Optimal and Better Transports

G. Maresch
jointly with M. Beiglböck, M. Goldstern and W. Schachermayer

November 1, 2010
Fields Institute, Toronto

Setting

- (X, μ), (Y, ν) Polish probability spaces

Setting

- (X, μ), (Y, ν) Polish probability spaces
- $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ cost function, usually satisfying regularity assumptions.

Setting

- (X, μ), (Y, ν) Polish probability spaces
- $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ cost function, usually satisfying regularity assumptions.
- The Kantorovich Problem is to minimize the functional

$$I_c[\pi] : \pi \mapsto \int_{X \times Y} c(x, y) d\pi(x, y)$$

over all transport plans $\pi \in \Pi(\mu, \nu)$.

Setting

- (X, μ), (Y, ν) Polish probability spaces
- $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ cost function, usually satisfying regularity assumptions.
- The Kantorovich Problem is to minimize the functional

$$I_c[\pi] : \pi \mapsto \int_{X \times Y} c(x, y) d\pi(x, y)$$

over all transport plans $\pi \in \Pi(\mu, \nu)$.

$\Pi(\mu, \nu) \ldots$ measures with X-marginal μ and Y-marginal ν.

Moving Croissants

Topics in Optimal Transportation

$X \ldots$ bakeries, $Y \ldots$ cafes

Cyclical Rerouting

Definition

A Borel set $\Gamma \subseteq X \times Y$ is called *c-monotone* if

$$\sum_{i=1}^{n} c(x_i, y_i) \leq \sum_{i=1}^{n} c(x_i, y_{\sigma(i)})$$

for every permutation $\sigma \in S(n)$.

Cyclical Rerouting

Definition

A Borel set $\Gamma \subseteq X \times Y$ is called *c-monotone* if

$$\sum_{i=1}^{n} c(x_i, y_i) \leq \sum_{i=1}^{n} c(x_i, y_{i+1})$$

for all pairs $(x_1, y_1), \dots, (x_n, y_n) \in \Gamma$. (where $y_{n+1} := y_1$)

Cyclical Rerouting

Definition

A Borel set $\Gamma \subseteq X \times Y$ is called *c-monotone* if

$$\sum_{i=1}^{n} c(x_i, y_i) \leq \sum_{i=1}^{n} c(x_i, y_{i+1})$$

for all pairs $(x_1, y_1), \ldots, (x_n, y_n) \in \Gamma$. (where $y_{n+1} := y_1$)

- A transport plan π is called c-monotone if there exists such a Γ with $\pi(\Gamma) = 1$.

Necessary and Sufficient?

X and Y finite: Suppose π is a transport plan on whose support c-monotonicity is violated.

Necessary and Sufficient?

X and Y finite: Suppose π is a transport plan on whose support c-monotonicity is violated.

$$\sum_{i=1}^{n} c(x_i, y_i) > \sum_{i=1}^{n} c(x_i, y_{i+1})$$

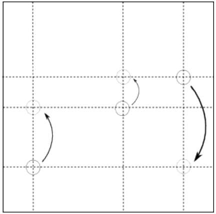
where $x_1, \ldots, x_n$ resp. $y_1, \ldots, y_n$ carry positive mass α.

Necessary and Sufficient?

X and Y finite: Suppose π is a transport plan on whose support c-monotonicity is violated.

$$\sum_{i=1}^{n} c(x_i, y_i) > \sum_{i=1}^{n} c(x_i, y_{i+1})$$

where $x_1, \ldots, x_n$ resp. $y_1, \ldots, y_n$ carry positive mass α.



'96, W. Gangbo and R. McCann

Improve π:

$$\pi^{\beta} := \pi + \alpha \sum_{i=1}^{n} \delta_{(x_i, y_{i+1})} - \alpha \sum_{i=1}^{n} \delta_{(x_i, y_i)}.$$

'96, W. Gangbo and R. McCann

Improve π:

$$I_c[\pi^\beta] = I_c[\pi] + \alpha \sum_{i=1}^{n} \Big(c(x_i, y_{i+1}) - c(x_i, y_i) \Big).$$

'96, W. Gangbo and R. McCann

Improve π:

$$I_c[\pi^\beta] = I_c[\pi] + \alpha \sum_{i=1}^{n} \Big(c(x_i, y_{i+1}) - c(x_i, y_i) \Big).$$

Proposition

Let X, Y compact spaces equipped with Borel probability measures ν, μ. Let $c : X \times Y \to \mathbb{R}_{\geq 0}$ be a continuous cost function. Then every optimal transport plan is c-monotone.

