Quantum Schubert Cells and Quantum Flag Varieties

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Milen Yakimov

Louisiana State University

Prime spectrum

Let A be a noetherian $\mathbb{C}$-algebra. $\mathrm{Spec}A$ is a topological space, consisting of all prime ideals P with the Zariski topology (means $P \in \mathrm{Spec}A$ if every two ideals I, J, $I.J \subset P$ implies $I \subset P$ or $J \subset P$). The closed sets are $V(I) = \{P \in \mathrm{Spec}A \mid I \subset P\}$ for all ideals I of A.

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Goodearl-Letzter machinery: Assume that a complex torus H acts by algebra automorphisms. Denote by $H - \mathrm{Spec}A$ the set of H invariant prime ideals of A. If $J \in \mathrm{Spec}A$ then

$$\Big(\cap_{h\in H}\, h.J\Big) \in H - \mathrm{Spec}A.$$

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Denote by A_I the localization of A/I by all nonzero homogeneous elements. Then $Z(A_I)$ is a (commutative) Laurent polynomial ring and

$$\mathrm{Spec}_I A \cong \mathrm{Spec} Z(A_I).$$

Relations to Poisson geometry

Let A be a an associative algebra over $\mathbb{C}$ with a $\mathbb{Z}_{\geq 0}$ filtration:

$$A_0 \subset A_1 \subset \ldots \subset A, \quad A = \cup_k A_k, \quad A_k . A_l \subset A_{k+l}.$$

If the associated graded $\mathrm{gr} A$ is commutative, then it inherits a canonical structure of a Poisson algebra:

$$\{a_k + A_{k-1}, a_l + A_{l-1}\} = a_k a_l - a_l a_k + A_{k+l-2}, \quad a_k \in A_k, a_l \in A_l,$$

note that $a_k a_l - a_l a_k \in A_{k+l-1}$. If in addition $\mathrm{gr} A$ has no nilpotent elements, then one obtains a canonical Poisson structure π on the affine variety $\mathrm{Spec}(\mathrm{gr} A)$.

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If A is equpped with a torus action H (preseving the filtration), then π is H-invariant. One wants to match $H - \mathrm{Spec} A$ with the H-orbits of symplectic leaves of $(\mathrm{Spec}(\mathrm{gr} A), \pi)$.

Quantum groups

The quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra with generators

$$X_i^{\pm}, K_i^{\pm 1},\ i = 1, \dots, r,$$

subject to the relations

$$K_i^{-1}K_i = K_iK_i^{-1} = 1,\ K_iK_j = K_jK_i,\ K_iX_j^{\pm}K_i^{-1} = q^{\pm c_{ij}}X_j^{\pm},$$

$$X_i^+X_j^- - X_j^-X_i^+ = \delta_{i,j}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-c_{ij}} \begin{bmatrix} 1 - c_{ij} \\ k \end{bmatrix}_q (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-c_{ij}-k} = 0,\ i \neq j.$$

Here r=rank of $\mathfrak{g}$, Cartan matrix (c_{ij}), $q \in \mathbb{C}$ is transcendental, $q_i = q^{d_i}$.

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It is a Hopf algebra. Its finite dimensional weight irreps are parametrized by the set of dominant integral weights P_+, $\lambda \in P_+ \mapsto V(\lambda)$.

There is a natural action of the related Braid group on $\mathcal{U}_q(\mathfrak{g})$ and $V(\lambda)$, $w \in W \mapsto T_w$.

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$\mathcal{U}_\pm$ the subalg. generated by $X_i^{\pm}$, $H = \langle K_1, \ldots, K_r \rangle$ the group of group-like elements.

DKP algebras

Fix $w \in W$. De Concini, Kac and Procesi defined a family of subalgebras $\mathcal{U}^w_\pm \subset \mathcal{U}_\pm$ which are deformations of $\mathcal{U}(\mathfrak{n}_+ \cap \mathrm{Ad}_w(\mathfrak{n}_-))$.

