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Scalar and Q-curvature of random Riemannian metrics

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- (M, g) is n-dimensional compact manifold, $n \geq 2$. *Riemann curvature tensor* is defined by $R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$; ∇- Levi-Civita connection. In local coordinates: $R_{ijkl} := \langle R(\partial_i, \partial_j)\partial_k, \partial_l \rangle$.
- *Ricci curvature:* $R_{jk} = g^{il} R_{ijkl}$. In geodesic normal coordinates, the volume element takes the form $dV_g = [1 - (1/6) R_{jk} x^j x^k + O(|x|^3)] dV_{Euclidean}$.
- *Scalar curvature:* $R = g^{ik} R_{ik}$. Geometric meaning: "excess volume:" as $r \to 0$,

$$\mathrm{vol}(B_M(x_0, r)) = \mathrm{vol}(B_{\mathbf{R}^n}(r)) \left[1 - \frac{R(x_0) r^2}{6(n+2)} + O(r^4) \right].$$

- *Sectional curvature:* X, Y - two linearly independent vector fields; sectional (=Gauss) curvature of the plane XY spanned by X_p, Y_p is $K_{XY} = \langle R(X,Y)X, Y \rangle / [||X||^2 ||Y||^2 - \langle X, Y \rangle^2]$.

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- **Main Question:** What is the *probability* that a random metric g satisfies certain curvature bounds? In particular, what is the probability that scalar curvature R has *constant sign* on M?
- Use *Laplacian* to define random metrics in a *conformal class* and to estimate that probability.
- Later, use *conformally covariant* operators to study analogous questions for Branson's *Q-curvature*.
- Techniques: conformal field theory; differential geometry; spectral theory of elliptic operators.
- Theory of excursion sets and extrema of Gaussian random fields on manifolds (Borell, Tsirelson, Ibragimov, Sudakov, Adler, Taylor).

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- Choose a *reference metric* g_0; scalar curvature R_0 has constant sign. Our random metrics will be concentrated *close* to g_0.
- **Question:** do such metrics exist in every conformal class?
- Dimension 2: *Uniformization theorem* shows that in every conformal class there exists a unique metric of constant Gauss curvature.
- Dimension $n \geq 3$: *Yamabe problem* (Yamabe, Trudinger, Aubin, Schoen): in every conformal class there exist metric(s) of constant scalar curvature R_0 (its sign is uniquely determined). If $R_0 \leq 0$, that metric is unique.

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- ***Q*-curvature** (Branson, Gover) arises in the study of *conformally covariant* differential operators (Paneitz, Fefferman, Graham, Jenne, Mason, Sparling et al).
- Existence of metrics with constant Q-curvature in conformal classes:
- $n = 4$: Chang-Yang (1995) and Djadli-Malchiodi (2008), true for *generic* conformal classes (Paneitz operator should not have certain numbers as eigenvalues).
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- (M, g_0) - compact, orientable. Consider metrics in conformal class of g_0 of the form $e^{af} \cdot g_0, a \geq 0$; we choose f to be a random (suitably regular) function on M.
- Δ_0 - Laplacian of g_0. Spectrum: $\Delta_0 \phi_j + \lambda_j \phi_j = 0, \;\; 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$.
- We expand f in random Gaussian Fourier series in ϕ_j as follows: $f(x) = -\sum_{j=1}^{\infty} a_j c_j \phi_j(x)$, where $a_j \sim \mathcal{N}(0,1)$ are i.i.d standard Gaussians, and c_j are positive real numbers. We assume that $c_j = F(\lambda_j)$, e.g. $e^{-b\lambda_j}$ or λ_j^{-s}. We shall later require that $c_j = O(\lambda_j^{-s})$ ("random Sobolev metrics").
- The *covariance function* $r_f(x,y) := \mathbb{E}[f(x)f(y)] = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)\phi_j(y)$, for $x, y \in M$. For $x \in M$, $f(x)$ is mean zero Gaussian of variance $r_f(x,x) = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)^2$.

