Harmonic Spinors on Homogeneous Spaces

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This is joint work with Leticia Barchini

I. Dirac Operators

- G a connected real semisimple Lie group.
- $\mathfrak{g}$ the Lie algebra of G; $\langle\ ,\ \rangle$ the Killing form.
- $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ a Cartan decomposition of $\mathfrak{g}$;
- K a maximal compact subgroup of G.
- the Killing form restricts to a positive definite form $\langle\ ,\ \rangle_{\mathfrak{s}}$ on $\mathfrak{s}$.

There is a spin representation of K arising from $\mathrm{Ad} : K \to SO(\mathfrak{s})$, call it S. (Maybe pass to cover.) Suppose E_μ is a finite dimensional irreducible representation of K. Let $\mathcal{E}_\mu \otimes S \to G/K$ be the equivariant vector bundle corresponding to $E_\mu \otimes S$.
There is a well-known Dirac operator

$$\mathcal{D} : C^\infty(G/K, \mathcal{E}_\mu \otimes S) \to C^\infty(G/K, \mathcal{E}_\mu \otimes S)$$

$$\mathcal{D} = \sum R(X_j) \otimes 1 \otimes \gamma(X_j),$$

where:
$\{X_j\}$ is an orthonormal basis of $\mathfrak{s}$
$\gamma(X_j)$ is Clifford multiplication, and
$R(X_j)$ is right translation on sections:
$(R(X)F)(g) := \frac{d}{dt} F(g \exp(tX))|_{t=0}$.
This is a G-invariant first order operator.

Theorem 1 (Parthasarathy 72, Atiyah-Schmid 77) Suppose that $\mathrm{rank}(K) = \mathrm{rank}(G)$. Then the L^2 harmonic space

$$\begin{aligned}\mathcal{H}_2(G/K, E_\mu) :=& \{F \in C^\infty(G/K, \mathcal{E}_\mu \otimes S) \,:\, \mathcal{D}F = 0 \\ & \text{and} \int_{G/K} ||F(g)||^2 \, dg < \infty\}\end{aligned}$$

is an irreducible unitary representation in the discrete series of G. Every discrete series representation is realized in this way.

Note. (a) $||-||$ is the norm in $E_\mu \otimes S$ for a K-invariant positive definite hermitian form on.

(b) $\mathcal{D}$ is an elliptic operator.

Question: What can one say for Dirac operators on more general G/H?

First Example.
Suppose that $H \subsetneq K$, $\operatorname{rank}(H) = \operatorname{rank}(G)$.
Write $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}, \mathfrak{q} = \mathfrak{h}^\perp$. Since H is compact we may choose an H-invariant positive definite form on $\mathfrak{q}$.

This gives a spin representation $S_\mathfrak{q}$ of H.

There is a Dirac operator with formula as above.

We may consider the L^2 harmonic space.

BUT this is the wrong Dirac operator.
An important formula of Parthasarathy for G/K is

$$\mathcal{D}^2 = R(\Omega_G) \otimes 1 \otimes 1 - (||\mu + \rho_\mathfrak{k}||^2 - ||\rho||^2);$$

this fails for G/H.

However, Kostant's "cubic" Dirac operator is the correct differential operator:
– There is a good formula for the square.
– There is an induction in stages formula:

"$\mathcal{D}_{G/H} = \mathcal{D}_{G/K} + \mathcal{D}_{K/H}$"

under the identification

$$C^\infty(G/H, \mathcal{E}_\mu \otimes S_\mathfrak{q}) \simeq C^\infty(G/K, C^\infty(K/H, \mathcal{E}_\mu \otimes S_{\mathfrak{q}\cap\mathfrak{k}}) \otimes S_\mathfrak{s})$$

This leads to the following statement:

Proposition 2 *$\mathcal{H}_2(G/H, E_\mu)$ is a discrete series representation.*

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Let's return to more arbitrary G/H.

