Bessel Models for General Admissible Induced Representations: The Compact Stabilizer Case

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2. *There exists a generic, unitary character χ on N, such that its stabilizer in M,*

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If $P \subset G$ is a parabolic subgroup satisfying 1. and 2., then we say that P is a Siegel parabolic subgroup.

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Given a Siegel parabolic subgroup, $P = MAN$, of G, and a generic character, χ, of N, with compact stabilizer, set

$$Wh_\chi(V) = \{\lambda : V \longrightarrow \mathbb{C} \,|\, \lambda(\pi(n)v) = \chi(n)\lambda(v)\}.$$

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This is the so called space of Bessel functionals (or Bessel models) of V.

Let G and $P = MAN$ be as before. Let (σ, V_σ) be an admissible, smooth, Fréchet representation of M, and let $\nu \in Lie(A)'_{\mathbb{C}}$.

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Given $\mu \in V'_\sigma$ set $\gamma_\mu(\nu) = \mu \circ J^{\chi}_{P,\sigma_\nu}$. Then, for ν as above, γ_μ defines a weakly holomorphic map into $(I^{\infty}_{M\cap K,\sigma|_{M\cap K}})'$.

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Theorem

1. *γ_μ has a weakly holomorphic continuation to all of $Lie(A)'_{\mathbb{C}}$*
2. *Given $\nu \in Lie(A)'_{\mathbb{C}}$ define*

$$\lambda_\mu(f_{P,\sigma_\nu}) = \gamma_\mu(\nu)(f), \qquad f \in I^{\infty}_{M\cap K,\sigma|_{M\cap K}}.$$

Then $\lambda_\mu \in Wh_\chi(I^{\infty}_{P,\sigma_\nu})$ and the map $\mu \mapsto \lambda_\mu$ defines an M_χ-equivariant isomorphism between V'_σ and $Wh_\chi(I^{\infty}_{P,\sigma_\nu})$.

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Fix a non-degenerate unitary character, $\chi_\circ$, of $\mathbb{R}$.

1. $G = Sp(n, \mathbb{R})$ realized as $2n \times 2n$ matrices g, such that $gJ_ng^T = J_n$, with

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   We can describe $\mathfrak{g} = Lie(G)$ as the Lie subalgebra of $M_{2n}(\mathbb{H})$ of matrices of the form

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   If we define M, A, N and χ in a similar way as before, $MA \cong GL(n, \mathbb{H})$ and $M_\chi \cong Sp(n)$, the quaternionic unitary group.

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   Example 4 corresponds to the octonions, $\mathbb{O}$. Here we replace $M_3(\mathbb{O})$ by $\mathfrak{m} \oplus \mathfrak{a} = E_{6,2} \oplus \mathbb{R}$, and take for X, Y elements of the exceptional Euclidean Jordan algebra (the 3×3 conjugate adjoint matrices over $\mathbb{O}$).

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If we now define χ as in the above examples, then M_χ is isomorphic to compact F_4.

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Then

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2. *Let* w_G *be the longest element of* W, w_M *the longest element of* W_M, *and set* $w^M = w_G w_M$. *Then*

$$P_\circ (w^M)^* P = P (w^M)^* N$$

and if $w \neq w^M$ *then*

$$\dim P_\circ w^* P < \dim P (w^M)^* N.$$

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Hence, there exist linear functionals $\varepsilon_1, ..., \varepsilon_n$ on $\mathfrak{a}_o = Lie(A_o)$ such that

$$\Phi^+ = \{\varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq n\} \cup \{2\varepsilon_1, ..., 2\varepsilon_n\}$$

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Let $X \in Lie(N)$ be such that $[H, X] = 2\varepsilon_i(H)X$, for all $H \in Lie(A_\circ)$. For such an X it can be checked that $d\chi(X) \neq 0$.

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Therefore $w^{-1} \cdot (\varepsilon_i + \varepsilon_j) \in -\Phi^+$ for all $i \leq j$, which implies that $w = w^M$. □

The Theory of the Transverse Symbol of Kolk-Varadarajan

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We will denote by

$$D'(X : E) := (C_c^\infty(X : E))'$$

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We will call any element in this space an E-distribution on X.

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Let

$$I_x^{(r)} = \mathrm{Diff}_x^{(r-1)} + V_x^{(r)}.$$

Choosing local coordinates at x it can be seen that $I_x^{(r)}$ actually is the stalk at x of a subsheaf $I^{(r)} \subset \mathrm{Diff}^{(r)}$.

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Hence we have a well-defined quotient sheaf

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We say that $T \in D'(X : E)$ has transverse order $\leq r$ at $x \in O$, if there exists an open neighborhood U of x in X, such that for all $f \in C_c^\infty(U : E)$, with the property that $Df|_{O \cap U} = 0$ for all $D \in \mathrm{Diff}^{(r)}(U)$, $T(f) = 0$.

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Let $D_O'^{(r)}(X : E)$ be the linear subspace of elements in $D'(X : E)$ which have transverse order $\leq r$ at all points of O.