'96, W. Gangbo and R. McCann

Improve π:

$$I_c[\pi^\beta] = I_c[\pi] + \alpha \sum_{i=1}^{n} \Big(c(x_i, y_{i+1}) - c(x_i, y_i) \Big).$$

Proposition

Let X, Y compact spaces equipped with Borel probability measures ν, μ. Let $c : X \times Y \to \mathbb{R}_{\geq 0}$ be a continuous cost function. Then every optimal transport plan is c-monotone.

'96, W. Gangbo and R. McCann

Improve π:

$$I_c[\pi^\beta] = I_c[\pi] + \alpha \sum_{i=1}^{n} \Big(c(x_i, y_{i+1}) - c(x_i, y_i) \Big).$$

Proposition

Let X, Y compact spaces equipped with Borel probability measures ν, μ. Let $c : X \times Y \to \mathbb{R}_{\geq 0}$ be a continuous cost function. Then every optimal transport plan is c-monotone.

Necessary ...

Proposition ('09, BGMS)

Let X and Y be Polish spaces equipped with Borel probability measures ν, μ. Let $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a measurable cost function. Then every optimal transport plan is c-monotone.

Necessary ...

Proposition ('09, BGMS)

Let X and Y be Polish spaces equipped with Borel probability measures ν, μ. Let $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a measurable cost function. Then every optimal transport plan is c-monotone.

- Makes use of deep duality results by H. Kellerer in the multimarginal setting.

Necessary ...

Proposition ('09, BGMS)

Let X and Y be Polish spaces equipped with Borel probability measures ν, μ. Let $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a measurable cost function. Then every optimal transport plan is c-monotone.

- Makes use of deep duality results by H. Kellerer in the multimarginal setting.
- Dichotomy of Borel subsets of $(X \times Y)^n$.

... but not sufficient

Example ('01, Ambrosio and Pratelli)

Let $X = Y = [0, 1]$ be the torus, equipped with Lebesgue measure.

... but not sufficient

Example ('01, Ambrosio and Pratelli)

Let $X = Y = [0, 1]$ be the torus, equipped with Lebesgue measure.

1. Let $\Gamma_1 = \{(x, x) : x \in X\}$,

... but not sufficient

Example ('01, Ambrosio and Pratelli)

Let $X = Y = [0, 1]$ be the torus, equipped with Lebesgue measure.

1. Let $\Gamma_1 = \{(x, x) : x \in X\}$,
2. Pick α irrational.
   Let $\Gamma_2 = \{(x, x + \alpha) : x \in X\}$.

... but not sufficient

Example ('01, Ambrosio and Pratelli)

Let $X = Y = [0, 1]$ be the torus, equipped with Lebesgue measure.

1. Let $\Gamma_1 = \{(x, x) : x \in X\}$,
2. Pick α irrational.
   Let $\Gamma_2 = \{(x, x + \alpha) : x \in X\}$.

Let $c = a$ on Γ_1, $c = b$ on Γ_2 and ∞ else.

... but not sufficient

Example ('01, Ambrosio and Pratelli)

Let $X = Y = [0,1]$ be the torus, equipped with Lebesgue measure.

1. Let $\Gamma_1 = \{(x,x) : x \in X\}$,
2. Pick α irrational.
   Let $\Gamma_2 = \{(x, x+\alpha) : x \in X\}$.

Let $c = a$ on Γ_1, $c = b$ on Γ_2 and ∞ else.

- Both Γ_1 and Γ_2 are c-monotone.

... but not sufficient

Example ('01, Ambrosio and Pratelli)

Let $X = Y = [0, 1]$ be the torus, equipped with Lebesgue measure.

1. Let $\Gamma_1 = \{(x, x) : x \in X\}$,
2. Pick α irrational.
   Let $\Gamma_2 = \{(x, x + \alpha) : x \in X\}$.

Let $c = a$ on Γ_1, $c = b$ on Γ_2 and ∞ else.