We assume that H is a connected reductive group so that the Killing form restricted to $\mathfrak{h} \times \mathfrak{h}$ is nondegenerate.
Write $\quad \mathfrak{g} = \mathfrak{h} + \mathfrak{q}, \quad \mathfrak{q} := \mathfrak{h}^\perp$.
Let $\langle\ ,\ \rangle_\mathfrak{q}$ be the restriction of the Killing form to $\mathfrak{q}$. (This is nondegenerate and defines a typically indefinite hermitian form.) We therefore have:

Clifford algebra $C\ell(\mathfrak{q})$ and
spin represention $S_\mathfrak{q}$ of $\mathfrak{h}$

constructed from $\langle\ ,\ \rangle_\mathfrak{q}$.
Kostant defined the 'cubic Dirac operator' as follows:
Let $\{X_j\}$ be a basis of $\mathfrak{q}$ so that $\langle X_j, X_k\rangle_\mathfrak{q} = \epsilon_j \delta_{jk}$, $\epsilon_j = \pm 1$.
$D \in \{\mathcal{U}(\mathfrak{g}) \otimes C\ell(\mathfrak{q})\}^\mathfrak{h}$ is defined by

$$D = \sum \epsilon_j X_j \otimes \gamma(X_j) - 1 \otimes \gamma(c),$$

where

$$c = \frac{1}{6} \sum \epsilon_i \epsilon_j \epsilon_k \langle X_i, [X_j, X_k]\rangle X_i X_j X_k.$$

(cubic term).
This gives a 'geometric Dirac operator': Let E_μ be a f.d. representation of $\mathfrak{h}$ so that $E \otimes S_\mathfrak{q}$ lifts to rep. of H and define

$$\mathcal{D} : C^\infty(G/H, \mathcal{E} \otimes S_\mathfrak{q}) \to C^\infty(G/H, \mathcal{E} \otimes S_\mathfrak{q})$$

by

$$\mathcal{D}F(g) = \sum \gamma(X_i)(r(X_i)F)(g) - (1 \otimes \gamma(c))(F(g)).$$

Facts:

- There is a good formula for $\mathcal{D}^2$, as above.
- Behaves well under induction in stages, as above.
- $\mathcal{D}$ is in fact a Dirac operator from a G-invariant connection. (I. Agricola, 2002.)
- If $H = K$, then $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k} \perp \mathfrak{s}$, so $c = 0$.

An *Harmonic Spinor* is a section F so that $\mathcal{D}F = 0$.
Assume that H is not compact and assume that G and H have the same (complex) rank.
– Given that the bundle has an indefinite hermitian form, what would we mean by an L_2 section???
– Can we define an L^2-harmonic space, and construct a unitary reprsentation???

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II. Harmonic spinors

Is the space of harmonic spinors nonzero?
Under a condition that the highest weight of E is 'sufficiently regular', there is a G-intertwining integral operator

$$\mathcal{P} : C^\infty(G/P, \mathcal{W}) \to C^\infty(G/H, \mathcal{E} \otimes S_\mathfrak{q})$$

with the property that

$$im(\mathcal{P}) \subset Ker(\mathcal{D}).$$

We will need an explicit formula: for $\mathcal{P}(\varphi)$.
Remark. There is an analogy here with the Poisson transform: harmonic functions on the disk are Poisson integrals of functions on the boundary.
Construction of $\mathcal{P}$:
The parabolic subgroup P is defined by choosing a maximal abelian

$$\mathfrak{a} \subset \mathfrak{h} \cap \mathfrak{s}.$$

Then

$$MA = Z_G(\mathfrak{a}).$$

$P = MAN$, with a little care in the choice of N.

Lemma 3 *If* $T \in \hom_{P\cap H}(W \otimes \mathbf{C}_{\rho_{\mathfrak{g}}-2\rho_{\mathfrak{h}}}, E_\mu \otimes S_{\mathfrak{q}})$ *then,*

$$(\mathcal{P}\varphi)(g) = \int_{H\cap K} \ell \cdot T(\varphi(g\ell))\, d\ell$$

is a G-intertwining operator.

Proof:

$$\begin{aligned}
&\hom_G(C^\infty(G/P, \mathcal{W}), C^\infty(G/H, \mathcal{E}_\mu \otimes S_{\mathfrak{q}})) \\
&\qquad \simeq \hom_H(C^\infty(G/P, \mathcal{W}), E_\mu \otimes S_{\mathfrak{q}}) \\
&\qquad \simeq \hom_H((E_\mu \otimes S_{\mathfrak{q}})', \mathcal{D}'(G/P, \mathcal{W}' \otimes \mathbf{C}_{2\rho})) \\
&\qquad \supset \hom_{H\cap P}((E_\mu \otimes S_{\mathfrak{q}})', C^\infty(H/H\cap P, \mathcal{W}' \otimes \mathbf{C}_{2\rho_{\mathfrak{h}}}))
\end{aligned}$$