For a reduced expression $w = s_{i_1} \dots s_{i_k}$ define the roots

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_k = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k}).$$

Let $\mathcal{U}^w_\pm$ be the subalgebras of $\mathcal{U}_q(\mathfrak{g})$, generated by the root vectors

$$X^\pm_{\beta_1} = X^\pm_{i_1}, X^\pm_{\beta_2} = T_{s_{i_1}}(X^\pm_{i_2}), \dots, X^\pm_{\beta_k} = T_{s_{i_1} \dots s_{i_{k-1}}}(X^\pm_{i_k}).$$

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Theorem [De Concini-Kac-Procesi]. The definition of the algebras $\mathcal{U}^w_\pm$ does not depend on the choice of a reduced decomposition of w. The algebras $\mathcal{U}^w_\pm$ have the PBW bases

$$(X^\pm_{\beta_k})^{n_k} \dots (X^\pm_{\beta_1})^{n_1}, \;\; n_1, \dots, n_k \in \mathbb{N}.$$

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Theorem [Heckenberger–Schneider]. All right coideal subalgebras of $\mathcal{U}_q(\mathfrak{b}_+)$ containing H are of the form $\mathcal{U}^w_+ \mathbb{C}[H]$.

An Example

Let $\mathfrak{g} = \mathfrak{sl}_{m+n}$ and $w = c^m$ where c is the Coxeter element $(12\ldots m+n)$. Think of $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$. Then $\mathcal{U}_-^w(\mathfrak{g})$ is isomorphic to the algebra of quantum matrices $R_q[M_{m,n}]$. The latter is the $\mathbb{C}$-algebra generated by x_{ij}, $1 \le i \le m$, $1 \le j \le n$, with relations

$$x_{ij}x_{lj} = qx_{lj}x_{ij}, \quad \text{for } i < l,$$

$$x_{ij}x_{ik} = qx_{ik}x_{ij}, \quad \text{for } j < k,$$

$$x_{ij}x_{lk} = x_{lk}x_{ij}, \quad \text{for } i < l, j > k,$$

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Plan. 1. Construct another realization of $\mathcal{U}_-^w$. Describe explicitly all H invariant prime ideals of $\mathcal{U}_-^w$. 2. Resolve a conjecture of Goodearl and Lenagan on existence of polynormal generating sequences, prove that $\mathrm{Spec}\mathcal{U}_-^w$ is normally separated and all algebras $\mathcal{U}^w$ are catenary. 3. Classify $H - \mathrm{Spec}$ of all quantum partial flag varieties.

All approaches so far were based on treating $\mathcal{U}_-^w$ as an iterated skew polynomial ring.

Group Poisson structures

For $w \in W$ we will put a quadratic Poisson structure π_w on the Schubert cell $X_w \subset G/B_+$, such that $(\mathbb{C}(X_w), \pi_w)$ is the associated graded of $\mathcal{U}_-^w$.

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Fix a pair of opposite Borel subgroups $B_\pm$ of G, $H = B_+ \cap B_-$ – a maximal torus of G.

- Let Δ_+ be the set of all positive roots of $\mathfrak{g} = \mathrm{Lie}\, G$,
- Fix two dual sets of root vectors, $\{e_\alpha\}_{\alpha \in \Delta_+}$, $\{f_\alpha\}_{\alpha \in \Delta_+}$, normalized by $\langle e_\alpha, f_\alpha \rangle = 1$, where $\langle ., . \rangle$ is the Killing form on $\mathfrak{g}$.

Define

$$\pi_G = \sum_{\alpha \in \Delta_+} L_{e_\alpha} \wedge L_{f_\alpha} - \sum_{\alpha \in \Delta_+} R_{e_\alpha} \wedge R_{f_\alpha}$$

called the standard Poisson structure on G. (Here L and R denote left and right invariant vector fields on G.)

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Example. $(SL_n(\mathbb{C}), \pi_{SL_n})$ embeds in $M_{n \times n}$ with

$$\sum_{i,k=1}^{n} \sum_{j,l=1}^{n} (\mathrm{sign}(k-i) + \mathrm{sign}(l-j)) x_{il} x_{kj} \frac{\partial}{\partial x_{ij}} \wedge \frac{\partial}{\partial x_{kl}}.$$

Poisson structures on flag varieties

Fix a parabolic subgroup $P \supset B_+$ of G. Under the map $p\colon G \to G/P$ the Poisson structures π_G can be pushed forward to a well defined Poisson structure $\pi_{G/P} = p_*(\pi_G)$ on G/P.