- Both Γ_1 and Γ_2 are c-monotone.
- Both Γ_1 or Γ_2 support a unique transport plan π_1 resp. π_2.

... but not sufficient

Example ('01, Ambrosio and Pratelli)

Let $X = Y = [0,1]$ be the torus, equipped with Lebesgue measure.

1. Let $\Gamma_1 = \{(x,x) : x \in X\}$,
2. Pick α irrational.
   Let $\Gamma_2 = \{(x, x+\alpha) : x \in X\}$.

Let $c = a$ on Γ_1, $c = b$ on Γ_2 and ∞ else.

- Both Γ_1 and Γ_2 are c-monotone.
- Both Γ_1 or Γ_2 support a unique transport plan π_1 resp. π_2.
- $I_c[\pi_1] = a$ and $I_c[\pi_2] = b$.

'96, W. Gangbo and R. McCann

Proposition

Let $X = Y = \mathbb{R}^n$ and $c : X \times Y \to \mathbb{R}_{\geq 0}$ strictly convex. Then every π which is c-monotone is optimal.

'01, L. Ambrosio and A. Pratelli

Proposition

Let X, Y be Polish spaces and $c : X \times Y \to \mathbb{R}_{\geq 0}$ l.s.c. fulfilling the moment conditions

$$\mu\left(\{x : \int_Y c(x,y)d\nu(y) < \infty\}\right) > 0,$$
$$\nu\left(\{y : \int_X c(x,y)d\mu(x) < \infty\}\right) > 0.$$

Then every π which is c-monotone is optimal.

'07, A. Pratelli; W. Schachermayer and J. Teichmann

Proposition

Let X, Y be Polish spaces and $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ a continuous cost function. Then every π which is c-monotone is optimal.

'07, A. Pratelli; W. Schachermayer and J. Teichmann

Proposition

Let X, Y be Polish spaces and $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ a continuous cost function. Then every π which is c-monotone is optimal.

Proposition

Let X, Y be Polish spaces and $c : X \times Y \to \mathbb{R}_{\geq 0}$ a finite and l.s.c. cost function. Then every π which is c-monotone is optimal.

'07, A. Pratelli; W. Schachermayer and J. Teichmann

Proposition

Let X, Y be Polish spaces and $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ a continuous cost function. Then every π which is c-monotone is optimal.

Proposition

Let X, Y be Polish spaces and $c : X \times Y \to \mathbb{R}_{\geq 0}$ a finite and l.s.c. cost function. Then every π which is c-monotone is optimal.

'07, A. Pratelli; W. Schachermayer and J. Teichmann

Proposition

Let X, Y be Polish spaces and $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ a continuous cost function. Then every π which is c-monotone is optimal.

Proposition

Let X, Y be Polish spaces and $c : X \times Y \to \mathbb{R}_{\geq 0}$ a finite and l.s.c. cost function. Then every π which is c-monotone is optimal.

M. Beiglböck, M. Goldstern, W. Schachermayer, G. M.

Theorem

Let X and Y be Polish spaces. Let $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be Borel measurable and such that the set $\{(x, y) : c(x, y) = \infty\}$ is

M. Beiglböck, M. Goldstern, W. Schachermayer, G. M.

Theorem

Let X and Y be Polish spaces. Let $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be Borel measurable and such that the set $\{(x, y) : c(x, y) = \infty\}$ is closed:

Then every finite c-monotone transport is optimal.

M. Beiglböck, M. Goldstern, W. Schachermayer, G. M.

Theorem

Let X and Y be Polish spaces. Let $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be Borel measurable and such that the set $\{(x, y) : c(x, y) = \infty\}$ is ~~closed~~ the union of a closed set and a $\mu \otimes \nu$-null set:

Then every finite c-monotone transport is optimal.

M. Beiglböck, M. Goldstern, W. Schachermayer, G. M.

Theorem

Let X and Y be Polish spaces. Let $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be Borel measurable and such that the set $\{(x, y) : c(x, y) = \infty\}$ is ~~closed~~ the union of a closed set and a $\mu \otimes \nu$-null set:

Then every finite c-monotone transport is optimal.