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- We shall later need another random field $h(x) := \Delta_0 f(x) = \sum_{j=1}^{\infty} a_j c_j \lambda_j \phi_j(x)$, with covariance function $r_h(x,y) = \sum_{j=1}^{\infty} c_j^2 \lambda_j^2 \phi_j(x)\phi_j(y)$.
- Standard Sobolev regularity properties of random Fourier series, Weyl's law for Δ_0 and Sobolev embedding theorem imply
- **Proposition 1:** If $c_j = O(\lambda_j^{-s}), s > n/2$, then $f \in C^0$ a.s; if $c_j = O(\lambda_j^{-s}), s > n/2+1$, then $f \in C^2$ a.s. Similarly, if $c_j = O(\lambda_j^{-s}), s > n/2+1$, then $h = \Delta_0 f \in C^0$ a.s; if $c_j = O(\lambda_j^{-s}), s > n/2+2$, then $h = \Delta_0 f \in C^2$ a.s.
- **Volume change:** Let $V_0 = \mathrm{vol}(M, g_0)$. If $g_1 := g_1(a) = e^{af} g_0$, then $dV_1 = e^{naf/2} dV_0$. One can show that $\lim_{a \to 0} \mathbb{E}[\mathrm{vol}(M, g_1(a))] = V_0$.

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$$R_1 = e^{-af} \left[R_0 - a(n-1)\Delta_0 f - \frac{a^2(n-1)(n-2)}{4} |\nabla_0 f|^2 \right] \quad (1)$$

- **Dimension two:** (M, g_0) - compact, orientable; g_0 has scalar curvature R_0. The gradient term vanishes in (1) when $n = 2$:

$$R_1 = e^{-af}[R_0 - ah] \quad (2)$$

- **Smoothness, dimension n:** If $R_0 \in C^0, c_j = O(\lambda_j^{-s}), s > n/2 + 1$ then $R_1 \in C^0$ a.s. If $R_0 \in C^2, c_j = O(\lambda_j^{-s}), s > n/2 + 2$ then $R_1 \in C^2$ a.s.
- **Key observation:** If $R_0 \neq 0$, then $\mathrm{Sgn}(R_1) = \mathrm{Sgn}(R_0)\mathrm{Sgn}(1 - ah/R_0)$, where $1 - ah/R_0$ is a "random wave".

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- **Smoothness, dimension n:** If $R_0 \in C^0, c_j = O(\lambda_j^{-s}), s > n/2 + 1$ then $R_1 \in C^0$ a.s. If $R_0 \in C^2, c_j = O(\lambda_j^{-s}), s > n/2 + 2$ then $R_1 \in C^2$ a.s.
- **Key observation:** If $R_0 \neq 0$, then $\mathrm{Sgn}(R_1) = \mathrm{Sgn}(R_0)\mathrm{Sgn}(1 - ah/R_0)$, where $1 - ah/R_0$ is a "random wave".

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- Let $g_1 = e^{af} g_0$. Then the scalar curvature R_1 of the new metric is given by

$$R_1 = e^{-af}\left[R_0 - a(n-1)\Delta_0 f - \frac{a^2(n-1)(n-2)}{4}|\nabla_0 f|^2\right] \quad (1)$$