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Special case: $P = B_+$. The H-orbits of symplectic leaves of $(G/B_+, \pi_{G/B})$ are the open Richardson varieties

$$R_{y_-,y_+} = B_- y_- \cdot B_+ \cap B_+ y_+ \cdot B_+ \subset G/B_+, \quad y_\pm \in W, y_- \leq y_+.$$

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Theorem. [Brown, Goodearl, Y.] The H–orbits of symplectic leaves of $(G/P, \pi_{G/P})$ are precisely the sets

$$S_P(y_-, y_+) = q(B_- y_- \cdot B_+ \cap B_+ y_+ \cdot B_+), \quad y_- \in W, y_+ \in W^{W_P}, y_- \leq y_+$$

where W^{W_P} is the set of min length repr. of the cosets W/W_P and $q\colon G/B_+ \to G/P$ is the canonical projection. (This is the Lusztig stratification of G/P.)

Note that $q\colon B_+ y_+ \cdot B_+ \to B_+ y_+ \cdot P$ is an isom. of (Poisson) affine spaces for $y_+ \in W^{W_P}$.

Total positivity: Rietsch, algebraic geometry (Frobenius splitting): Knutson, Lam, and Speyer.

Poisson side I

The codimension of a symplectic leaf in the open Richardson variety R_{y_-,y_+} is

$$\dim \ker(1 + y_+^{-1} y_-) = \dim E_{-1}(y_+^{-1} y_-).$$

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We will interpret DKP algebras as quantized algebras of functions on Schubert cells $(B_+ w \cdot B_+, \pi|_{B_+ w \cdot B_+})$, where $\pi := \pi_{G/B_+}$. First restrict the Poisson structure π to the translated open Schubert cell $wB_- \cdot B_+$. Note that $B_+ w \cdot B_+ \subset wB_- \cdot B_+$.

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Theorem. The T-orbits of symplectic leaves of the translated open Schubert cell $(wB_- \cdot B_+, \pi)$ are

$$S(y_-, y_+) = wB_- \cdot B_+ \cap R_{y_-,y_+} = wB_- \cdot B_+ \cap B_- y_- \cdot B_+ \cap B_+ y_+ \cdot B_+$$

parametrized by pairs $(y_-, y_+) \in W \times W$ such that $y_- \leq w \leq y_+$.

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Identify

$$\mathbb{C}[wB_- \cdot B_+] \cong \mathbb{C}[wB_- B_+]^{B_+} = \{c^\lambda_{\xi, v_\lambda} / c^\lambda_w \mid \lambda \in P_+, \xi \in V(\lambda)^*\},$$

$c^\lambda_{\xi,v}$ denotes the matrix coefficient of $v \in V(\lambda)$ and $\xi \in V(\lambda)^*$: for $g \in G$, $c^\lambda_{\xi,v}(g) = \langle \xi, gv \rangle$. Moreover v_λ is a h.w.v. of $V(\lambda)$, ξ_λ is a dual vector and $c^\lambda_w = c^\lambda_{w\xi_\lambda, v_\lambda}$.

Poisson side II

Denote $\mathfrak{n}_\pm = \mathrm{Lie}\, U_\pm$. For $y \in W$, define the ideals

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Schubert varieties are linearly defined (Kempf-Ramanathan):

$$\oplus_{\lambda \in P_+} H^0(G/B_+, \mathcal{L}_\lambda) \to \oplus_{\lambda \in P_+} H^0(X_y, \mathcal{L}_\lambda)$$

is surjective and its kernel is generated by elements in deg 1. So the ideal of $\overline{S_w(y_-, y_+)}$ $\subset wB_- \cdot B_+$ is generated by

$$\bigcup_j \{c^{\omega_j}_{\xi, v_{\omega_j}}/c^{\omega_j}_w \mid \xi \in (\mathcal{U}(\mathfrak{n}_-) y_- v_{\omega_j} \cap \mathcal{U}(\mathfrak{n}_+) y_+ v_{\omega_j})^\perp\}$$

Poisson str. on Schubert cells

Denote $U_+^w = U_+ \cap wU_- w^{-1}$, identify $j_w : U_+^w \cong B_+ w \cdot B_+$.

Demazure modules $V_w(\lambda) = \mathcal{U}(\mathfrak{b}_+) w v_\lambda = \mathcal{U}(\mathfrak{n}_+^w) w v_\lambda$. Then $\eta \in V_w(\lambda)^* \mapsto d_\eta^{w,\lambda} \in \mathbb{C}[U_+^w]$, $d_\eta^{w,\lambda}(u) = \langle \eta, u\dot{w} v_\lambda \rangle$, $u \in U_+^w$. One has

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Theorem. (1) The T-orbits of symplectic leaves of the Schubert cells (U_+^w, π) are

$$S_w(y) = j_w^{-1}(R_{y,w}) = U_+^w \cap B_- y B_+ w^{-1},$$

parametrized by $y \in W^{\leq w}$.

