Homogenization, inverse problems and optimal control via selfdual variational calculus

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November 2010

Limitations of classical calculus of variations

- Many basic elliptic PDEs can be written in the form

$$\partial\Phi(u) = p \tag{1}$$

where Φ is a convex lower semi-continuous functional on an infinite dimensional function space H. e.g.„ to solve

$$\begin{cases} -\mathrm{div}(\partial\varphi(\nabla u(x)) + F'(u(x)) & = & p(x) \quad \text{on} \quad \Omega \subset \mathbb{R}^n, \\ u & = & 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

where φ (resp., F) is convex on $\mathbf{R}^n$ (resp., $\mathbf{R}$), it suffices to minimize on $H_0^1(\Omega)$ the convex functional

$$\Phi(u) = \int_\Omega \Big\{\varphi(\nabla u(x)) + F(u(x)) - p(x)u(x)\Big\}\,dx.$$

This is a typical Euler-Lagrangian equation.

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- But what about the following Dirichlet BVP?

$$\begin{cases} -\,\mathrm{div}(T(\nabla u(x)) + F'(u(x)) + \Sigma_{i=1}^n a_i(x)\frac{\partial u}{\partial x_i} & = & p(x) \ \text{ on } \ \Omega\,, \\ u & = & 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

where T is a vector field not derived from a potential?

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  3. Control theory problems
  4. Homogenization of such equations.
- Indicate why all this should be developed on the Wasserstein manifold.

Basic example of selfdual variational calculus

$$\begin{cases} -\Delta u + |u|^{p-2}u + \Sigma_{i=1}^n a_i(x)\frac{\partial u}{\partial x_i} & = & f \text{ on } \Omega\,, \\ u & = & 0 \text{ on } \partial\Omega. \end{cases}$$

Assuming $\mathrm{div}(\mathbf{a}) = 0$ **on** Ω, then it suffices to minimize, on the same $H_0^1(\Omega)$, the new convex functional

$$I(u) = \Psi(u) + \Psi^*(\mathbf{a}\cdot\nabla u), \text{ where}$$

$$\Psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{p}\int_\Omega |u|^p dx + \int_\Omega f u dx$$

and ψ^* is its Fenchel-Legendre transform. Note that

$$I(u) = \Psi(u) + \Psi^*(\mathbf{a}\cdot\nabla u) - \langle u, \mathbf{a}\cdot\nabla u\rangle \geq 0$$

since by Legendre duality: $\psi(x) + \psi^*(p) - \langle x, p\rangle \geq 0$, and

$$\psi(x) + \psi^*(p) - \langle x, p\rangle = 0 \quad \text{iff} \quad p \in \partial\psi(x).$$

and so if $I(u) = 0$, then

$$\mathbf{a}\cdot\nabla u = \partial\Psi(u) = -\Delta u + |u|^{p-1}u + f$$

Key concept: Selfdual Lagrangians

1. Selfdual Lagrangians: $L : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ is convex lsc in both variables and

$$L^*(p,x) = L(x,p) \text{ for all } (p,x) \in X^* \times X.$$

In this case, $L(x,p) - \langle x,p\rangle \geq 0$ for every $(x,p) \in X \times X^*$, and

$$L(x,p) - \langle x,p\rangle = 0 \text{ if and only if } (p,x) \in \partial L(x,p)$$

2. Selfdual Vector Field: $F : X \to X^*$ such that there is L selfdual Lagrangian with $F = \bar{\partial} L$, i.e.,

$$F(x) = \bar{\partial} L(x) := \{p \in X^*; L(x,p) - \langle x,p\rangle = 0\}$$

$$= \{p \in X^*; (p,x) \in \partial L(x,p)\}.$$

3. The Completely Selfdual Equations.

$$p = \bar{\partial} L(x) \quad \text{or} \quad (p,x) = \partial L(x,p).$$

Basic examples of selfdual Lagrangians:

1. If φ is **convex lower semi-continuous** on X, then

$$L(x,p) = \varphi(x) + \varphi^*(-p)$$

is a selfdual Lagrangian on $X \times X^*$ and $\bar{\partial} L(x) = \partial\varphi(x)$.

