# Triangulation and discretizations of metric measure spaces 

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Technion, Haifa
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## Motivation

## Point of view : Of a "newcomer", an "amateur", of (Discrete)

Differential Geometry background.
Therefore, interested in :

- Geometry

Beyond the "BIG" Theorems: Bishop-Gromov,
Bonnet-Myers, Brunn-Minkowski, Sobolev and Poincare
Inequalities, etc.....

- Triangulations
- Discretizations
- Geodesics
- Applications
- Information Geometry
- Manifold Learning
- Sampling Theory


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## Motivation - cont.

- Image Processing and Analysis



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## Efficient packings

## Definition

Let $p_{1}, \ldots, p_{n_{0}}$ be points $\in M^{n}$, satisfying the following conditions:
(1) The set $\left\{p_{1}, \ldots, p_{n_{0}}\right\}$ is an $\varepsilon$-net on $M^{n}$, i.e. the balls $\beta^{n}\left(p_{k}, \varepsilon\right), k=1, \ldots, n_{0}$ cover $M^{n}$;
(2) The balls (in the intrinsic metric of $\left.M^{n}\right) \beta^{n}\left(p_{k}, \varepsilon / 2\right)$ are pairwise disjoint.


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Then the set $\left\{p_{1}, \ldots, p_{n_{0}}\right\}$ is called a minimal $\varepsilon$-net and the packing with the balls $\beta^{n}\left(p_{k}, \varepsilon / 2\right), k=1, \ldots, n_{0}$, is called an efficient packing.
The set $\left\{(k, I) \mid k, I=1, \ldots, n_{0}\right.$ and $\left.\beta^{n}\left(p_{k}, \varepsilon\right) \cap \beta^{n}\left(p_{l}, \varepsilon\right) \neq \emptyset\right\}$ is
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## Efficient packings - cont.

- In the following $M^{n}=\left(M^{n}, g\right)$ is a closed, connected $n$-dimensional Riemannian manifold with sectional curvature $k_{M}$ bounded from below by $k$, diam $M^{n}$ bounded from above by $D$, and $\operatorname{Vol} M^{n}$ bounded from below by $v$.
- Efficient packings have then the following important properties, which we list below
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Lemma (Grove-Petersen, 1988)
There exists $n_{1}=n_{1}(n, k, D)$, such that if $\left\{p_{1}, \ldots, p_{n_{0}}\right\}$ is a minimal $\varepsilon$-net on $M^{n}$, then $n_{0} \leq n_{1}$.

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> Let $M_{1}^{n}, M_{2}^{n}$, be manifolds having the same bounds $k=k_{1}=k_{2}$ and $D=D_{1}=D_{2}$ (see above) and let $\left\{p_{1}, \ldots, p_{n_{0}}\right\}$ and $\left\{q_{1}, \ldots, q_{n_{0}}\right\}$ be minimal $\varepsilon$-nets with the same intersection pattern, on $M_{1}^{n}, M_{2}^{n}$, respectively. Then there exists a constant $n_{3}=n_{3}(n, k, D, C)$, such that if $d\left(p_{i}, p_{j}\right)<C \cdot \varepsilon$, then $d\left(q_{i}, q_{j}\right)<n_{3} \cdot \varepsilon$.

This properties provide us with a simple triangulation method of closed, connected Riemannian manifolds. Indeed, we can construct a simplicial complex having as vertices the centers of the balls $\beta^{n}\left(p_{k}, \varepsilon\right)$, as follows

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## Efficient packings - cont.

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## Efficient packings - cont.

- One can ensure that the triangulation will be convex and that its simplices are convex, by choosing $\varepsilon=\operatorname{ConvRad}\left(M^{n}\right)-$ the convexity radius of $M^{n}$ :
$\operatorname{ConvRad}\left(M^{n}\right)=\inf \left\{r>0 \mid \beta^{n}(x, r)\right.$ is convex, for all $\left.x \in M^{n}\right\}$ (Note that: $\operatorname{ConvRad}\left(M^{n}\right) \geq \frac{1}{2} \operatorname{InjRad}\left(M^{n}\right)$, where $\operatorname{InjRad}\left(M^{n}\right)$ denotes the injectivity radius of $M^{n}$

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## Grove-Petersen

Note that by a classical result of Cheeger, there is a universal positive lower bound for $\operatorname{InjRad}(M)$ in terms of $k, D$ and $v$, where $v, k$ are the lower bounds for the volume and sectional curvature, respectively and $D$ is an upper bound on the diameter of $M^{n}$.