(2) The vanishing ideal of $\overline{S_w(y)}$ is:

$$\mathcal{V}(\overline{S_w(y)}) = \{ d_\eta^{w,\lambda} \mid \eta \in (\mathcal{U}(\mathfrak{n}_+) w v_\lambda \cap \mathcal{U}(\mathfrak{n}_-) y v_\lambda)^\perp \subset V_w(\lambda)^* \}.$$

(3) $\overline{S_w(y)}$ is generated by the above sets for $\lambda = \omega_1, \dots, \omega_r$.

Quantum Schubert cells

Define the quantized coordinate ring $R_q[U_+^w]$ of the Schubert cell $B_+ w \cdot B_+$ as the subset of $(\mathcal{U}_+)^*$ consisting of all matrix coefficients $d_\eta^{w,\lambda}(x) := \langle \eta, x T_w v_\lambda \rangle$ for $\eta \in V_w(\lambda)^*$.
Multiplication:

$$d_{\eta_1}^{w,\lambda_1} d_{\eta_2}^{w,\lambda_2} = q^{\langle \lambda_2, \lambda_1 - w^{-1}\mu_1 \rangle} d_\eta^{w,\lambda_1+\lambda_2},$$

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The universal R-matrix associated to w is given by

$$\mathcal{R}^w = \prod_{j=k,\dots,1} \exp_{q_{i_j}} \left((1 - q_{i_j})^{-2} X_{\beta_j}^+ \otimes X_{\beta_j}^- \right), \quad \exp_{q_i}(y) = \sum_{n=0}^{\infty} q_i^{n(n+1)/2} \frac{y^n}{[n]_{q_i}!}.$$

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Theorem. $R_q[U_+^w] \cong \mathcal{U}_-^w$ under

$$d_\eta^{w,\lambda} \mapsto (d_\eta^{w,\lambda} \otimes \mathrm{id})\mathcal{R}^w.$$

DKP algebras

Theorem. [Y.] Fix $w \in W$. For each $y \in W^{\leq w}$ define

$$I_w(y) = \{(d_\eta^{w,\lambda} \otimes \mathrm{id})(\mathcal{R}^w) \mid \lambda \in P_+, \eta \in (\mathcal{U}_+ T_w v_\lambda \cap \mathcal{U}_- T_y v_\lambda)^\perp\}.$$

Then:

(a) $I_w(y)$ is an H-invariant prime ideal of $\mathcal{U}_-^w$ and all H-invariant prime ideals of $\mathcal{U}_-^w$ are of this form.

(b) The correspondence $y \in W^{\leq w} \mapsto I_w(y)$ is an isomorphism from the poset $W^{\leq w}$ to the poset of H invariant prime ideals of $\mathcal{U}_-^w$ ordered under inclusion; that is $I_w(y) \subseteq I_w(y')$ for $y, y' \in W^{\leq w}$ if and only if $y \leq y'$.

(c) $I_w(y)$ is generated as a right ideal by

$$(d_\eta^{w,\omega_i} \otimes \mathrm{id})(\mathcal{R}^w) \quad \text{for} \quad \eta \in (\mathcal{U}_+ T_w v_{\omega_i} \cap \mathcal{U}^- T_y v_{\omega_i})^\perp, i = 1, \ldots, r,$$

where $\omega_1, \ldots, \omega_r$ are the fundamental weights of $\mathfrak{g}$.

The proof uses theorems of Gorelik and Joseph (ring theoretic results along the lines of the results of Ramanathan and Kempf–Ramanathan).

Algebras of quantum matrices

$R_q[M_{m,n}]$ is the $\mathbb{C}$-algebra generated by x_{ij}, $1 \le i \le m$, $1 \le j \le n$, with relations

$$\begin{aligned} x_{ij}x_{lj} &= q x_{lj} x_{ij}, \quad \text{for } i < l, \\ x_{ij}x_{ik} &= q x_{ik} x_{ij}, \quad \text{for } j < k, \\ x_{ij}x_{lk} &= x_{lk} x_{ij}, \quad \text{for } i < l, j > k, \\ x_{ij}x_{lk} - x_{lk}x_{ij} &= (q - q^{-1}) x_{ik} x_{lj}, \quad \text{for } i < l, j < k, \end{aligned}$$

$\mathbb{Z}^{m+n}$ acts on $R_q[M_{m,n}]$, by $(a_1, \ldots, a_m, b_1, \ldots, b_n) \cdot x_{ij} = q^{a_i - b_j} x_{ij}$.