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2. If $\Gamma : X \to X^*$ is **skew-symmetric** (i.e., $\Gamma^* = -\Gamma$), then

$$L(x,p) = \varphi(x) + \varphi^*(-\Gamma x + p)$$

   is a selfdual Lagrangian on $X \times X^*$ and $\bar{\partial} L = \Gamma + \partial\varphi$
   i.e., superposition of a dissipative and conservative vector fields) is derived from a selfdual Lagrangian (potential!)

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3. Solving $p \in \bar{\partial} L(x) = \Gamma(x) + \partial \varphi(x)$ amounts to showing that 0 is the infimum of

$$I_p(x) = L(x,p) - \langle x,p \rangle = \varphi(x) + \varphi^*(-\Gamma x + p) - \langle x,p \rangle.$$

Important:

- $\bar{\partial}L$ is NOT necessarily a differential, yet it is derived from a potential in the sense that a solution can be obtained by minimizing $I(x) = L(x,p) - \langle x,p\rangle$ and by showing that $\inf_{x\in X} I(x) = 0$ equal to zero!

Theorem

Let L be a selfdual Lagrangian on a reflexive Banach space $X \times X^*$, let $p \in X^*$ be such that $(0,p) \in Dom(L)$. If the functional $I_p(x) = L(x,p) - \langle x,p\rangle$ is coercive on X, then there exists $u \in X$ such that

$$I_p(u) = \min_{u\in X} I_p(u) = 0 \textit{ and } p \in \bar{\partial}L(u).$$

Unexpected surprise: All maximal monotone operators are selfdual vector fields and vice-versa

(i) Let L be a proper selfdual Lagrangian L on a reflexive Banach space $X \times X^*$, then the vector field $x \to \bar{\partial} L(x)$ is maximal monotone.
(ii) Conversely, if $\beta : D(\beta) \subset X \to 2^{X^*}$ is a maximal monotone operator with a non-empty domain, then there exists a selfdual Lagrangian L_β on $X \times X^*$ such that $\beta = \bar{\partial} L_\beta$.

Not surprising –in retrospect– but many advantages:

- Maximal monotone operators can be reduced to convex analysis in phase space.

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Not surprising –in retrospect– but many advantages:

- Maximal monotone operators can be reduced to convex analysis in phase space.
- Equations involving MM vector fields are variational.
- Analogue of Rockafellar's theorem for cyclically monotone operators.

Solving variationally non-potential equations

Variational resolution of two typical equations involving a maximal monotone vector field $\beta : X \to X^*$.

First associate to β a selfdual Lagrangian $L : X \times X^* \to \mathbf{R}$ such that $\beta = \bar{\partial} L_\beta$.

(1) Solving $p \in \beta(u)$ amounts to minimizing on X the functional

$$I_p(u) = L_\beta(u, p) - \langle u, p \rangle.$$

(2) Solving $-\mathrm{div}(\beta(\nabla u(x))) = p(x)$ *on* Ω, $u = 0$ on $\partial\Omega$, amounts to minimizing on $H^1_0(\Omega)$ the functional

$$I_p(u) := \inf_{\substack{f \in L^2(\Omega;\mathbb{R}^N) \\ -\mathrm{div}(f) = p}} \int_\Omega \Big[L_\beta\Big(\nabla u(x), f(x)\Big) - \langle u(x), p(x) \rangle_{\mathbb{R}^N} \Big] \, dx$$

Because $I_p(u) = \mathcal{L}(u, p) - \langle u, p \rangle$ where

$$\mathcal{L}(u, p) := \inf\{ \int_\Omega L\Big(x, \nabla u(x), f(x)\Big) \, dx; f \in L^2(\Omega; \mathbb{R}^N), -\mathrm{div}(f) = p\},$$

is a selfdual Lagrangian Lagrangian on $H^1_0(\Omega) \times H^{-1}(\Omega)$

Nonlinear inverse problems

Given $u_0 \in H_0^1(\Omega)$, find a vector field β in a given class of maximal monotone maps $\mathcal{C}$ such that u_0 is a solution of

$$-\mathrm{div}(\beta(\nabla u(x)) = p(x), \quad u = 0 \text{ on } \partial\Omega \tag{2}$$

Least square approach: Minimize

$$\int_\Omega |u(x) - u_0(x)|^2 dx$$

over all $u \in H_0^1(\Omega)$, $\beta \in \mathcal{C}$, such that $-\mathrm{div}(\beta(\nabla u)) = p$ on Ω.
The constraint set is not easily tractable.