Moreover, we have the following Finiteness Theorem

Let $M_{1}^{n}, M_{2}^{n}$ be two manifolds having the same upper diameter bound D, as well as the same lower bounds $k$ and $v$, on their curvatures and volumes, respectively. Then there exists $\varepsilon=\varepsilon(n, k, D, v)$ such that, if $M_{1}$ and $M_{2}$ have minimal packings with identical intersection patterns, then they are homotopy equivalent.

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Since only volumes of balls arguments are employed, one can replace the last condition by the more general one $\operatorname{Ric}_{M} \geq(n-1) k$.

The basic tool for proving the Lemmas is

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## Theorem (Bishop-Gromov)

Let $\left(M^{n}, g\right)$ be a complete Riemannian manifold satisfying. $\operatorname{Ric}_{M} \geq(n-1) k$. Then, for any $x \in M=M^{n}$, the function

$$
\varphi(r)=\frac{\operatorname{Vol} B(x, r)}{\int_{0}^{r} S_{K}^{n}(t) d t},
$$

is nonincreasing (as function of $r$ ), where

## Bishop-Gromov - cont.

$$
S_{K}^{n}(r)= \begin{cases}\left(\sin \sqrt{\frac{K}{n-1}} r\right)^{n-1} & \text { if } K>0 \\ r^{n-1} & \text { if } K=0 \\ \left(\sinh \sqrt{\frac{|K|}{n-1}} r\right)^{n-1} & \text { if } K<0\end{cases}
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(Here $S_{K}^{n}$ is the model space form.)

## Apology

Warning!
We do not introduce here the definition(s) of (weak) $\mathrm{CD}(\mathrm{K}, \mathrm{N})$ spaces, because

- They are technically involved and lengthy. ...and
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## Adaptation to $M M S p$

## Theorem (Generalized Bishop-Gromov Inequality, Lott-Villani 2009, <br> Sturm 2006)

Let $M$ be a Riemannian manifold equipped with a reference measure $\nu=e^{-V}$ Vol and satisfying a curvature-dimension condition $C D(K, N), K \in \mathbb{R}, 1<N<\infty$. Then, for any $x \in M$, the function

$$
\varphi(r)=\frac{\nu[B(x, r)]}{\int_{0}^{r} S_{K}^{N}(t) d t},
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## Adaptation to $M M S p$ - cont.

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## Remark

A similar - but technically more involved - resulta holds for weak $\mathrm{CD}(\mathrm{K}, \mathrm{N})$ spaces.
a. Sturm, 2006

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## Lemma

Let $(X, d, \nu)$ be a compact weak $\mathrm{CD}(K, N)$ space, $N<\infty$, such that $\operatorname{Supp} \nu=X$ and such that $\operatorname{diam} X \leq D$. Then there exists $n_{1}=n_{1}(K, N, D)$, such that if $\left\{p_{1}, \ldots, p_{n_{0}}\right\}$ is a minimal $\varepsilon$-net in $X$, then $n_{0} \leq n_{1}$.

## Remark

Note that, since $N<\infty$, the condition Supp $=X$ imposes no real restriction on X. a

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$\mid\left\{j \mid j=1, \ldots, n_{0}\right.$ and $\left.\beta^{n}(x, \varepsilon) \cap \beta^{n}\left(p_{j}, \varepsilon\right) \neq \emptyset\right\} \mid \leq n_{2}$, for any minimal $\varepsilon$-net $\left\{p_{1}, \ldots, p_{n_{0}}\right\}$.


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## Lemma

Let $\left(M_{1}^{n}, d_{1}, \nu_{1}\right)$ and $\left(M_{2}^{n}, d_{2}, \nu_{2}\right)$ be as in the first Lemma, and let $\left\{p_{1}, \ldots, p_{n_{0}}\right\}$ and $\left\{q_{1}, \ldots, q_{n_{0}}\right\}$ be minimal $\varepsilon$-nets with the same intersection pattern, on $M_{1}^{n}, M_{2}^{n}$, respectively. Then there exists a constant $n_{3}=n_{3}(N, K, D, C)$, such that if $d_{1}\left(p_{i}, p_{j}\right)<C \cdot \varepsilon$, then $d_{2}\left(q_{i}, q_{j}\right)<n_{3} \cdot \varepsilon$.