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- We next want to estimate the probability of the event $\{\mathrm{Sgn}(R_1) = \mathrm{Sgn}(R_0)\}$. We restrict ourselves to surfaces of genus $\gamma \neq 1$, since on $\mathbf{T}^2, \int_M R_1 = 0$ by Gauss-Bonnet theorem, hence R_1 has to change sign.
- Recall that the *reference* metric g_0 is chosen so that $\forall x, R_0(x) \neq 0$. Denote by $P_1(a)$ the probability $\mathrm{Prob}\{\mathrm{Sgn}(R_0 - ah) = \mathrm{Sgn}(R_0)\}$, and by $P_2(a) = 1 - P_1(a)$ the complementary probability $\mathrm{Prob}\{\exists x \in M : 1 - ah(x)/R_0(x) < 0\}$. Let $||\psi|| := \sup_{x \in M} \psi(x)$. Clearly,

$$P_2(a) = \mathrm{Prob}\{||h/R_0|| > 1/a\} := \mathrm{Prob}\{||v|| > 1/a\},$$

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- We shall estimate $P_2(a)$ in the limit $a \to 0$. Geometrically, this implies that a.s. $g_1(a) \to g_0$, so $P_2(a) \to 0$. We want to estimate the *rate*.
- To prove the first estimate, valid for any surface but not optimal, we shall use Borell-TIS inequality:
- **Proposition 2** (Borell, Tsirelson, Ibragimov, Sudakov, 1975-76): Let v be a centered Gaussian process, a.s. bounded on M, and $\sigma_v^2 := \sup_{x\in M} \mathbb{E}[v(x)^2]$. Then $E\{||v||\} < \infty$, and there exists a constant α depending only on $\mathbb{E}\{||v||\}$ so that for $\tau > E\{||v||\}$ we have

$$\mathrm{Prob}\{||v|| > \tau\} \le e^{\alpha\tau - \tau^2/(2\sigma_v^2)}.$$

- We assume that $R_0 \in C^0$, $s > 2$, so by Proposition 1, $v \in C^0(M)$ a.s. and Proposition 2 applies. In our situation, $\tau = 1/a \to \infty$ as $a \to 0$, so $P_2(a) \le \exp[C_2/a - 1/(2a^2\sigma_v^2)]$.

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- To estimate $P_2(a)$ *from below* choose $x_0 \in M$ where the variance $r_v(x,x)$ attains its supremum σ_v^2. Clearly, $\mathrm{Prob}(||v|| > 1/a) \geq \mathrm{Prob}(v(x_0) > 1/a)$.
- The random variable $v(x_0)$ is Gaussian with mean 0 and variance σ_v. Accordingly, $\mathrm{Prob}(v(x_0) > 1/a) = \Psi(1/a\sigma_v)$, where we denote the error function $\Psi(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-t^2/2} dt$. Combining the estimates from above and below and using the standard estimates for Ψ, we get
- **Theorem 3:** Assume that $R_0 \in C^0, c_j = O(\lambda_j^{-s}), s > 2$. Then $\exists C_1 > 0, C_2 > 0$ such that

$$(C_1 a) e^{-1/(2a^2\sigma_v^2)} \leq P_2(a) \leq e^{C_2/a - 1/(2a^2\sigma_v^2)},$$

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- **Random analytic metrics.** Choose the coefficients $c_j = e^{-\lambda_j T/2}/\lambda_j$. Then a simple calculation shows that

$$r_h(x, x) = e^*(x, x, T),$$

where $e^*(x, x, T)$ is the heat kernel, *without the constant term.* Accordingly,

$$r_v(x, x, T) = e^*(x, x, T)/(R_0(x))^2.$$

- **Small T asymptotics** of $e^*(x, x, T)$ imply that as $T \to 0^+$,

$$\sigma_v^2 \sim \frac{1}{(4\pi T)^{n/2} \inf_{x \in M}(R_0(x))^2}.$$

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  Let g_0 and g_1 be two distinct reference metrics on M, normalized to have equal area, such that R_0 and R_1 have constant sign, $R_0 \equiv const$ and $R_1 \not\equiv const$. Then $\exists a_0 > 0, T_0 > 0$ (that depend on g_0, g_1) such that for any $0 < a < a_0$ and for any $0 < t < T_0$, we have $P_2(a, T, g_1) > P_2(a, T, g_0)$.
- **Proof:** By Gauss-Bonnet, $\int_M R_0 dV_0 = \int_M R_1 dV_1$. Since $\mathrm{vol}(M, g_0) = \mathrm{vol}(M, g_1)$, and since by assumption $R_0 \equiv const$ and $R_1 \not\equiv const$, it follows that $b_0 := \min_{x \in M}(R_0(x))^2 > \min_{x \in M}(R_1(x))^2 := b_1$. Accordingly, as $T \to 0^+$, we have