Penalized least square

Let $\mathcal{L} = \{L$ selfdual on $\mathbf{R}^n \times \mathbf{R}^n$; $\bar{\partial} L = \beta$ for some $\beta \in \mathcal{C}\}$.
For each $\epsilon > 0$, minimize the functional

$$
\begin{aligned}
\mathcal{P}_\epsilon(L, u, f) &= \int_\Omega |u(x) - u_0(x)|^2 dx \\
&\quad + \frac{1}{\epsilon} \int_\Omega \left\{ L\Big(\nabla u, f\Big) - p(x) u(x) \right\} dx
\end{aligned}
$$

on the class $\mathcal{T} := \Big\{(L, u, f) \in \mathcal{L} \times H_0^1(\Omega) \times L^2(\Omega);\ \mathrm{div}\, f = p\Big\}$

- $\mathcal{P}_\epsilon$ is convex and lsc in all variables.
- If $\mathcal{L}$ is a convex "compact" class of selfdual Lagrangians, there exists a minimizer $(L_\epsilon, u_\epsilon, f_\epsilon) \in \mathcal{T}$
- If ϵ is small enough, the non-negative penalization has to be small at $(L_\epsilon, u_\epsilon, f_\epsilon)$, and a weak cluster point (L_0, u_0, f_0) is a solution with $\beta_0 := \bar{\partial} L_0$ being the optimal maximal monotone operator, since the penalty term has to be zero.

Optimal control: Cheapest temperature source control for a desired temperature profile

Consider the heat equation

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= f(x,t) \quad \text{in } \Omega \times [0,1] \\ u(x,t) &= 0 \quad \text{on } \partial\Omega \times [0,1] \\ u(x,0) &= g(x) \quad \text{in } \Omega \end{aligned} \tag{3}$$

Let $u_0(x,t)$, with $u_0(x,0) = g(x)$ be a desired temperature profile to be achieved over Ω along $[0,1]$.
Need to control the temperature by specifying the heat source f over the domain Ω, assume the cost of maintaining such temperature is given by $C(f) = \int_0^1 \|f\|_2^2 \,\mathrm{d}t$.
We want to minimize the cost of f and achieve the closest possible behaviour to the profile u_0, i.e., we want to minimize

$$\int_0^1 \int_\Omega (|u(x,t) - u_0(x,t)|^2 + |f(x,t)|^2) dx \,\mathrm{d}t$$

among all possible solutions u of (3) for some f.

Selfdual variational formulation of heat equation

For a given f, Equation (3) is solved by minimizing

$$\begin{aligned} J_f(u) &= \frac{1}{2}\int_0^1 \int_\Omega \Big(|\nabla u(t,x)|^2 + \Big|\nabla(-\Delta)^{-1}\big(f(x) - \dot{u}(t,x)\big)\Big|^2 - 2f(x)u(t,x)\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_\Omega |g(x)|^2\,\mathrm{d}x - 2\int_\Omega u(0,x)g(x)\,\mathrm{d}x + \frac{1}{2}\int_\Omega (|u(0,x)|^2 + |u(1,x)|^2)\,\mathrm{d}x. \end{aligned}$$

over $A^2[[0,T];H_0^1(\Omega)]$.
The control problem amounts to minimize for each $\epsilon > 0$,

$$\int_0^1 \int_\Omega (|u(x,t) - u_0(x,t)|^2 + |f(x,t)|^2) dx\,\mathrm{d}t + \frac{1}{\epsilon} J_f(u)$$

over $A^2[[0,T];H_0^1(\Omega)] \times L^2([0,T]\times\Omega)$, to find (u_ϵ, f_ϵ), then let ϵ go to zero.

