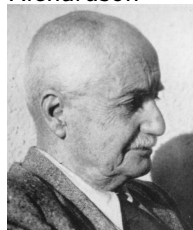
$$\chi^\lambda(\mu) = \sum_T (-1)^{\text{ht}(T)}$$

where the sum is over all border strip tableaux of shape λ and type μ .

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Early History - Further work

- ▶ Francis Murnaghan (1937) *On representations of the symmetric group*



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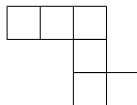
- ▶ Tadasi Nakayama (1941) *On some modular properties of irreducible representations of a symmetric group*

Border Strips

A *border strip* of size r is a connected skew partition consisting of r boxes and containing no 2×2 squares.

Example

$(4, 3, 3)/(2, 2)$ is a border strip of size 6:

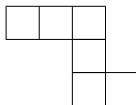


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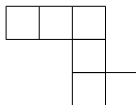
$$\text{ht}(\lambda/\mu) = \# \text{ vertical dominos in } \lambda/\mu$$

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$$\text{ht} \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline & & \square \\ \hline & & \square \\ \hline & \square & \square \\ \hline \end{array} \right) = 2$$

The Murnaghan-Nakayama rule

Theorem

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sum over all λ such that λ/μ a border strip of size r .

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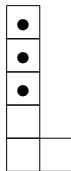
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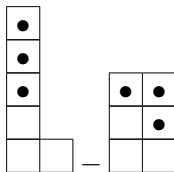
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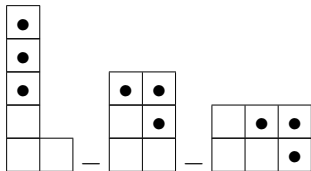
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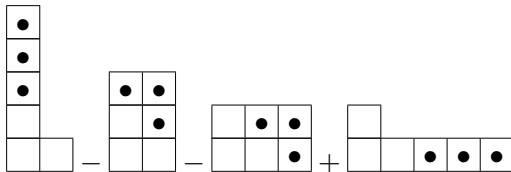
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Example

$$p_3 s_{2,1} = s_{2,1,1,1,1} - s_{2,2,2} - s_{3,3} + s_{5,1}$$



Border strip tableaux

Definition

A *border strip tableau* of shape λ is a filling of λ satisfying:

- ▶ Restriction to any single entry is a border strip
- ▶ Restriction to first k entries is partition shape for every k

Type of a border strip tableau: ($\#$ of boxes labelled i);

Height of a border strip tableau: sum of heights of border strips

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$$T =$$

1	3	3	
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$$\text{type}(T) = (4, 1, 5)$$

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$$p_{\mu} = \sum_{\lambda} \chi^{\lambda}(\mu) s_{\lambda} \quad \text{where} \quad \chi^{\lambda}(\mu) = \sum_T (-1)^{\text{ht}(T)}$$

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$$p_{2,1} = -s_{1,1,1}$$

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The affine Murnaghan-Nakayama rule

Theorem (B-Schilling-Zabrocki, 2010)

For $r \leq k$,

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Mike Zabrocki

k -Schur functions

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$$s_{\lambda}^{(k)}(x; t) = \sum_{T \in A_{\lambda}^{(k)}} t^{ch(T)} s_{sh(T)}$$

k -Schur functions

I will use the definition due to Lapointe and Morse in 2004:



$$h_r s_{\lambda}^{(k)}(x) = \sum_{\mu} s_{\mu}^{(k)}(x)$$

where the sum is over those μ such that $\mathfrak{c}(\mu)/\mathfrak{c}(\lambda)$ is a horizontal strip.

Partitions and cores

k -bounded partitions: First part $\leq k$

$k + 1$ -cores: No hook length $= k + 1$

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Bijection: Slide rows with big hooks

Partitions and cores

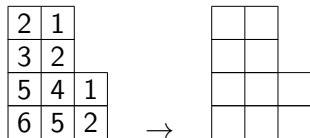
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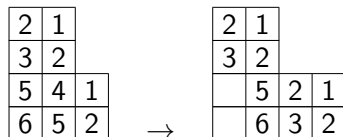
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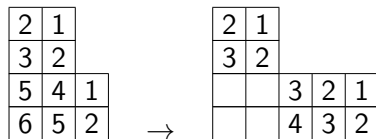
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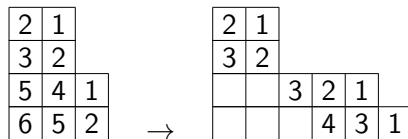
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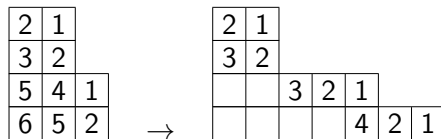
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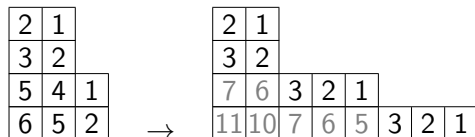
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k -conjugate