(distributions supported on the closed submanifold $H/H\cap P$)

$$\begin{aligned}
&\simeq \hom_{H\cap P}((E_\mu \otimes S_{\mathfrak{q}})', \mathcal{W}' \otimes \mathbf{C}_{2\rho_{\mathfrak{h}}}) \\
&\simeq \hom_{H\cap P}(\mathcal{W} \otimes \mathbf{C}_{-2\rho_{\mathfrak{h}}}, E_\mu \otimes S_{\mathfrak{q}}).
\end{aligned}$$

□

The representation of P on W:

$$\begin{aligned}
&\text{Action of } A : \nu = (\text{highest wt. of } E) + \rho_{\mathfrak{h}} + \rho_{\mathfrak{g}}, \\
&\text{Action of } N : \text{trivial}, \\
&\text{Action of } M : \text{discrete series realized as a space of} \\
&\qquad\qquad\qquad L_2 \text{ harmonic spinors on } M/M\cap H.
\end{aligned}$$

Note that

$$W \subset C^\infty(M/M\cap H, U_{\mu'} \otimes S_{\mathfrak{m}\cap\mathfrak{q}}),$$

for some $M\cap K$-representation $U_{\mu'}$.
$\pi_0 : U_{\mu'} \otimes S_{\mathfrak{m}\cap\mathfrak{q}} \to V_0$ is a projection to the subspace V_0 of $\mathfrak{h}\cap\mathfrak{n}$-invariants in $E_{\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})} \subset E_\mu \otimes S_{\mathfrak{q}}$.
Then $T(w) = \pi_0(w(e))$ gives the G-intertwining operator

$$(\mathcal{P}\varphi)(g) = \int_{H\cap K} \ell \cdot \pi_0(\varphi(g\ell)(e))\, d\ell.$$

Theorem 4 (Mehdi-Z.) Under the 'sufficiently dominant' condition

$$\mathcal{P}\varphi \in ker(\mathcal{D}).$$

III. Square integrability.

In general $E \otimes S_{\mathfrak{q}}$ does not have a positive definite H-invariant hermitian form, so it is not clear what we mean by a square integrable section of $\mathcal{E} \otimes S_{\mathfrak{q}}$. To get started we have:

Lemma 5 *Suppose an irreducible finite dimensional representation of H has highest weight λ such that $\langle \lambda, \alpha \rangle \in \mathbf{R}$, $\alpha \in \Delta(\mathfrak{g})$. Then the representation has a positive definite hermitian form $\langle\ ,\ \rangle_{\mathrm{pos}}$ with the property that*

$$\langle h \cdot v, w \rangle_{\mathrm{pos}} = \langle v, \theta(h^{-1}) \cdot w \rangle_{\mathrm{pos}},$$

for $h \in H$.

Not every finite dimensional representation has an H-invariant hermitian form. (There is one, for example, in case H and $H \cap K$ have the same complex rank.) The form is typically of indefinite signature.

Lemma 6 *If E has an H-invariant hermitian form $\langle\ ,\ \rangle$, then there is a constant C so that*

$$|\langle v, w \rangle| \leq C ||v||_{\mathrm{pos}} ||w||_{\mathrm{pos}},$$

for all $v \in S_{\mathfrak{q}} \otimes E$.

Note that $||F(g)||_{\mathrm{pos}}$ is not a function on G/H.
We use the Mostow decomposition

$$G = K \exp(\mathfrak{q} \cap \mathfrak{s}) \exp(\mathfrak{h} \cap \mathfrak{s}),$$

$$g = k(g) \exp(X(g)) \exp(Y(g)).$$

It is easy to check that

$$||F(k(g) \exp(X(g))||^2_{\mathrm{pos}}$$

is a function on G/H. (Note that $||\cdot||_{\mathrm{pos}}$ is $K \cap H$-invariant by Lemma 5.)

Definition 7 *Let $\mathcal{H}_2(G/H, E)$ be the space of harmonic spinors F so that*

$$||F||^2_{pos} := \int_{G/H} ||F(k(g) \exp(X(g))||^2_{pos}\, dg$$

is finite.

(This definition comes from Rawnsley-Schmid-Wolf, where is was used in the context of differential forms in an indefinite metric setting.)

It follows from Lemma 6 that

$$\langle F_1, F_2\rangle_{\mathrm{inv}} := \int_{G/H} \langle F_1(g), F_2(g)\rangle \, dg$$

is finite for all $F_1, F_2 \in \mathcal{H}_2(G/H, E)$, so defines a G-invariant hermitian form on $\mathcal{H}_2(G/H, E)$.

The following theorem holds (under the condition that the highest weight of E is sufficiently regular).