$$\frac{\sigma_v^2(g_1, T)}{\sigma_v^2(g_0, T)} \asymp \frac{b_0}{b_1} > 1.$$

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- **Large T asymptotics:**
  λ_1 - the smallest nonzero eigenvalue of $-\Delta_0$. Let $m = m(\lambda_1)$ be the multiplicity of λ_1, and let

$$F := \sup_{x \in M} \frac{\sum_{j=1}^m \phi_j(x)^2}{R_0(x)^2}. \tag{3}$$

- One can show that

$$\lim_{T \to \infty} \frac{\sigma_v^2(T)}{F e^{-\lambda_1 T}} = 1.$$

- **Theorem 5.** Let g_0 and g_1 be two metrics (of equal area) on a compact surface M, such that R_0 and R_1 have constant sign, and such that $\lambda_1(g_0) > \lambda_1(g_1)$. Then there exist $a_0 > 0$ and $0 < T_0 < \infty$ (that depend on g_0, g_1), such that for all $a < a_0$ and $T > T_0$ we have $P_2(a, T; g_0) < P_2(a, T; g_1)$.

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- To summarize: Small $T \Rightarrow$ metrics with $R_0 \equiv const$ extremal.
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- Genus 0: (S^2, *round*) extremal for *both* small T and large T (Hersch). **Conjecture:** extremal for *all* T.
- Genus $\gamma \geq 2$: Small $T \Rightarrow$ hyperbolic metrics extremal.
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- To summarize: Small $T \Rightarrow$ metrics with $R_0 \equiv const$ extremal.
- Large $T \Rightarrow$ metrics with the largest λ_1 extremal.
- Genus 0: $(S^2, round)$ extremal for *both* small T and large T (Hersch). **Conjecture:** extremal for *all* T.
- Genus $\gamma \geq 2$: Small $T \Rightarrow$ hyperbolic metrics extremal.
- Large T: By a 1974 theorem of S.T. Yau, hyperbolic metrics *never* maximize λ_1 in their conformal class.
- Genus 2: Metrics maximizing λ_1 for surfaces of genus 2 of fixed area are branched coverings of the round S^2 (J, Levitin, Nigam, Nadirashvili, Polterovich).
- **Question:** Which metrics are extremal for intermediate T?

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- We next indicate how to obtain a better estimate for $P_2(a)$ for $M = S^2$. Recall that there is a unique conformal class $[g_0]$ on S^2, where g_0 is the round metric, which we normalize to have $R_0 \equiv 1$.
- The isometry group acts transitively on (S^2, g_0), so the random fields $f(x), h(x), v(x)$ are *isotropic* and in particular have *constant variance*. That allows us to apply results of Adler and Taylor and obtain more precise *asymptotic* estimates for $P_2(a)$.
- Note that for surfaces of genus $\gamma \geq 2$ (where $R_0 < 0$), the variance $r_v(x, x)$ is *not* constant, so the results of A-T do not apply.
- Also, the assumptions on h are more restrictive: to apply A-T we need $v \in C^2(S^2)$ a.s; to apply Borell-TIS, we only need $v \in C^0(S^2)$ a.s.