The k -conjugate of a k -bounded partition λ is found by

$$\lambda \rightarrow \mathfrak{c}(\lambda) \rightarrow \mathfrak{c}(\lambda)' \rightarrow \lambda^{(k)}$$

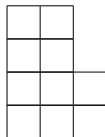
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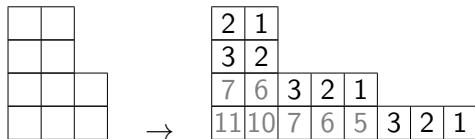
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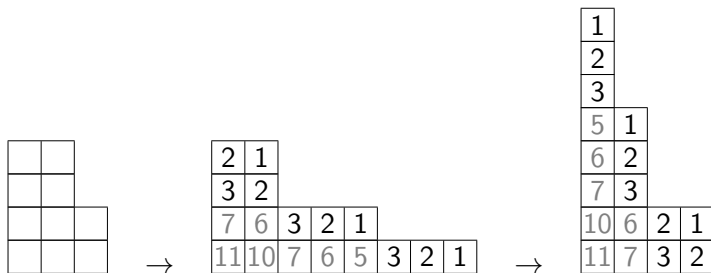
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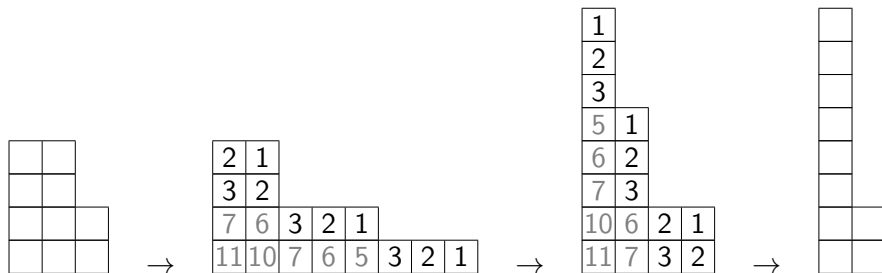
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content

When $k = \infty$, the *content* of a cell in a diagram is
(column index) $-$ (row index)

Example

-3	-2		
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For $k < \infty$ we use the *residue* mod $k + 1$ of the associated core

Example

1	2						
2	3						
3	0	1	2	3			
0	1	2	3	0	1	2	3

k -connected

A skew $k + 1$ core is k -connected if the residues are a proper subinterval of the numbers $\{0, \dots, k\}$, considered on a circle.

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A skew $k + 1$ core is *k-connected* if the residues are a proper subinterval of the numbers $\{0, \dots, k\}$, considered on a circle.

Example

A 3-connected skew core:

0									
1	2								
2	3	0							
3	0	1	2	3	0				
0	1	2	3	0	1	2	3	0	

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0									
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0	1	2	3	0	1	2	3	0	

A skew core which is not 3-connected:

0									
1	2								
2	3	0							
3	0	1	2	3	0				
0	1	2	3	0	1	2	3	0	

k -border strips

The skew of two k -bounded partitions λ/μ is a k -border strip of size r if it satisfies the following conditions:

- ▶ (size) $|\lambda/\mu| = r$
- ▶ (containment) $\mu \subset \lambda$ and $\mu^{(k)} \subset \lambda^{(k)}$
- ▶ (connectedness) $\mathfrak{c}(\lambda)/\mathfrak{c}(\mu)$ is k -connected
- ▶ (first ribbon condition) $\text{ht}(\lambda/\mu) + \text{ht}(\lambda^{(k)}/\mu^{(k)}) = r - 1$
- ▶ (second ribbon condition) $\mathfrak{c}(\lambda)/\mathfrak{c}(\mu)$ contains no 2×2 squares

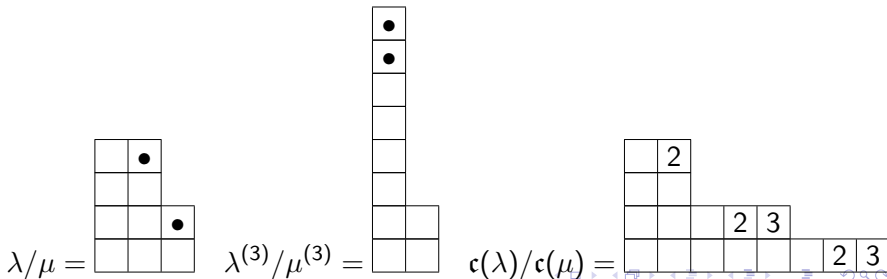
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Example

$k = 3, r = 2$



k -ribbons at ∞

At $k = \infty$ the conditions

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Proposition

At $k = \infty$ the first four conditions imply the fifth.