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Corollary. [Y.] The $\mathbb{Z}^{m+n}$-invariant prime ideals of $R_q[M_{m,n}]$ are parametrized by $y \in S_{m+n}^{\le w_{m,n}}$. The ideal corresponding to y is generated by the sets of quantum minors

$$\Delta^q_{w_m^\circ(p_1(I)),(\overline{m+1,m+k}\setminus p_2(I))-m}$$

for $k \in \overline{1,n}$, $I \subset \overline{1,m+n}$, $|I| = k$, $I \le c^m(\overline{1,k})$, $I \not\ge y(\overline{1,k})$ and

$$\Delta^q_{w_m^\circ(p_1(I)\setminus\overline{1,k-n}),(\overline{m+1,m+n}\setminus p_2(I))-m}$$

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The Goodearl–Lenagan conjecture I

An ideal I of R has a polynormal generating sequence $y_1, \ldots, y_k$ if the set generates I and for all $i = 1, \ldots, k$ the image of y_i in $R/\langle y_1, \ldots, y_{i-1} \rangle$ is normal.

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The standard R-matrix identities in $R_q[G]$ imply

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where $u_{\pm\alpha} \in (\mathcal{U}_\pm)_{\pm\alpha}$.

If $\eta \in (V_w(\lambda_i))^*_\mu$ set $ht(\eta) = \langle \mu, \omega_1^\vee + \dots + \omega_r^\vee \rangle$.

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Theorem. [Y.] Fix an H-prime $I_y(w)$ of $\mathcal{U}^w_-$, $y \in W^{\leq w}$. Consider any linear ordering of the generating set from the previous theorem with the property that, if $\eta_1, \eta_2 \in (V(\omega_k)_w)^*$ and $ht(\eta_1) \leq ht(\eta_2)$, then $(d^{w,\omega_k}_{\eta_1} \otimes \mathrm{id})(\mathcal{R}^w)$ comes before $(d^{w,\omega_k}_{\eta_2} \otimes \mathrm{id})(\mathcal{R}^w)$. Any such sequence is a polynormal generating set of $I_y(w)$.

The Goodearl–Lenagan conjecture II

We obtain the following constructive proof of the Goodearl–Lenagan conjecture:

Corollary. Consider the $\mathbb{Z}^{m+n}$-invariant prime ideals of $R_q[M_{m,n}]$ corresponding to $y \in S_{m+n}^{\leq w_{m,n}}$ and a linear order on the generating set from the previous theorem with the property that, if $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_k\}$ satify $i_1 + \ldots + i_k \leq j_1 + \ldots + j_k$, then Δ_I comes before Δ_J. Any such sequence is a polynormal generating set of the prime ideal.

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One says that $\mathrm{Spec} A$ is normally separated if for all $P \subset Q$, $P, Q \in \mathrm{Spec} A$ there exists a nonzero normal element on Q/P.

One says that $H - \mathrm{Spec} A$ is graded normally separated if for all $P \subset Q$, $P, Q \in H - \mathrm{Spec} A$ there exists a nonzero homogeneous normal element on Q/P.

Corollary. The H-primes of $\mathcal{U}_-^w$ are graded normally separated.

$\mathrm{Spec}\,\mathcal{U}_-^w$ is normally separated

Theorem [Goodearl]. Assume that R is right noetherian. If $H - \mathrm{Spec} R$ is graded normally separated then $\mathrm{Spec} R$ is normally separated.

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Theorem [Levasseur–Stafford]. All iterated skew polynomial rings are Auslander–Gorenstein and Cohen–Macauley.

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Dimensions of strata of primes I

Recall that the stratum of $\mathrm{Spec}\mathcal{U}_{-}^{w}$ over each H-prime $I_y(w)$ in the Goodearl–Letzter stratification is homeomorphic to the spectrum of a Laurent polynomial ring. The latter is the center of the localization of $\mathcal{U}_{-}^{w}/I_y(w)$ by all nonzero homogeneous elements.

