The magnitude of a metric space

Work of T. Leinster, S. Willerton, and M. M.

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Metric spaces are examples of enriched categories. Leinster’s formalism extends to the setting of enriched categories.

Definition of magnitude (Leinster)

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- "Generic" spaces A possess weightings.
- Different weightings yield the same $|A|$.
- If Z_A is invertible, then $|A| = \sum_{a,b \in A} (Z_A^{-1})_{ab}$.

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- Let $A = K_{3,2}$ with edges of length r. Then $\lim_{r \to \infty} |A| = 5$, but for some small values of r, $|A|$ is undefined or even negative.

Interpretations of magnitude, I (Willerton)

Think of each point $a \in A$ as an organism that

- can regulate the amount of heat w_a which it emits/absorbs,
- wishes to be at temperature 1,
- feels heat from $b \in A$ as $e^{-d(a,b)} w_b$.

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Then

- A weighting is a distribution of heat production/absorption that puts each organism at the desired temperature.
- The magnitude of A is the net heat production when this is achieved.

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We can define a notion of entropy for a probability distribution p on A which takes the metric into account:

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In theoretical ecology,

- points in A represent species,
- distances represent differences between species,
- probabilities represent relative frequencies of species,
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If A possesses a nonnegative weighting w, then $p = \frac{w}{\sum_{a \in A} w_a}$ maximizes diversity and

$$|A| = \max_p H_A(p).$$

Infinite spaces

There are at least three natural ways to extend the definition of magnitude to infinite spaces, each with their own problems.

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- Define a weight measure for A to be a signed Borel measure such that $\int_A e^{-d(a,b)} dw(b) = 1$ for every $a \in A$. Let $|A| = w(A)$.

  It's not clear how generically weight measures exist.

Positive definite spaces

A metric space X is positive definite if the matrix $\left[e^{-d(x_i,x_j)}\right]_{1\leq i,j\leq N}$ is (strictly) positive definite for every choice of distinct points $x_1, \ldots, x_N \in A$.

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Classical results of Bochner, Lévy, Schoenberg, etc. (plus some additional work) show that the following spaces are positive definite.

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- rS^n with the geodesic metric, $r > 0$.

Properties of positive definite spaces

Theorem (M.)

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Magnitude is lower semicontinuous on the class of compact positive definite metric spaces, equipped with the Gromov-Hausdorff distance; it is continuous for subsets of $\mathbb{R}$.

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$$|A| = \prod_{k=1}^{n}\left(1 + \frac{r_k}{2}\right) = \sum_{k=0}^{n} 2^{-k} V_k(A) = \sum_{k=0}^{n} \frac{V_k(A)}{k!\,\mathrm{vol}_k(B_1^k)}.$$

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- Let A be the ternary Cantor set of length r. Then

$$|A| = 1 + \frac{1}{2} \sum_{k=1}^{\infty} 2^k \tanh\left(\frac{r}{2 \cdot 3^k}\right).$$

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- With the geodesic metric,

$$\left|rS^n\right| = 2\left(1 + e^{-\pi r}\right) \prod_{k=1}^{n/2} \left(1 + \left(\frac{r}{n - 2k + 1}\right)^2\right)$$

for n even. (There is a similar formula for odd n.)

Magnitude and dimension, I

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For a metric space (A, d) and $t > 0$, let $tA = (A, t \cdot d)$. Define

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The previous examples show that

- $\dim_{\mathrm{Mag}}(\text{finite space}) = 0$,
- $\dim_{\mathrm{Mag}}(\text{interval}) = 1$,
- $\dim_{\mathrm{Mag}}(\text{Cantor set}) = \log_3 2$,
- $\dim_{\mathrm{Mag}}(S^n) = n$,

all of which agree with Hausdorff dimension.

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Let X be an n-dimensional positive definite metric vector space. If $A \subseteq X$ then

$$|A| \geq \frac{\mathrm{vol}_n(A)}{n!\,\mathrm{vol}_n(B_X)}.$$

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Corollary

If $A \subseteq \ell_p^n$ is compact with nonempty interior for $0 < p \leq 2$, then $\dim_{\mathrm{Mag}}(A) = n$.

Magnitude and intrinsic volumes, I

The results on the last slide and the formula for the magnitude of cuboids in ℓ_1^n suggest the following conjecture.

Conjecture (Leinster, Willerton)

If $A \subseteq \ell_2^n$ is convex and compact, then

$$|A| = \sum_{k=0}^{n} \frac{V_k(A)}{k! \, \mathrm{vol}_k(B_2^k)},$$

where $V_0, \ldots, V_n$ are intrinsic volumes and $\omega_k = \mathrm{vol}_k(B_2^k)$.

This is also supported by numerical calculations and heuristics.

Magnitude and intrinsic volumes, II

Theorem (Willerton)

Suppose A is an n-dimensional homogeneous Riemannian manifold with the geodesic metric. Then

$$|tA| = \frac{V_n(tA)}{n!\omega_n} + \frac{(n+1)V_{n-2}(tA)}{3(n-1)!\omega_{n-2}} + O\big(t^{n-4}\big) \quad \textit{as } t \to \infty.$$

In particular, $\dim_{\mathrm{Mag}}(A) = n$*.*

Corollary

Suppose A is a homogeneous Riemannian surface. Then

$$|tA| = \frac{\mathrm{area}(tA)}{2\pi} + \chi(A) + O\big(t^{-2}\big) \quad \textit{as } t \to \infty.$$

Thank you.