Theorem 8 (Barchini-Z.) If G/H is a semisimple symmetric space, then $\mathcal{H}_2(G/H, E) \neq \{0\}$. If, in addition, E has an invariant hermitian form, then $\mathcal{H}_2(G/H, E)$ carries a G-invariant hermitian form.

The proof is to use the explicit form of $\mathcal{P}$ to show that $F = \mathcal{P}\varphi$ is square integrable (for K-finite φ).

The proof involves numerous changes of variables and identities for Iwasawa and Mostow decompositions. Here are a few intermediate steps.

Lemma 9 *For $\varphi \in C^\infty(G/P, \mathcal{W})$ and $g \in G$, $||\mathcal{P}\varphi(g)||^2_{pos} =$*

$$\int_{K\cap H} \int_{\overline{N}\cap H} \langle \pi_0(\varphi(g\ell)(e)), \pi_0(\varphi(g\ell\overline{n}_H)(e))\rangle_{\mathrm{pos}} \, d\overline{n}_H d\ell.$$

In case G/H is semisimple symmetric, $H = G^\sigma$, we have

Lemma 10 *If $\sigma\theta(g) = g$, then $\overline{H}(g) = -H(g)$ and $\overline{m}(g) = m(g)$.*

This relates the Iwasawa decompositions

$$\begin{aligned} G &= K \exp(\mathfrak{m} \cap \mathfrak{s}) N A \\ g &= \kappa(g) m(g) n(g) \mathrm{e}^{H(g)} \end{aligned}$$

and

$$\begin{aligned} G &= K \exp(\mathfrak{m} \cap \mathfrak{s}) \overline{N} A \\ g &= \overline{\kappa}(g) \overline{m}(g) \overline{n}(g) \mathrm{e}^{\overline{H}(g)} \end{aligned}$$

Combining the two lemmas (and numerous integration formulas and changes of variables) we get

$$\int_{G/H} ||\mathcal{P}\varphi(k(g)\exp(X(g))||^2_{\text{pos}}\, dg$$

is bounded by a constant multiple of

$$\int_{\overline{N}} \mathrm{e}^{-\nu(H(\overline{n}))} d\overline{n},$$

which is known to converge (by our choice of ν).

IV. Dolbeault Cohomology.

Consider an elliptic coadjoint orbit G/L.
G/L has a G-invariant complex structure.
The anti-holomorphic tangent space at eL may be identified with a subalgebra $\mathfrak{u}$, and $\mathfrak{l} + \mathfrak{u}$ is a parabolic subalgebra of $\mathfrak{g}$.

Suppose χ is a unitary character of L, $\mathcal{L}_\chi \to G/L$ the corresponding homogeneous holo. line bundle.
This gives the Dolbeault complex

$$\overline{\partial} : C^\infty(G/L, \mathbf{C}_\chi \otimes \wedge^m \overline{\mathfrak{u}}) \to C^\infty(G/L, \mathbf{C}_\chi \otimes \wedge^{m+1} \overline{\mathfrak{u}}),$$

and the formal adjoint of $\overline{\partial}$

$$\overline{\partial}^* : C^\infty(G/L, \mathbf{C}_\chi \otimes \wedge^{m+1} \overline{\mathfrak{u}}) \to C^\infty(G/L, \mathbf{C}_\chi \otimes \wedge^m \overline{\mathfrak{u}}).$$

(This is the formal adjoint with respect to the natural invariant hermitian form on $\mathbf{C}_\chi \otimes \wedge^\bullet \overline{\mathfrak{u}}$.)
Well-known basic (but nontrivial) facts:

(a) $H^m(G/L, \mathcal{L}_\chi)$ is an admissible representation of G, and is a maximal globalization.

(b) If $\langle \chi + \rho, \beta \rangle > 0, \beta \in \Delta(\mathfrak{g})$, then
– vanishing in degrees $m \neq s := \dim_{\mathbf{C}}(K/K \cap L)$
– $H^s(G/L, \mathcal{L}_\chi)$ is irreducible with Harish-Chandra module equal to a unitarizable '$A_\mathfrak{q}(\lambda)$' (cohomologically induced $(\mathfrak{g}, K)$-module).

Rawnsley, Schmid and Wolf suggested a way to construct the irreducible UNITARY representations of G:
Define an L^2 space of strongly harmonic forms:

$\overline{\partial}\omega = 0, \overline{\partial}^* \omega = 0$ and L^2 as described earlier.