The ribbon statistic at $k = \infty$

Let λ/μ be connected of size r , and

$$\text{ht}(\lambda/\mu) + \text{ht}(\lambda'/\mu') = \#\text{vert. dominos} + \#\text{horiz. dominos} = r - 1$$

Then λ/μ is a ribbon

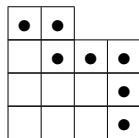
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$$3 + 3 = 6$$

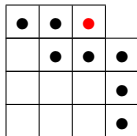
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Example



$$(3 + \textcolor{red}{1}) + (3 + \textcolor{red}{1}) = \textcolor{red}{8} \neq 7$$

Recap for general k

For $r \leq k$,

$$p_r s_\mu^{(k)} = \sum_{\lambda} (-1)^{\text{ht}(\lambda/\mu)} s_\lambda^{(k)}$$

where the summation is over all λ such that λ/μ satisfies

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For $r \leq k$,

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Conjecture

The first four conditions imply the fifth.

This has been verified for all $k, r \leq 11$, all μ of size ≤ 12 and all λ of size $|\mu| + r$.

The non-commutative setting

Theorem (Fomin-Greene, 1998)

Any algebra with a linearly ordered set of generators u_1, \dots, u_n satisfying certain relations contains an homomorphic image of Λ .

Example

The type A nilCoxeter algebra. Generators s_1, \dots, s_{n-1} . Relations

- ▶ $s_i^2 = 0$
- ▶ $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
- ▶ $s_i s_j = s_j s_i$ for $|i - j| > 2$.

Sergey Fomin



Curtis Greene

The affine nilCoxeter algebra

The affine nilCoxeter algebra A_k is the \mathbb{Z} -algebra generated by u_0, \dots, u_k with relations

- ▶ $u_i^2 = 0$ for all $i \in [0, k]$
- ▶ $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ for all $i \in [0, k]$
- ▶ $u_i u_j = u_j u_i$ for all i, j with $|i - j| > 1$

All indices are taken modulo $k + 1$ in this definition.

A word in the affine nilCoxeter algebra is called *cyclically decreasing* if

- ▶ its length is $\leq k$
- ▶ each generator appears at most once
- ▶ if u_i and u_{i-1} appear, then u_i occurs first (as usual, the indices should be taken mod k).

As elements of the nilCoxeter algebra, cyclically decreasing words are completely determined by their support.

Example

$k = 6$

$$(u_0 u_6)(u_4 u_3 u_2) = (u_4 u_3 u_2)(u_0 u_6) = u_4 u_0 u_3 u_6 u_2 = \cdots$$

Noncommutative \mathbf{h} functions

For a subset $S \subset [0, k]$, we write u_S for the unique cyclically decreasing nilCoxeter element with support S .

For $r \leq k$ we define

$$\mathbf{h}_r = \sum_{|S|=r} u_S$$

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This immediately implies that the algebra $\mathbb{Q}[\mathbf{h}_1, \dots, \mathbf{h}_k] \cong \mathbb{Q}[h_1, \dots, h_k]$ where the latter functions are the usual homogeneous symmetric functions.

Noncommutative symmetric functions

We can now define non-commutative analogs of symmetric functions by their relationship with the \mathbf{h} basis.

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$$\mathbf{s}_\lambda^{(k)} \text{ by the } k\text{-Pieri rule}$$

k -Pieri rule

The k -Pieri rule is

$$\mathbf{h}_r \mathbf{s}_\lambda^{(k)} = \sum_{\mu} \mathbf{s}_\mu^{(k)}$$

where the sum is over all k -bounded partitions μ such that μ/λ is a horizontal strip of length r and $\mu^{(k)}/\lambda^{(k)}$ is a vertical strip of length r . This can be re-written as

$$\mathbf{h}_r \mathbf{s}_\lambda^{(k)} = \sum_{|S|=r} \mathbf{s}_{u_S \cdot \lambda}^{(k)}$$

The action on cores

There is an action of A_k on $k + 1$ -cores given by

$$u_i \cdot c = \begin{cases} 0 & \text{no addable } i\text{-residue} \\ c \cup \text{all addable } i\text{-residues} & \text{otherwise} \end{cases}$$

Example

$$k = 4$$

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	2	3						
	3	0	1	2	3			
$s_2 s_0 \cdot$	0	1	2	3	0	1	2	3

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Multiplication rule

A corollary of the k -Pieri rule is that if \mathbf{f} is any non-commutative symmetric function of the form

$$\mathbf{f} = \sum_u c_u u$$

then

$$\mathbf{f} \mathbf{s}_\lambda^{(k)} = \sum_u c_u \mathbf{s}_{u \cdot \lambda}^{(k)}$$

Hook words

Fomin and Greene define a *hook word* in the context of an algebra with a totally ordered set of generators to be a word of the form

$$u_{a_1} \cdots u_{a_r} u_{b_1} \cdots u_{b_s}$$

where

$$a_1 > a_2 > \cdots > a_r > b_1 \leq b_2 \leq \cdots \leq b_s$$

To extend this notion to A_k which has a *cyclically* ordered set of generators, we only consider words whose support is a proper subset of $[0, \cdots, k]$.