Remark

Transport plans are usually supported on a $\mu \otimes \nu$-null set.

Towards Better Optimality

Definition

A Borel set $\Gamma \subseteq X \times Y$ is *strongly c-monotone* iff there exist Borel measurable functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ and $\psi : Y \to \mathbb{R} \cup \{-\infty\}$ such that

Towards Better Optimality

Definition

A Borel set $\Gamma \subseteq X \times Y$ is *strongly c-monotone* iff there exist Borel measurable functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ and $\psi : Y \to \mathbb{R} \cup \{-\infty\}$ such that

1. $\varphi(x) + \psi(y) \leq c(x, y)$ for all $(x, y) \in X \times Y$,

Towards Better Optimality

Definition

A Borel set $\Gamma \subseteq X \times Y$ is *strongly c-monotone* iff there exist Borel measurable functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ and $\psi : Y \to \mathbb{R} \cup \{-\infty\}$ such that

1. $\varphi(x) + \psi(y) \leq c(x, y)$ for all $(x, y) \in X \times Y$,
2. $\varphi(x) + \psi(y) = c(x, y)$ for all $(x, y) \in \Gamma$.

Consistency

- A transport plan $\pi \in \Pi(\mu, \nu)$ is strongly c-monotone if it is concentrated on a strongly c-monotone Borel set Γ.

Consistency

- A transport plan $\pi \in \Pi(\mu, \nu)$ is strongly c-monotone if it is concentrated on a strongly c-monotone Borel set Γ.
- Strong monotonicity implies ordinary monotonicity.

Consistency

- A transport plan $\pi \in \Pi(\mu, \nu)$ is strongly c-monotone if it is concentrated on a strongly c-monotone Borel set Γ.
- Strong monotonicity implies ordinary monotonicity.

Proof.

$$\sum_{i=1}^{n} c(x_{i+1}, y_i)$$

Consistency

- A transport plan $\pi \in \Pi(\mu, \nu)$ is strongly c-monotone if it is concentrated on a strongly c-monotone Borel set Γ.
- Strong monotonicity implies ordinary monotonicity.

Proof.

$$\sum_{i=1}^{n} c(x_{i+1}, y_i) \geq \sum_{i=1}^{n} \varphi(x_{i+1}) + \psi(y_i)$$

Consistency

- A transport plan $\pi \in \Pi(\mu, \nu)$ is strongly c-monotone if it is concentrated on a strongly c-monotone Borel set Γ.
- Strong monotonicity implies ordinary monotonicity.

Proof.

$$\sum_{i=1}^{n} c(x_{i+1}, y_i) \geq \sum_{i=1}^{n} \varphi(x_{i+1}) + \psi(y_i) = \sum_{i=1}^{n} \varphi(x_i) + \psi(y_i)$$

Consistency

- A transport plan $\pi \in \Pi(\mu, \nu)$ is strongly c-monotone if it is concentrated on a strongly c-monotone Borel set Γ.
- Strong monotonicity implies ordinary monotonicity.

Proof.

$$\sum_{i=1}^{n} c(x_{i+1}, y_i) \geq \sum_{i=1}^{n} \varphi(x_{i+1}) + \psi(y_i) = \sum_{i=1}^{n} \varphi(x_i) + \psi(y_i) = \sum_{i=1}^{n} c(x_i, y_i)$$

$\square$

Consistency

- A transport plan $\pi \in \Pi(\mu, \nu)$ is strongly c-monotone if it is concentrated on a strongly c-monotone Borel set Γ.
- Strong monotonicity implies ordinary monotonicity.

Proof.

$$\sum_{i=1}^{n} c(x_{i+1}, y_i) \geq \sum_{i=1}^{n} \varphi(x_{i+1}) + \psi(y_i) = \sum_{i=1}^{n} \varphi(x_i) + \psi(y_i) = \sum_{i=1}^{n} c(x_i, y_i)$$

$\square$

- Strong monotonicity implies optimality.

. . . but not necessary.

Example

- Let $X = Y = [0, 1]$ equipped with Lebesgue measure λ.

... but not necessary.