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Then

$$a_\lambda d^{w,\mu}_\xi = q^{-\langle (y+w)\lambda, \nu - w\mu\rangle} d^{w,\mu}_\xi a_\lambda, \quad \forall \xi (\in V_w(\mu))^*_\nu$$

in $(\mathcal{U}_-^w/I_y(w))[a_\lambda^{-1}, \lambda\in P_+]$.

Dimensions of strata of primes II

Therefore the center of the localization of $\mathcal{U}_-^w/I_y(w)$ by all nonzero homogeneous elements contains the Laurent polynomial ring spanned by

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If its dimension is greater, then we pass to an integral form of the algebra over $\mathbb{Z}[q, q^{-1}]$ and specialize at $q = 1$. That would imply that the center of the Poisson field of rational functions on the open Richardson variety $R_{y,w}$ has trascendence degree strictly greater than

$$\dim \ker(1 + y^{-1}w) = \dim E_{-1}(y^{-1}w)$$

which is a contradiction.

Quantum partial flag varieties I

Choose a set of simple roots $I \subset \overline{1,r}$ and consider the standard parabolic subgroup $P_I \supset B_+$. Consider the multicone:

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over G/P_I. Its coordinate ring is quantized to the subalgebra $R_q[G/P_I]$ of the restricted dual of $\mathcal{U}_q(\mathfrak{g})$ spanned by the matrix coefficients

$$c^{\lambda}_{\xi, v_\lambda}, \quad \lambda = \sum_{i \notin I} n_i \omega_i, n_i \in \mathbb{Z}_{\geq 0}, \xi \in V(\lambda)^*, v_\lambda - \text{h.w.v. of } V(\lambda).$$

The construction is due to Lakshmibai–Reshetikhin and Soibelman.

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Problem. Classify the H-invariant prime ideals of $R_q[G/P_I]$ not containing the augmentation ideal.

Only two cases were previously known: full flag varieties Gorelik and Grassmannians Launois–Lenagan–Rigal 2008.

Quantum partial flag varieties II

Denote by $H - \mathrm{Spec}_+(R_q[G/P_I])$ the set of H-invariant prime ideals of $R_q[G/P_I]$ not containing the augmentation ideal. Denote the quantum Schubert cell ideals:

$$Q(w)_I^+ = \mathrm{Span}\{c^{\lambda}_{\xi, v_\lambda} \mid \lambda = \sum_{i \notin I} n_i \omega_i, \xi \in V(\lambda)^*, \xi \perp \mathcal{U}_+ T_w v_\lambda\}, \quad w \in W^{W_I}.$$

They are $\mathcal{U}_q(\mathfrak{b}_+)$ invariant prime ideals of $R_q[G/P_I]$.

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We have the decomposition:

$$H - \mathrm{Spec}_+(R_q[G/P_I]) = \sqcup_{w \in W^{W_I}} X_I^w$$

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Denote $c^\lambda_w = c^\lambda_{T_w\xi_\lambda,v_\lambda}, c^I_w = \{c^\lambda_w \mid \lambda = \sum_{i\notin I} n_i\omega_i\}$.

Proposition. For all $w \in W^{W_I}$ the algebras

$$\big((R_q[G/P_I]/Q(w)_I^+)[(c^I_w)^{-1}]\big)^H \quad \text{and} \quad \mathcal{U}_-^w$$

are isomorphic and for each $\mathcal{I} \in X_I^w$, $\mathcal{I} \cap c^I_w = \emptyset$. (Similar strategy to the one for the isomorphism between the 2 realizations of DKP algebras.)

Quantum partial flag varieties III

Theorem. [Y.] For an arbitrary partial flag variety G/P_I the H-invariant prime ideals of $R_q[G/P_I]$ (not containing the augmentation ideal) are parametrized by

$$\{(y_-, y_+) \in W \times W^{W_I} \mid y_- \leq y_+\}.$$

Denote by $\mathcal{I}^I_{y_-, y_+}$ the ideal corresponding to (y_-, y_+).

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Conjecture. Let $y_-, y'_- \in W$, $y_+, y'_+ \in W^{W_I}$, $y_- \leq y_+$, $y'_- \leq y'_+$. Then $\mathcal{I}^I_{y_-,y_+} \subseteq \mathcal{I}^I_{y'_-,y'_+}$ if and only if there exits $z \in W_I$ such that

$$y_- \geq y'_- z \text{ and } y_+ \leq y'_+ z.$$

The conjecture is open even for Grassmannians.