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- Since Δ_0 on (S^2, g_0) is highly degenerate, we normalize our random Fourier series differently.
- $\mathcal{E}_m$ - space of spherical harmonics of degree m, dimension $N_m = 2m+1$; the corresponding eigenvalue is $E_m = m(m+1)$. Let $B_m = \{\eta_{m,k}\}_{k=1}^{N_m}$ be an orthonormal basis of $\mathcal{E}_m$.
- Let $f(x) = -\sqrt{|\mathcal{S}^2|} \sum_{m \geq 1,\, k} \frac{\sqrt{c_m}}{E_m \sqrt{N_m}} a_{m,k} \eta_{m,k}(x)$, where $a_{m,k}$ are standard Gaussian i.i.d. and $c_m > 0$ are (suitably decaying) constants satisfying $\sum_{m=1}^{\infty} c_m = 1$.
- It follows that $v = h = \sqrt{|\mathcal{S}^2|} \sum_{m \geq 1,\, k} \frac{\sqrt{c_m}}{\sqrt{N_m}} a_{m,k} \eta_{m,k}(x)$, It has unit variance, and covariance is given by $r_h(x,y) := \mathbb{E}[h(x)h(y)] = \sum_{m=1}^{\infty} c_m P_m(\cos(d(x,y)))$, where P_m is the Legendre polynomial.

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- We next estimate the probability of the event $\{||R_1 - R_0||_\infty < u\}, u > 0$; we shall do that for $g_1 = e^{af} g_0$, in the limit $a \to 0$. The result below hold for any compact orientable surface, including $\mathbf{T}^2$.
- To state the result, we define a new random field w on M:

$$w = \Delta_0 f + R_0 f = h + R_0 f.$$

We denote its covariance function by $r_w(x, y)$, and we define $\sigma_w^2 = \sup_{x \in M} r_w(x, x)$. Note that on flat $\mathbf{T}^2$, $R_0 = 0$ and therefore $w \equiv h$.

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  Theorem 7: Assume that the random metric is chosen so that the random fields f, h, w are a.s. C^0. Let $a \to 0$ and $u \to 0$ so that $(u/a) \to \infty$. Then

$$\log \mathrm{Prob}(\|R_1 - R_0\|_\infty > u) \sim -\frac{u^2}{2a^2\sigma_w^2}.$$

- The proof uses Borell-TIS inequality. The condition $(u/a) \to \infty$ ensures that the application of Borell-TIS gives an asymptotic result for $\log \mathrm{Prob}(\|R_1 - R_0\|_\infty > u)$.
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- We now explain the difficulties that arise when trying to extend Theorems 3, 4, 5 to dimension $n > 2$.
- **Main difficulty:** the gradient term $-a^2(n-1)(n-2)|\nabla_0 f|^2/4$ in the formula (1)

$$R_1 e^{af} = R_0 - a(n-1)\Delta_0 f - a^2(n-1)(n-2)|\nabla_0 f|^2/4$$

  no longer vanishes. Accordingly, the random field $R_1 e^{af}$ is no longer Gaussian, making its study more difficult.
- We obtain the following (weaker) generalization of Theorem 3.

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- $M^n, n \geq 3$ - compact. Assume that the scalar curvature $R_0 \in C^0$ of g_0 has constant sign. Let $g_1 = e^{af} g_0$, and let c_j satisfy $c_j = O(\lambda_j^{-s}), s > n/2 + 1$, so that $R_1 \in C^0$ a.s. Let$v = (\Delta_0 f)/R_0 = h/R_0$. As usual, we let $\sigma_v^2 = \sup_{x \in M} r_v(x,x)$. If $R_0 > 0$, let

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- **Theorem 8:**
  Assume that $\forall x \in M.\ R_0(x) < 0$. Then there exists $\alpha > 0$ so that

$$P_2(a) = O\left(\exp\left(\frac{\alpha}{a} - \frac{1}{2a^2(n-1)^2\sigma_v^2}\right)\right).$$

