Positive definite functions and stable random vectors.

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n-dimensional versions.

Definition (Eaton, 1981)

A random vector $X = (X_1, ..., X_n)$ is an *n-dimensional version* of a random variable Y if there exists a function $\gamma : \mathbb{R}^n \to [0, \infty)$, such that for every $a \in \mathbb{R}^n$ the random variables

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γ is an even homogeneous of degree 1 non-negative (and equal to zero only at zero) continuous function on $\mathbb{R}^n$. This means that $\gamma = \|\cdot\|_K$ is the Minkowski functional of some origin symmetric star body K in $\mathbb{R}^n$.

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2) Characterize all K for which $\|\cdot\|_K$ can appear as the standard of an n-dimensional version

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By Bochner's theorem, this means that the function $f(\|\cdot\|_K)$ is positive definite. Recall that a complex valued function f defined on $\mathbb{R}^n$ is called *positive definite* on $\mathbb{R}^n$ if, for every finite sequence $\{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ and every choice of complex numbers $\{c_i\}_{i=1}^m$, we have

$$\sum_{i=1}^m \sum_{j=1}^m c_i \bar{c}_j f(x_i - x_j) \geq 0.$$

Classes $\Phi(K)$

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In particular, $\|\cdot\|_K$ appears as the standard of an n-dimensional version if and only if the class $\Phi(K)$ is non-trivial, i.e. contains at least one non-constant function.

P.Levy (1920's): stable processes

For any finite dimensional subspace $(\mathbb{R}^n, \|\cdot\|)$ of L_q with $0 < q \le 2$, the function $g = \exp(-\|\cdot\|^q)$ is positive definite on $\mathbb{R}^n$, and any random vector $X = (X_1, ..., X_n)$ in $\mathbb{R}^n$, whose characteristic functional is g, is an n-dimensional version.

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Answer (Misiewicz, 1989, for $q = \infty$; K.,1991, for $2 < q < \infty$): if $n \ge 3$, not positive definite for any $p > 0$, if $n = 2$, positive definite iff $p \in (0, 1]$.

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Connection with embeddings in L_p

Bretagnolle, Dacunha-Castelle, Krivine (1966): a normed space embeds isometrically in L_q, $0 < q \le 2$ if and only if the function $\exp(-\|\cdot\|^q)$ is positive definite.

Characterization of the classes $\Phi(K)$

- Schoenberg: $f \in \Phi(\ell_2^n)$ iff
$$f(t) = \int_0^\infty \Omega_n(tr)\ d\lambda(r),$$
$f \in \Phi(\ell_2)$ iff
$$f(t) = \int_0^\infty \exp(-t^2 r^2)\ d\lambda(r)$$
- Bretagnolle, Dacunha-Castelle, Krivine: same for $\Phi(\ell_q)$, $0 < q < 2$, $\Phi(\ell_q)$ trivial if $q > 2$
- Cambanis, Keener, Simons: same for $\Phi(\ell_1^n)$
- Richards, Gneiting: partial results for $\Phi(\ell_q^n)$, $0 < q < 2$
- Aharoni, Maurey, Mityagin: $\Phi(K)$ is trivial if
$$\lim_{n\to\infty} \|e_1 + \ldots + e_n\|/\sqrt{n} = 0$$
- Misiewicz: $\Phi(\ell_\infty^n)$ is trivial if $n \geq 3$
- Lisitsky, Zastavny (independently): same for $\Phi(\ell_q^n)$, $q > 2$.

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L_p-conjecture (Misiewicz, 1987)

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Supporting argument

It is so under additional condition that $\mathbb{E}|Y|^p < \infty$. In fact,

$$\mathbb{E}|(X,a)|^p = \|a\|_K^p \mathbb{E}|Y|^p.$$

L_0-conjecture (Lisitsky, 1997)

If $\Phi(K)$ non-trivial, then $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0, i.e. there exist a finite Borel measure μ on the sphere S^{n-1} and a constant $C \in \mathbb{R}$ so that, for every $x \in \mathbb{R}^n$,

$$\log \|x\|_K = \int_{S^{n-1}} \log |(x, \xi)| d\mu(\xi) + C.$$

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Embedding in L_0 introduced in [Kalton, K., Yaskin, Yaskina, 2007]

Easy under additional condition $\mathbb{E}|\log|Y|| < \infty$.

Main Theorem.