## Limits

- The existence of the triangulation follows now immediately, since, by definition, weak $\mathrm{CD}(K, N)$ spaces are geodesic. Moreover, in nonbranching ${ }^{1}$ spaces, the geodesics connecting two vertices of the triangulation are unique a.e.
- Unfortunately, the lower bound on the sectional curvature is essential for the proof of the Grove-Petersen Homotopy Theorem. Therefore, one cannot formulate a similar theorem for weak $\mathrm{CD}(K, N)$ spaces. However, it is possible for weighted manifolds, by imposing the additional constraint on sectional curvature

1. i.e. any two geodesics $\gamma_{1}, \gamma_{2}:[0, t] \rightarrow X$ that coincide on a subinterval $\left[0, t_{0}\right], 0<t_{0}<t$, coincide on $[0, t]$.

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- The existence of the triangulation follows now immediately, since, by definition, weak $\mathrm{CD}(K, N)$ spaces are geodesic. Moreover, in nonbranching ${ }^{1}$ spaces, the geodesics connecting two vertices of the triangulation are unique a.e.
- Unfortunately, the lower bound on the sectional curvature is essential for the proof of the Grove-Petersen Homotopy
Theorem. Therefore, one cannot formulate a similar theorem for weak $\mathrm{CD}(K, N)$ spaces. However, it is possible for weighted manifolds, by imposing the additional constraint on sectional curvature :

[^0]
## Limits - cont.

## Theorem

Let $\left(M_{1}^{n}, d_{1}, \nu_{1}\right),\left(M_{2}^{n}, d_{2}, \nu_{2}\right), \nu_{i}=e^{-V_{i}} d V o l, V_{i} \in \mathcal{C}^{2}(\mathbb{R}), i=1,2$ be smooth, compact metric measure spaces satisfying $\mathrm{CD}(K, N)$ for some $K \in \mathbb{R}$ and $1<N<\infty$, and such that $\operatorname{diam} M_{i}^{n}<D, \operatorname{Vol} M_{i}^{n}<v, i=1,2$ and, moreover, having the same lower bound $k$ on their sectional curvatures. Then there exists $\varepsilon=\varepsilon(N, K, k, D, v)$ such that, if $M_{1}^{n}, M_{2}^{n}$ have minimal packings with identical intersection patterns, they are homotopy equivalent.

We can strengthen the simple result above, to render a "geometrically nice" triangulation, namely we can formulate the following

Proposition
Any smooth, compact metric measure space ( $M^{n}, d, v$ ) satisfying $\mathrm{CD}(K, N)$ admits a $\varphi^{*}$-thick triangulation, where

## Recall that thick (or fat) triangulations are defined as follows

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## Definition

Let $\tau \subset \mathbb{R}^{n} ; 0 \leq k \leq n$ be a $k$-dimensional simplex. The thickness $\varphi$ of $\tau$ is defined as being :

$$
\varphi=\varphi(\tau)=\inf _{\substack{\sigma<\tau \\ \operatorname{dim} \sigma=j}} \frac{\operatorname{Vol}_{j}(\sigma)}{\operatorname{diam}^{j} \sigma} .
$$

The infimum is taken over all the faces of $\tau, \sigma<\tau$, and $\operatorname{Vol}_{j}(\sigma)$ and $\operatorname{diam} \sigma^{j}$ stand for the Euclidian $j$-volume and the diameter of $\sigma$ respectively. (If $\operatorname{dim} \sigma=0$, then $\operatorname{Vol}_{j}(\sigma)=1$, by convention.) A simplex $\tau$ is $\varphi_{0}$-thick, for some $\varphi_{0}>0$, if $\varphi(\tau) \geq \varphi_{0}$. A triangulation (of a submanifold of $\mathbb{R}^{n}$ ) $\mathcal{T}=\left\{\sigma_{i}\right\}_{i \in \mathrm{I}}$ is $\varphi_{0}$-thick if all its simplices are $\varphi_{0}$-thick. A triangulation $\mathcal{T}=\left\{\sigma_{i}\right\}_{i \in \mathbf{I}}$ is thick if there exists $\varphi_{0} \geq 0$ such that all its simplices are $\varphi_{0}$-thick.