Theorem (Kolk-Varadarajan)

Let X be a C^∞ manifold with a left action of H, let (π, E) be a smooth Fréchet representation of a normal subgroup H' of H, and let $O \subset X$ be an H-orbit of X.

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1. *Assume that the action of H' can be extended to an action of H. If there exists $y \in O$, such that*

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($\mathbb{C}_y$ is just $\mathbb{C}$, with an H'-action given by the character $\chi_y = \frac{\delta_{H'}}{\delta_{H_y'}}$).

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$$T \in D_O'^{(r)}(X : E)/D_O'^{(r-1)}(X : E),$$

there exists $\mu_y \in (M_y^{(r)} \otimes E' \otimes \mathbb{C}_y')^{H_y}$ such that

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Furthermore the map $f \otimes v \mapsto f_{P,\sigma,\nu,v}$ from $C_c^\infty(G)\bar{\otimes} V_\sigma$ to U_{P,σ_ν} is surjective.

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Let

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Given $\lambda \in Wh_\chi(I_{P,\sigma_\nu})$, define $\bar{\lambda} \in D'(P(w^M)^*N : V_\sigma)$ by

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Hence, according to part ii) of Kolk-Varadarajan theorem, there exist $\mu_\lambda \in V'_\sigma$ such that

$$\begin{aligned}
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We will denote the map $\lambda \mapsto \mu_\lambda$ by Φ_{P,σ_ν}.

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If $\lambda \in Wh_\chi(I^\infty_{P,\sigma_\nu})$ *and* $\lambda_{|U_{P,\sigma_\nu}} = 0$ *then* $\lambda = 0$*.*

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The map

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proof

Using Casselman subrepresentation theorem, we can reduce the proof to the case where σ is an induced representation.

proof

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Proceeding as before, we can define a distribution

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Now, if we can prove that

$$D'_{P_\circ w^* K_M N}(G : F \otimes \mathbb{C}_\chi)^{N_\circ \times N} = (0) \qquad \forall w \in W^M, \quad w \neq w^M,$$

then, the standard Bruhat theoretic argument shows that $\bar{\lambda}$, and hence λ, is equal to 0.

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Now observe that $K_M = M_\chi$. Hence we can extend the action of $N_\circ \times N$ on $F \otimes \mathbb{C}_\chi$ to an action of $P_\circ \times K_M N$.

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Therefore, from part 1. of Kolk-Varadarajan theorem, we just need to show that

$$(M_{w^*}^{(r)} \otimes (F \otimes \mathbb{C}_\chi)')^{(N_\circ \times N)_{w^*}} = (0), \qquad \forall r \geq 0.$$

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But this follows from the fact $N_\circ$ acts unipotently on F' and that the restriction of χ to $(w^*)^{-1} N_\circ w^* \cap N$ is non-trivial.

Tensoring with Finite Dimensional Representations

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Theorem

1. *The map*
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Tensoring with Finite Dimensional Representations

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Corollary

Let (τ, F) be an irreducible representation of M_χ, and let $Wh_{\chi,\tau}(I^\infty_{P,\sigma_\nu})$ be the set of maps $T : I^\infty_{P,\sigma_\nu} \to F$ such that $T(\pi_{P,\sigma_\nu}(nm)\phi) = \chi(n)\tau(m)T(\phi)$ for all $n \in N$, $m \in M_\chi$. Then

$$\dim Wh_{\chi,\tau}(I^\infty_{P,\sigma_\nu}) = \dim Hom_{M_\chi}(V_\sigma, F)$$

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Observe that $\dim A = 1$ and, furthermore, there exists $H \in Lie(A)$ such that, if we set

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Observe that there is natural isomorphism of G-modules,

$$\begin{array}{ccc} I^{\infty}_{P,\sigma_\nu} \otimes F & \cong & I^{\infty}_{P,\sigma_\nu \otimes \eta} \\ \phi & \rightarrow & \hat{\phi} \\ \check{\phi} & \leftarrow & \phi, \end{array}$$

Lemma (Wallach)

Let $\mathfrak{g} = Lie(G)$. *There exists an element* $\Gamma \in U(\mathfrak{g})^{M_\chi}$ *such that*

1. *The map*

$$\Gamma : Wh_\chi(I^\infty_{P,\sigma_\nu}) \otimes F' \longrightarrow Wh_\chi(I^\infty_{P,\sigma_\nu} \otimes F)$$

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Then it's clear, from the above lemma, that $\check{\Gamma}$ defines an M_χ-equivariant isomorphism.