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- where

$$B = \frac{2 + \kappa - \sqrt{\kappa^2 + 4\kappa}}{\sigma_2 n(n-1)(n-2)}.$$

and

$$\kappa = \frac{4\sigma_v^2(n-1)}{\sigma_2 n(n-2)}.$$

- In dimension $n > 3$, after a conformal change of variables, Laplacian acquires a gradient term. Problem: construct (possibly higher order) elliptic operators so that after a conformal change of variables, the gradient term vanishes.
- Example: $n = 4$; *Paneitz operator*

$$P_4 = \Delta_g^2 + \delta[(2/3)R_g g - 2\mathrm{Ric}_g]d.$$

- General theory of such *conformally covariant operators*: Fefferman, Graham, Zworski, Jenne, Mason, Sparling, Chang, Yang et al.

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- M - compact, orientable manifold of even dimension $n \geq 4$. Consider conformally covariant elliptic operator P of order n.
- $P = \Delta^{n/2} +$ *lower order terms*. P is self-adjoint (Graham, Zworski, Fefferman). Under a conformal transformation of metric $\tilde{g} = e^{2\omega} g$, the operator P changes as follows: $\widetilde{P} = e^{-n\omega} P$. No lower order terms!
- There exist lower order operators with similar properties (GJMS operators of Graham- Jenne- Mason- Sparling). For even n, P has the largest possible order (*dimension critical*).

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- M has even dimension n. Q-curvature for $n = 4$ was defined by Paneitz:

$$Q_g = -\frac{1}{12}\left(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|^2\right).$$

- $n \geq 6$: Q-curvature - local scalar invariant associated to the operator P_n. It was introduced by T. Branson; alternative constructions were provided Fefferman, Graham, Hirachi using the *ambient metric* construction.
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- Important properties of Q-curvature: it is equal to $1/(2(n-1))\Delta^{n/2}R$ modulo nonlinear terms in curvature. Under a conformal transformation of variables $\tilde{g} = e^{2\omega}g$ on M^n, the Q-curvature transforms as follows:

$$P\omega + Q = \tilde{Q}e^{n\omega}. \quad (4)$$

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- **Proposition 9:** (M, g) compact, $n \geq 4$ even, Assume that M satisfies the following "generic" assumptions:
  i) $n = 4$: $\ker P_n = \{const\}$, and $\int_M QdV \neq 8\pi^2 k, k = 1, 2, \ldots$
  ii) $n \geq 6$: $\ker P_n = \{const\}$, and $\int_M QdV \neq (n-1)!\omega_n k, k = 1, 2, \ldots$, where $(n-1)!\omega_n = \int_{S^n} QdV$, the integral of Q-curvature for the round $\mathcal{S}^n$.

  Then there exists a metric g_Q on M in the conformal class of g with constant Q-curvature. If $n = 4, \int_M QdV < 8\pi^2$, $P_4 \geq 0$ and $\ker P_4 = \{const\}$, then g_Q is unique.
- If g has positive R and $M \neq \mathcal{S}^4$, then the assumption $\int_M QdV < 8\pi^2$ is satisfied; if in addition $\int_M Q \geq 0$, then the assumptions $P_4 \geq 0$ and $\ker P_4 = \{const\}$ are also satisfied.

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- It is possible to generalize Theorems 3, 7 for Q-curvature:
- Strategy:
  i) Consider (M, g_0) such that Q_0 has constant sign;
  ii) Consider the conformal perturbation $g_1 = e^{2af} g_0$ where a is a positive number; expand f in a series of eigenfunctions of P_n.
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- Improve estimates for the scalar curvature in higher dimensions.
- Consider “rough” metrics that arise in 2D quantum gravity.
- Study the case when $a \nrightarrow 0$.
- Study Ricci and sectional curvatures in high dimensions.
- Consider the space of all metrics, not just those in a conformal class.
- Study differential geometry of random metrics, e.g. distance between two points, diameter etc.
- Study geodesic and frame flows and their ergodicity; existence of conjugate points; entropy etc.
- Δ: small eigenvalues, heat kernel asymptotics.
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