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## This type of triangulations allows us, amongst others, to construct ${ }^{2}$ quasimeromorphic mappings :


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## Definition

Let $M^{n}, N^{n}$ be oriented, Riemannian $n$-manifolds.
(1) $f: M^{n} \rightarrow N^{n}$ is called quasiregular (qr) iff
(1) $f$ is locally Lipschitz (and thus differentiable a.e.); and
(2) $0<\left|f^{\prime}(x)\right|^{n} \leq K J_{f}(x)$, for any $x \in M^{n}$;
where $\left|f^{\prime}(x)\right|=\sup _{|h|=1}\left|f^{\prime}(x) h\right|$, and where $J_{f}(x)=\operatorname{det} f^{\prime}(x)$;
(2) quasimeromorphic (qm) iff $N^{n}=\mathbb{S}^{n}$.

The smallest number $K$ that satisfies condition (b) above is called the outer dilatation of $f$.
2. using the so called Alexander trick

Emil Saucan
Triangulation and discretizations of metric measure s

## Remark

Usually $\mathbb{S}^{n}$ is identified with $\widehat{\mathbb{R}^{n}}=\mathbb{R}^{n} \cup\{\infty\}$ endowed with the spherical metric.

## Corollary

Any smooth, compact metric measure space ( $M^{n}, d, \nu$ ) satisfying $\mathrm{CD}(K, N)$ admits a non-constant quasimeromorphic mapping $f: M^{n} \rightarrow \mathbb{S}^{n}$.

## Remark

This result can be extended to $m$ dimensional Alexandrov spaces with curvature $\geq K$.
(Recall that Alex $[K] \subset C D((m-1) K, m)) \cdot{ }^{a}$ )
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## First Application : Information Geometry

The method of triangulation and qm mapping of weighted Riemannian manifolds above represents a generalization of a result classical in Information Geometry:

- Let $A$ be a finite set, let $f_{i}(x), i=1,2$ be bounded distributions on $A$, and let $p_{i}(x)=\frac{f_{i}(x)}{f_{i(x)}}$, viewed as probability densities on
- The relative information between $p_{1}$ and $p_{2}$ (or the Kullback-Leibler divergence) is defined as



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- The relative information between $p_{1}$ and $p_{2}$ (or the Kullback-Leibler divergence) is defined as

$$
K L\left(p_{1} \| p_{2}\right)=\sum_{A} p_{1} \log \left(\frac{p_{1}}{p_{2}}\right)
$$

## First Application : Information Geometry - cont.

- $K L\left(p_{1} \| p_{2}\right)$ represents a generally accepted measure of the divergence between the two given probabilities, but, unfortunately, it fails to be a metric.
- However, it induces a Riemannian metric on $P(A)$ - the manifold of probability densities on $A$, namely the Fisher information metric


## where $p \in P(A)$ is given and $\Delta$ represents an infinitesimal

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- However, it induces a Riemannian metric on $P(A)$ - the manifold of probability densities on $A$, namely the Fisher information metric :

$$
g_{\text {Fischer }, p}(\Delta)=K L(p, p+\Delta)=\sum_{A} \frac{\Delta(x)^{2}}{p(x)},
$$

where $p \in P(A)$ is given and $\Delta$ represents an infinitesimal perturbation.

## First Application : Information Geometry - cont.

- It turns out that the Fisher information can be written as Riemannian metric in the following form : $g_{\text {Fischer, }}=\left(g_{i j}\right)$, where



## First Application : Information Geometry - cont.

- It turns out that the Fisher information can be written as Riemannian metric in the following form : $g_{\text {Fischer, }}=\left(g_{i j}\right)$, where

$$
g_{i j}=E_{p}\left(\frac{\partial \mathbf{I}}{\partial \theta^{i}}, \frac{\partial \mathbf{I}}{\partial \theta^{j}}\right),
$$

where $\theta^{1}, \ldots, \theta^{k}, k=|A|$ represent the coordinates on $\sigma_{0}$, $\mathbf{I}=\log p$ is the so called $\log$-likelihood and $E_{p}(f g)$ denotes the expectation of $f g, E_{p}(f g)=\int f g d p$.