Let K be an origin symmetric star body in $\mathbb{R}^n$, $n \geq 2$ and suppose that there exists an even non-constant continuous function $f : \mathbb{R} \mapsto \mathbb{R}$ such that $f(\|\cdot\|_K)$ is a positive definite function on $\mathbb{R}^n$. Then the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0.

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Corollary.

If a function γ is the standard of an n-dimensional version of a random variable, then there exists an origin symmetric star body K in $\mathbb{R}^n$ such that $\gamma = \|\cdot\|_K$ and the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0.

Embedding in L_0.

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- If $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0, it also embeds in L_p for every $-n < p < 0$.
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- There are many examples of normed spaces that embed in L_0, but don't embed in any $L_p,\ p \in (0,2)$. For example, the spaces $\ell_q^3,\ q > 2$ have this property.
- Every three dimensional **normed** space embeds in L_0.

Second derivative test (SDT)

Let $n \geq 4$ and let $X = (\mathbb{R}^n, \|\cdot\|)$ be an n-dimensional **normed** space with normalized basis $e_1, \dots, e_n$ so that:

(i) For every fixed $(x_2, \dots, x_n) \in \mathbb{R}^{n-1} \setminus \{0\}$,

$$\|x\|'_{x_1}(0, x_2, \dots, x_n) = \|x\|''_{x_1^2}(0, x_2, \dots, x_n) = 0$$

(ii) There exists a constant C so that, for every $x_1 \in \mathbb{R}$ and every $(x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ with $\|x_2 e_2 + \dots + x_n e_n\| = 1$, one has

$$\|x\|''_{x_1^2}(x_1, x_2, \dots, x_n) \leq C.$$

(iii) Convergence in the limit

$$\lim_{x_1 \to 0} \|x\|''_{x_1^2}(x_1, x_2, \dots, x_n) = 0$$

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ℓ_q^n, $q > 2$, $n \geq 4$ have this property: $|x_1|^{q-2} = 0$ when $x_1 = 0$.

q-sums

For normed spaces X and Y and $q \in \mathbb{R}$, $q \geq 1$, the q-sum $(X \oplus Y)_q$ of X and Y is defined as the space of pairs $\{(x,y) : x \in X, y \in Y\}$ with the norm

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Orlicz spaces

Orlicz function M is a non-decreasing convex function on $[0,\infty)$ such that $M(0) = 0$ and $M(t) > 0$ for every $t > 0$. The norm $\|\cdot\|_M$ of the n-dimensional Orlicz space ℓ_M^n is defined implicitly by the equality

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Orlicz spaces ℓ_M^n with $n \geq 4$ and $M''(0) = 0$ satisfy SDT.

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We say that a distribution is negative outside of the origin in $\mathbb{R}^n$ if $\langle f, \phi \rangle \le 0$ for any $\phi \ge 0$ with compact support outside of the origin.

Fourier characterization of embedding in L_0

Theorem ([KKYY])

Let K be an origin symmetric star body in $\mathbb{R}^n$. The space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if and only if the Fourier transform of $\log \|x\|_K$ is a negative distribution outside of the origin in $\mathbb{R}^n$.

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Idea of Proof

$$\log\|x\|_K = \int_{S^{n-1}} \log|(x,\xi)| d\mu(\xi) + C.$$

Let ϕ be a non-negative even test function with support outside of the origin.

$$\langle(\log\|x\|)^\wedge, \phi\rangle = \langle\log\|x\|, \hat{\phi}(x)\rangle \quad \textit{need to prove} \le 0$$

$$= \int_{S^{n-1}} \int_{\mathbb{R}^n} \log|(x,\xi)| \hat{\phi}(x)\ dx\ d\mu(\xi) + C \int_{\mathbb{R}^n} \hat{\phi}(x) dx$$

$$= \int_{S^{n-1}} \langle \log|t|, \int_{(x,\xi)=t} \hat{\phi}(x)\ dx\rangle\ d\mu(\xi)$$

$$= -(2\pi)^n \int_{S^{n-1}} \int_{\mathbb{R}} |t|^{-1} \phi(t\xi)\ dt\ d\mu(\xi) \le 0. \quad \square$$

Main Theorem.

Let K be an origin symmetric star body in $\mathbb{R}^n$, $n \geq 2$ and suppose that there exists an even non-constant continuous function $f : \mathbb{R} \mapsto \mathbb{R}$ such that $f(\|\cdot\|_K)$ is a positive definite function on $\mathbb{R}^n$. Then the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0.

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Assume $f(0) = 1$. By Bochner's theorem, $\hat{\mu} = f(\|\cdot\|)$ for some probability measure μ on $\mathbb{R}^n$.