Then attempt to prove:
(A) each (K-finite) cohomology class contains an L^2 strongly harmonic form.
(B) the invariant hermitian form (as discussed earlier) is positive semi-definite, with null space $im(\overline{\partial})$.

Then $\langle\ ,\ \rangle_{\text{inv}}$ will be positive definite on cohomology, completion will be a unitary representation.
The success of this method has is limited to cases where G/L is semisimple symmetric or there is a chain $L = L_1 \subset L_2 \subset \cdots \subset L_k = G$ with each L_{i+1}/L_i symmetric with invariant complex structure. (RSW, Barchini-Z.)

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Now consider the Dirac operator $\mathcal{D}$ associated to a bundle on G/L (elliptic orbit as above).

$\mathfrak{g} = \mathfrak{l} + \mathfrak{q}, \quad \mathfrak{q}_{\mathbf{C}} = \mathfrak{u} + \overline{\mathfrak{u}}.$

Facts:
– $S_{\mathfrak{q}} \simeq \wedge^\bullet \overline{\mathfrak{u}} \otimes \mathbf{C}_{\rho(\mathfrak{u})} = \sum_m \wedge^m \overline{\mathfrak{u}} \otimes \mathbf{C}_{\rho(\mathfrak{u})}$.
– $\mathcal{D} = \overline{\partial} + \overline{\partial}^*$.
So, if $F \in C^\infty(G/L, \mathbf{C}_{\chi+\rho(\mathfrak{u})} \otimes \wedge^m \overline{\mathfrak{u}})$ and $\mathcal{D}F = 0$, then F is strongly harmonic.

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For the remainder of the lecture we look at an example where we can use the Dirac operator to obtain L^2 strongly harmonic forms representing Dolbeault cohomology classes (G/L not symmetric).

V. An Example.

Consider the following example.

$$
\begin{aligned}
&G = Sp(n, \mathbf{R}),\\
&H = Sp(p, \mathbf{R}) \times Sp(q, \mathbf{R}), \quad n = p + q,\\
&L = U(p) \times Sp(q, \mathbf{R}).
\end{aligned}
$$

Then $L \subset H \subset G$, each symmetric in the next,
BUT G/H has no invariant complex structure.

(Note that $\mathfrak{l}_{\mathbf{C}}+\mathfrak{u}$ is a max parabolic subalg. of $\mathfrak{g}_{\mathbf{C}}$.)
Write $\mathfrak{g}=\mathfrak{l}+\mathfrak{q}=\mathfrak{h}+\mathfrak{q}_1$.
Look for a G-intertwining operator

$$\mathcal{P}: C^\infty(G/P,\mathcal{W}) \to C^\infty(G/L,\mathbf{C}_\chi\otimes S_{\mathfrak{q}}).$$

Since

$$C^\infty(G/H,\mathcal{H}_2(H/L,\mathbf{C}_\chi\otimes S_{\mathfrak{h}\cap\mathfrak{q}})\otimes S_{\mathfrak{q}_1}) \subset C^\infty(G/L,\mathbf{C}_\chi\otimes S_{\mathfrak{q}})$$

it is enough to look for

$$\mathcal{P}: C^\infty(G/P,\mathcal{W}) \to C^\infty(G/H,\mathcal{H}_2(H/L,\mathbf{C}_\chi\otimes S_{\mathfrak{h}\cap\mathfrak{q}})\otimes S_{\mathfrak{q}_1}).$$

$P=MAN$,
$\mathfrak{a}=\text{diagonal}(0,...,0,a_{p+1},\dots,a_n|0,\dots,0,-a_{p+1},\dots,-a_n)$,
The same principal as before applies: we look for

$$T\in \hom_{P\cap H}(W\otimes\mathbf{C}_{-2\rho_{\mathfrak{h}}},\mathcal{H}_2(H/L,\mathbf{C}_\chi)\otimes S_{\mathfrak{q}_1})$$

But this is fairly easy, since

$$M\cap H/M\cap K\simeq H/L \quad (\simeq Sp(p,\mathbf{R})/U(p)).$$

Proposition 11 *For χ dominant*

$$\begin{aligned} im(\mathcal{P}) &\subset \mathcal{H}_2(G/H,\mathcal{H}_2(H/L,\mathbf{C}_\chi)) \\ &\subset \mathcal{H}_2(G/L,\mathbf{C}_\chi) \end{aligned}$$

In fact the image of $\mathcal{P}$ is contained in L^2 harmonic forms of type $(0,s)$, and the invariant form is positive semidefinite on the image of $\mathcal{P}$.