## First Application : Information Geometry - cont.

- The correspondence $p(x) \mapsto u(x)=2 \sqrt{p(x)}$ maps the probability simplex $\sigma_{0}=\left\{p(x) \mid x \in A, p(x)>0, \sum_{A} p(x)=1\right\}$, onto the first orthant of the sphere $S=\sum_{A} u(x)^{2}=4$. This mapping preserves the geometry, in the sense that the geodesic distance between $p, q \in \sigma_{0}$ ), measured in the Fisher metric, equal the spherical distance between their images (under the mapping above). Moreover, geodesics are mapped to great circles.


## First Application : Information Geometry - cont.

To summarize :The quasimeromorphic mapping of a weighted Riemannian manifold ( $M^{n}, d, \nu$ ) onto the $n$-dimensional unit sphere $\mathbb{S}^{n}$, represents a generalization of the considerents above in two manners

- It allows for the mapping with controlled and bounded distortion (i.e. qm) of a more general class of Riemannian manifolds (with arbitrary metrics) endowed with a variety of (probability) measures, and not just of the standard statistical model ;
- It permits the reduction to the study of the geometry of the standard simplex in $\mathbb{S}^{n}$, of the geometry of the whole information manifold, and not just of the probability simplex.


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## Kanai

We concentrate on unbounded spaces and generalize the work of Kanai on discretizations of Riemannian manifolds to MMSp.

- As before, we consider $\varepsilon$-nets $\mathcal{N}$, with the further proviso that they are maximal with respect to inclusion.
- We call the graph $G(\mathcal{N})$ obtained as above (i.e the 1-skeleton of the simplicial complex constructed) a discretization of $X$, with separation $\varepsilon$ and covering radius $\varepsilon$, (or a $\varepsilon$-separated net).
- We say that $G(\mathcal{N})$ has bounded geometry iff there exists $\rho_{0}>0$, such that $\rho(p) \leq \rho_{0}$, for any vertex $p \in \mathcal{N}$, where $\rho(p)$ denotes the degree of $p$.


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## Kanai - cont.

## Definition (Rough isometry)

Let $(X, d)$ and $(Y, \delta)$ be two metric spaces, and let $f: X \rightarrow Y$ (not necessarily continuous). $f$ is called a rough isometry iff
(1) There exist $a \geq 1$ and $b>0$, such that

$$
\frac{1}{a} d\left(x_{1}, x_{2}\right)-b \leq \delta\left(f\left(x_{1}\right), f\left(x_{2}\right)\right) \leq a d\left(x_{1}, x_{2}\right)+b
$$

(2) there exists $\varepsilon_{1}$ such that

$$
\bigcup_{x \in X} B\left(f(x), \varepsilon_{1}\right)=Y ;
$$

(that is $f$ is $\varepsilon_{1}-f u l l$.)

## Remark

(1) Rough isometry represents an equivalence relation.
(2) If $\operatorname{diam}(X), \operatorname{diam}(Y)$ are finite, then $X, Y$ are roughly isometric.

## We have the following theorem, that extends a result of Kanai (1985), on Riemannian manifolds

> Theorem (Rough Isometry)
> Let $(X, d, \nu)$ be a weak $C D(K, N)$ space and let $G$ be a discretization of $X$. Then $(X, d)$ and $(G, d)$, where $d$ is the combinatorial metric, are roughly isometric.

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## Theorem (Rough Isometry)

Let $(X, d, \nu)$ be a weak $\mathrm{CD}(K, N)$ space and let $G$ be a discretization of $X$. Then $(X, d)$ and $(G, \mathrm{~d})$, where d is the combinatorial metric, are roughly isometric.