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Given:

$$\int_{\mathbb{R}^n} f(t\|x\|)\hat{\phi}(x)\ dx = \langle (f(t\|\cdot\|))^\wedge, \phi(x)\rangle \geq 0, \quad \forall t > 0$$

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$$\int_{\mathbb{R}^n} f(t\|x\|)\hat{\phi}(x)\ dx = \langle (f(t\|\cdot\|))^{\wedge}, \phi(x)\rangle \geq 0, \quad \forall t > 0$$

Need to prove:

For every $\phi \geq 0$ supported in $\mathbb{R}^n \setminus \{0\}$

$$\langle (\log \|x\|)^{\wedge}, \phi\rangle = \int_{\mathbb{R}^n} \log \|x\| \hat{\phi}(x)\ dx \leq 0.$$

Function $g(\varepsilon)$, $\varepsilon \in (0,1/2)$

$$g(\varepsilon) = \int_{\mathbb{R}^n} \left(\int_0^1 t^{-1+\varepsilon} f(t\|x\|) dt + \int_1^\infty t^{-1-\varepsilon} f(t\|x\|) dt \right) \hat{\phi}(x) dx$$

(....) bounded by $2/\varepsilon$. By the Fubini theorem, $g(\epsilon) \geq 0, \quad \forall \epsilon.$

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$$=u(\varepsilon)+v(\varepsilon)+w(\varepsilon).$$

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which implies

$$\int_{\mathbb{R}^n} \log\|x\|\hat{\phi}(x)dx \le 0. \quad \square$$

$\lim_{\varepsilon \to 0} w(\varepsilon) = 0$

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K star body, so $c|x|_2 \le \|x\| \le d|x|_2$, and $\|x\|^{-3/2}$ is locally integrable in $\mathbb{R}^n$, $n \ge 2$.

Lemma

Let h be a bounded integrable continuous at 0 function on $[0,A]$, $A>0$. Then

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^A t^{-1+\varepsilon}h(t)dt=\lim_{\varepsilon\to 0}\varepsilon\int_0^\varepsilon t^{-1+\varepsilon}h(t)dt=h(0).$$

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$$u(\varepsilon) = \int_{\mathbb{R}^n} \frac{\|x\|^{-\varepsilon} - 1}{\varepsilon} \left(\varepsilon \int_0^{\|x\|} t^{-1+\varepsilon} f(t)\ dt \right) \hat{\phi}(x) dx$$

$$\longrightarrow - \int_{\mathbb{R}^n} \log \|x\| \hat{\phi}(x)\ dx.$$

What is the problem with v?

$\lim_{\varepsilon\to 0} v(\varepsilon)$

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Since $|\psi(\varepsilon)| \leq \varepsilon^{\varepsilon}$, the alternative is that $\lim_{\varepsilon\to 0}\psi(\varepsilon) = 1$. Suppose that $\lim_{\varepsilon\to 0}\psi(\varepsilon) = 1$.

End of Proof

Lemma (Vakhania, Tarieladze, Chobanyan)

If μ is a probability measure on $\mathbb{R}^n$ and γ is the standard Gaussian measure on $\mathbb{R}^n$, then for every $t > 0$

$$\mu\{x \in \mathbb{R}^n : \ |x|_2 > 1/t\} \leq 3 \int_{\mathbb{R}^n} (1 - \hat{\mu}(ty))\, d\gamma(y),$$

where $|\cdot|_2$ is the Euclidean norm on $\mathbb{R}^n$.

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Contradiction to Hypothesis $\lim_{\varepsilon \to 0} \psi(\varepsilon) = 1$.

Let μ be the measure satisfying $\hat{\mu} = f(\|\cdot\|)$. Integrating by t we get

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As $\varepsilon \to 0$, the left-hand side converges to $\mu(\mathbb{R}^n \setminus \{0\})$. The right-hand side converges to 0. We get $\mu(\mathbb{R}^n \setminus \{0\}) = 0$, which means that f is a constant function - contradiction.

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μ, ν-probability measures, $0 < u < 1$

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$$\int (1 - \hat{\mu}) d\nu = \int (1 - \hat{\nu}) d\mu \ge (1 - u) \; \mu\{x : \; \hat{\nu}(x) < u\}$$

Let $\nu = \gamma$ - standard Gaussian measure on $\mathbb{R}^n$, then $\hat{\nu}(x) = e^{-|x|_2^2}$. Put $u = e^{-1}$.

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Now dilate μ and use $1/(1 - e^{-1}) < 3$.