Example

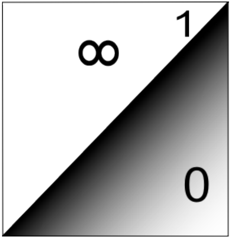
- Let $X = Y = [0, 1]$ equipped with Lebesgue measure λ.
- Take $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ to be infinite above the diagonal, 1 on the diagonal and 0 below.

. . . but not necessary.

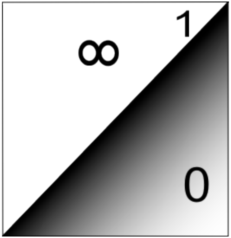
Example

- Let $X = Y = [0,1]$ equipped with Lebesgue measure λ.
- Take $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ to be infinite above the diagonal, 1 on the diagonal and 0 below.
- Let $\Gamma = \{(x,x) : x \in [0,1]\}$ be the diagonal in $X \times Y$.

Counterexample

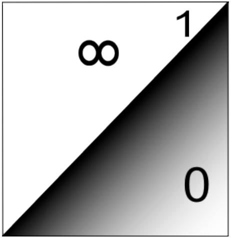


Counterexample



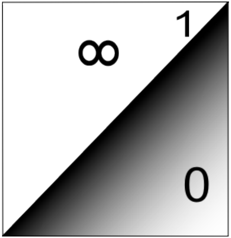
- Γ is c-monotone,

Counterexample



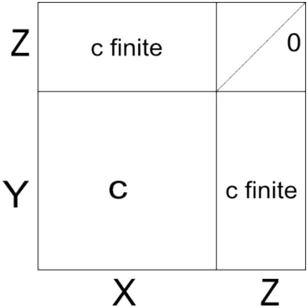
- Γ is c-monotone,
- Γ is not strongly c-monotone,

Counterexample

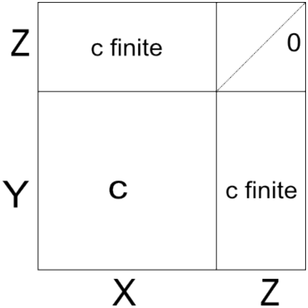


- Γ is c-monotone,
- Γ is not strongly c-monotone,
- Id: $X \to Y$ induces an optimal transport, which is concentrated Γ.

Robust Optimality



Robust Optimality



Theorem

For a finite transport plan robust optimality is equivalent to strong c-monotonicity.

The almost finite situation

When we put all our results together we get:

Theorem

Let $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be Borel measurable and $\mu \otimes \nu$-a.s. finite. For a finite transport plan π t.f.a.e.

The almost finite situation

When we put all our results together we get:

Theorem

Let $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be Borel measurable and $\mu \otimes \nu$-a.s. finite. For a finite transport plan π t.f.a.e.

1. *π is c-monotone.*

The almost finite situation

When we put all our results together we get:

Theorem

Let $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be Borel measurable and $\mu \otimes \nu$-a.s. finite. For a finite transport plan π t.f.a.e.

1. *π is c-monotone.*
2. *π is strongly c-monotone.*

The almost finite situation

When we put all our results together we get:

Theorem

Let $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be Borel measurable and $\mu \otimes \nu$-a.s. finite. For a finite transport plan π t.f.a.e.

1. *π is c-monotone.*
2. *π is strongly c-monotone.*
3. *π is optimal.*

The almost finite situation

When we put all our results together we get:

Theorem

Let $c : X \times Y \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be Borel measurable and $\mu \otimes \nu$-a.s. finite. For a finite transport plan π t.f.a.e.

1. *π is c-monotone.*
2. *π is strongly c-monotone.*
3. *π is optimal.*
4. *π is robustly optimal.*

Thank you for your attention!

- W. Gango and R. McCann, *The geometry of optimal transportation*, 1996, Acta Mathematica
- L. Ambrosio and A. Pratelli, *Existence and stability results in the L^1-theory of optimal transportation*, 2001, in: Lecture Notes in Mathematics
- A. Pratelli, *On the sufficiency of c-cyclical monotonicity for optimality of transport plans*, 2007, Mathematische Zeitschrift
- M. Beiglböck, G. Maresch, M. Goldstern and W. Schachermayer, *Optimal and Better Transport Plans*, 2009, Journal of Functional Analysis