## Adaptation - cont.

Main tool in the proof of the theorem :

```
Lemma
Let ( }X,d,\nu)\mathrm{ be a (weak) CD (K,N) space, K < 0,N< , and
let \mathcal{N}\mathrm{ be a }\varepsilon\mathrm{ -separated net. Then}
```


for any $x \in X$ and $r$

## Adaptation - cont.

Main tool in the proof of the theorem :

## Lemma

Let $(X, d, \nu)$ be a (weak) $\mathrm{CD}(K, N)$ space, $K \leq 0, N<\infty$, and let $\mathcal{N}$ be a $\varepsilon$-separated net. Then
(1)

$$
|\mathcal{N} \cap B(x, r)| \leq \frac{\int_{0}^{2 r+\varepsilon / 2} S_{K}^{N}(t) d t}{\int_{0}^{\varepsilon / 2} S_{K}^{N}(t) d t},
$$

(2)

$$
\rho(p) \leq \frac{\int_{0}^{4 r+\varepsilon / 2} S_{K}^{N}(t) d t}{\int_{0}^{\varepsilon / 2} S_{K}^{N}(t) d t} ;
$$

for any $x \in X$ and $r>0$.

## Kanai - cont.

## Definition (Volume growth)

For $x \in X$ and $r>0$ we denote the "volume" growth function by :

$$
\begin{equation*}
\mathcal{V}(x, r)=\nu[B(x, r)] . \tag{3.1}
\end{equation*}
$$

We say that $X$ has exponential (volume) growth iff

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \sup \frac{\log \mathcal{V}(x, r)}{r}>0 \tag{3.2}
\end{equation*}
$$

and polynomial (volume) growth iff there exists $k>0$ such that

$$
\begin{equation*}
\mathcal{V}(x, r) \leq C \cdot r^{k} \tag{3.3}
\end{equation*}
$$

( $C=$ const.) for all sufficiently large $r$.

## Kanai - cont.

## Kanai's main result in this direction is the following :

> Lemma (Roughly isometric graphs have identical growth rate)
> Let $G$ and $\Gamma$ be connected, roughly isometric graphs with bounded geometry. Then G has polynomial (exponential) growth iff $\Gamma$ has polynomial (exponential) growth.

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#### Abstract

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which we generalize as follows :

## Adaptation

Theorem (Weak spaces have the same growth as their

## discretizations)

Let $(X, d, \nu)$ be a weak $\mathrm{CD}(K, N)$ space, $K \leq 0, N<\infty$, satisfying the following non-collapsing condition :
(*) There exist $r_{0}, \mathcal{V}_{0}>0$ such that $\mathcal{V}\left(x, r_{0}\right) \geq \mathcal{V}_{0}$, for all $x \in X$.
Let $G$ be a discretization of $X$. Then $X$ has polynomial (exponential) volume growth iff $G$ has polynomial (exponential) volume growth.

## Adaptation - cont.

## Corollary

Let $X_{1}, X_{2}$ be weak $\mathrm{CD}(K, N)$ spaces, $K \leq 0, N<\infty$, satisfying condition (*) above. Then, if $X_{1}, X_{2}$ are roughly isometric, then they have the same volume growth type.

Remark
The results above hold, a fortiori, for smooth metric measure spaces as well.

## Adaptation - cont.

## Corollary

Let $X_{1}, X_{2}$ be weak $\mathrm{CD}(K, N)$ spaces, $K \leq 0, N<\infty$, satisfying condition ( $*$ ) above. Then, if $X_{1}, X_{2}$ are roughly isometric, then they have the same volume growth type.

## Remark

The results above hold, a fortiori, for smooth metric measure spaces as well.

## Adaptation - cont.

"Ingredients" in the proof of the theorem :

- The previous theorem
- Estimates on $\mathcal{V}(x, r)$, more precisely :

where $\mathcal{B}$ denotes the ball in the combinatorial metric of $G$.


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"Ingredients" in the proof of the theorem :

- The previous theorem
- Estimates on $\mathcal{V}(x, r)$, more precisely :

$$
\left|\mathcal{B}\left(x, a d\left(p_{1}, p_{2}\right)+b\right)\right| \geq \mathcal{V}(x, r) \geq \frac{\int_{0}^{r} S_{K}^{N}(t) d t}{\int_{0}^{r_{0}} S_{K}^{N}(t) d t} \mathcal{V}_{0}
$$

where $\mathcal{B}$ denotes the ball in the combinatorial metric of $G$.

## Finis

## Thank you for your attention!

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[^0]:    1. i.e. any two geodesics $\gamma_{1}, \gamma_{2}:[0, t] \rightarrow X$ that coincide on a subinterval $\left[0, t_{0}\right], 0<t_{0}<t$, coincide on $[0, t]$.