Let $\nu \in Lie(A)'_{\mathbb{C}}$ be such that Φ_{P,σ_ν} is an isomorphism. Let $\tilde{\Gamma}$ be the map that makes the following diagram commute

$$\begin{tikzcd}
Wh_\chi(I^\infty_{P,\sigma_\nu}) \otimes F' \arrow[r, "\check{\Gamma}"] \arrow[dd, "\Phi_{P,\sigma_\nu} \otimes id"] & Wh_\chi(I^\infty_{P,\sigma_\nu \otimes \eta}) \arrow[d, "\Phi_{P,\sigma_\nu\otimes\eta}"] \\
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Proposition

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In particular, if we choose a representation (η, F), such that the action of M on F_r is trivial, then $\sigma_\nu \otimes \bar{\eta}_0 \cong \sigma_{\nu-r}$, and hence

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Proposition

There exists and isomorphism

$$\phi : Wh_\chi(I^\infty_{P,\sigma_\nu\otimes\eta}) \longrightarrow \bigoplus_{j=0}^{r} W^{j+1}|_{I^\infty_{P,\sigma_\nu\otimes\eta_j}}$$

such that the following diagram is commutative:

Proof (of theorem)

$$
\begin{tikzcd}
Wh_\chi(I^\infty_{P,\sigma_\nu\otimes\eta}) \arrow[r, "\phi"] \arrow[dddr, "\Phi_{P,\sigma_\nu\otimes\eta}"] & \oplus_{j=0}^r W^{j+1}|_{I^\infty_{P,\sigma_\nu\otimes\eta_j}} \arrow[d] \\
 & \oplus_{j=0}^r Wh_\chi(I^\infty_{P,\sigma_\nu\otimes\bar\eta_j}) \arrow[d, "\Phi_{P,\sigma_\nu\otimes\bar\eta_j}"] \\
 & \oplus_{j=0}^r V'_{\sigma\otimes\bar\eta_j} \arrow[d, "X_j/X_{j+1}\cong F_{r-2j}"] \\
 & (V_\sigma\otimes F)'
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Proceeding by induction, it can now be shown that Φ_{P,σ_ν} is an isomorphism for all $\nu \in Lie(A_\circ)'_{\mathbb{C}}$.

The Holomorphic Continuation of Certain Jacquet Integrals

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defines a K_M*-equivariant isomorphism for all* $\nu \in \mathfrak{a}'_{\mathbb{C}}$*.*
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We have already seen that Φ_{P,σ_ν} is an isomorphism for all $\nu \in \mathfrak{a}'_\mathbb{C}$.

We will now show that the map $\nu \mapsto \mu \circ J^\chi_{P,\sigma_{nu}}$ is weakly holomorphic for all $\nu \in Lie(A_\circ)'_\mathbb{C}$.

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Let $\{v_j\}_{j=1}^{m}$ be a basis of F, and let $\{l_j\}_{j=1}^{m}$ be its dual basis.

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Let $\{v_j\}_{j=1}^m$ be a basis of F, and let $\{l_j\}_{j=1}^m$ be its dual basis. Then we can find $\eta_j \in Wh_\chi(I^{\infty}_{P,\sigma_\nu})$, $j = 1, \ldots, m$, and $\psi \in I_{P,\sigma_\nu \otimes \eta}$ such that

$$\begin{aligned}
\gamma_\mu(\nu - r)(\phi) &= \lambda(\phi_{P,\sigma,\nu-r}) = \check{\Gamma}(\sum \eta_j \otimes l_j)(\psi) \\
&= \Gamma(\sum \eta_j \otimes l_j)(\check{\psi}) \\
&= (\sum \eta_j \otimes l_j)(\Gamma^T \check{\psi}).
\end{aligned}$$

Proof (of theorem)

$$\gamma_\mu(\nu - r)(\phi) = (\sum \eta_j \otimes l_j)(\Gamma^T \check{\psi}).$$

Now since $\Gamma^T \check{\psi} \in I^{\infty}_{P,\sigma_\nu} \otimes F$,

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This is the desired shift equation which shows that γ_μ is weakly holomorphic everywhere.

Further Research

Bessel-Plancherel Measure for M_χ compact

Theorem

Let G, $P = MAN$, χ and M_χ be as before.

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Let G, $P = MAN$, χ and M_χ be as before. Then the spectral decomposition of $L^2(N\backslash G;\chi)$ with respect to the action of $M_\chi \times G$ is given by

$$L^2(N\backslash G;\chi) \cong \int_{G^\wedge} \bigoplus_{\tau \in M_\chi^\wedge} Wh_{\chi,\tau}(\pi) \otimes \tau^* \otimes \pi \, d\mu(\pi)$$

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where μ is the usual Plancherel measure of G.

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Let G and $P = MAN$ be as before, and let χ be a generic character of N.

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Theorem

1. *γ_μ extends to a weakly holomorphic map from $\mathfrak{a}'_{\mathbb{C}}$ to $Hom(I^\infty_{M\cap K,\sigma|_{M\cap K}}, V_\tau)$.*
2. *Given $\nu \in \mathfrak{a}'_{\mathbb{C}}$ define*

$$\lambda_\mu(f_{P,\sigma,\nu}) = \gamma_\mu(\nu)(f), \qquad f \in I^\infty_{M\cap K,\sigma|_{M\cap K}}.$$

Then $\lambda_\mu \in Wh_{\chi,\tau}(I^\infty_{P,\sigma,\nu})$ and the map $\mu \mapsto \lambda_\mu$ defines an isomorphism between $Hom_{M_\chi}(H_{\sigma^{w_M}}, V_\tau)$ and $Wh_{\chi,\tau}(I^\infty_{P,\sigma,\nu})$.

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where ν and μ are the usual Plancherel measures of M_χ and G, respectively.