Hook words

There is a *canonical order* on any proper subset of $[0, k]$ given by thinking of the smallest (in integer order) element which does not appear as the smallest element of the circle.

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Example

For $\{0, 1, 3, 4, 6\} \subset [0, 6]$, we have the order

$$2 < 3 < 4 < 5 < 6 < 0 < 1$$

Hook words in A_k have (support = proper subset) and form

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Hook word representations are unique; therefore the number of *ascents* in a hook word is well-defined as $s - 1$.

The non-commutative rule

Theorem (B-Schilling-Zabrocki, 2010)

$$\mathbf{p}_r \mathbf{s}_{\mu}^{(k)} = \sum_w (-1)^{\text{asc}(w)} \mathbf{s}_{w \cdot \mu}^{(k)}$$

where the sum is over all words in the affine nilCoxeter algebra satisfying

- ▶ (size) $\text{len}(w) = r$
- ▶ (containment) $w \cdot \mu \neq 0$
- ▶ (connectedness) w is a k -connected word
- ▶ (ribbon condition) w is a hook word

Sketch of non-commutative proof

Compute expansion of \mathbf{s}_{hook} into words using

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$$s_{r-i,1^i} = h_{r-i}e_i - h_{r-i+1}e_{i-1} + \cdots + (-1)^i h_r$$

and description of \mathbf{h} (resp. \mathbf{e}) as sums of cyclically increasing (resp. cyclically decreasing) words.

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Pair words of opposite sign to conclude

$$\mathbf{s}_{r-i,1^i} = \sum_w w$$

where the sum is over all hook words of size r with exactly i ascents.

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$$p_r = \sum_{i=0}^{r-1} (-1)^i s_{r-i,1^i} \quad \text{to conclude} \quad \mathbf{p}_r = \sum_w (-1)^{\text{asc}(w)} w$$

where the sum is over all (not necessarily connected) hook words of length r .

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A sign-reversing involution (Fomin and Greene) restricts the sum to connected hook-words. The multiplication rule

$$\mathbf{p}_r \mathbf{s}_\lambda^{(k)} = \sum_w (-1)^{\text{asc}(w)} \mathbf{s}_{w \cdot \lambda}^{(k)}$$

completes the proof.

Sketch of commutative proof

Characterize the image of the map $(w \rightarrow w \cdot \mu = \lambda)$:
conditions on words: conditions on shapes:

▶ (size)
 $\text{len}(w) = r$

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Iteration

Iterating the rule

$$p_r s_\lambda^{(k)} = \sum_{\mu} (-1)^{\text{ht}(\mu/\lambda)} s_\mu^{(k)}$$

gives

$$p_\lambda = \sum_T (-1)^{\text{ht}(T)} s_{sh(T)}^{(k)} = \sum_{\mu} \bar{\chi}_\lambda^{(k)}(\mu) s_\mu^{(k)}$$

where the sum is over all k -ribbon tableaux, defined analogously to the classical case.

Duality

In the classical case, the inner product immediately gives

$$p_\lambda = \sum_{\mu} \chi_\lambda(\mu) s_\mu \iff s_\mu = \sum_{\lambda} \frac{1}{z_\lambda} \chi_\lambda(\mu) p_\lambda$$

In the affine case we have

$$p_\lambda = \sum_{\mu} \bar{\chi}_\lambda^{(k)}(\mu) s_\mu^{(k)} \iff \mathfrak{S}_\mu^{(k)} = \sum_{\lambda} \frac{1}{z_\lambda} \bar{\chi}_\lambda^{(k)} p_\lambda$$

We would like the inverse matrix

$$s_\lambda^{(k)} = \sum_{\mu} \frac{1}{z_\mu} \chi_\lambda^{(k)}(\mu) p_\mu$$

Back to Frobenius

For V any S_n representation, we can find the decomposition into irreducible submodules with

$$\sum_{\mu} \frac{1}{z_{\mu}} \chi_V(\mu) p_{\mu} = \sum_{\lambda} c_{\lambda} s_{\lambda}$$

So finding

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would potentially allow one to verify that a given representation had a character equal to k -Schur functions.



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Full paper available at [arXiv:1004.8886](https://arxiv.org/abs/1004.8886)