A basic homogenization problem

We consider the conductivity equation with a given heat source u_n^* in a heterogenous medium defined by the non-homogeneous conductivity vector field β.

$$\begin{cases} \tau_n(x) & \in & \beta(\frac{x}{\epsilon_n}, \nabla u_n(x)) & x \in \Omega, \\ -\mathrm{div}(\tau_n(x)) & = & u_n^*(x) & x \in \Omega, \\ u_n(x) & = & 0 & x \in \partial\Omega, \end{cases} \tag{4}$$

where Ω is a bounded domain of $\mathbb{R}^N$, and $\beta : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a measurable map on $\Omega \times \mathbb{R}^N$ such that:

- $\beta(x, \cdot)$ is maximal monotone on $\mathbb{R}^N$ for almost all $x \in \Omega$
- $\beta(., \xi)$ is Q-periodic for an open non-degenerate parallelogram Q in $\mathbb{R}^n$.

This problem has been investigated in recent years by many authors: Francfort, Murat, Tartar, Damlamian, Meunier, Van Shaftingen, Braides, Chiado Piat, Dal Maso, Defranscheshi.

Representation of a family of maximal monotone fields

$$\begin{cases} \tau(x) & \in & \beta(x, \nabla u(x)) & a.e.\ x \in \Omega, \\ -\mathrm{div}(\tau(x)) & = & p(x) & a.e.\ x \in \Omega. \end{cases}$$

The class $M_{\Omega,p}(\mathbb{R}^N)$ introduced by Chiado Piat, Dal Maso, Defranscheshi consists of all possibly multi-valued functions $\beta : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ with closed values, which satisfy:
(i) β is measurable with respect to $\mathcal{L}(\Omega) \times \mathcal{B}(\mathbb{R}^N)$ and $\mathcal{B}(\mathbb{R}^N)$ where $\mathcal{L}(\Omega)$ is is the σ-field of all measurable subsets of Ω and $\mathcal{B}(\mathbb{R}^N)$ is the σ-field of all Borel subsets of $\mathbb{R}^N$.
(ii) For a.e. $x \in \Omega$, the map $\beta(x, .) : \mathbb{R}^N \to \mathbb{R}^N$ is maximal monotone.
(iii) There exist non-negative constants m_1, m_2, c_1 and c_2 such that for every $\xi \in \mathbb{R}^N$ and $\eta \in \beta(\xi)$,

$$\langle \xi, \eta \rangle_{\mathbb{R}^N} \geq \max\left\{ \frac{c_1}{p} |\xi|^p - m_1, \frac{c_2}{q} |\eta|^q - m_2 \right\}, \tag{5}$$

Selfdual Lagrangians associated to maximal monotone operators

(1) If $\beta \in M_{\Omega,p}(\mathbb{R}^N)$ for $p > 1$, then there exists a state-dependent selfdual Lagrangian $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ such that $\beta(x,.) = \bar{\partial} L(x,.)$ for a.e. $x \in \Omega$, and for all $a, b \in \mathbb{R}^N$,

$$(*) \quad C_0(|a|^p + |b|^q - n_0(x)) \leq L(x,a,b) \leq C_1(|a|^p + |b|^q + n_1(x))$$

where C_0 and C_1 are two positive constants and $n_0, n_1 \in L^1(\Omega)$.

(2) Conversely, if $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a state-dependent selfdual Lagrangian satisfying (**), then $\bar{\partial} L(x,.) \in M_{\Omega,p}(\mathbb{R}^N)$.

Lifting Self-dual Lagrangians from $\mathbf{R}^n \times \mathbf{R}^n$ to $W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$

Suppose L is a state-dependent selfdual Lagrangian on $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$ such that for all $a, b \in \mathbb{R}^N$,

$$(**) \quad C_0(|a|^p + |b|^q - n_0(x)) \leq L(x, a, b) \leq C_1(|a|^p + |b|^q + n_1(x))$$

where $C_0, C_1 > 0$ and $n_0, n_1 \in L^1(\Omega)$. Then the Lagrangian defined on $W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$ by

$$F(u, u^*) := \inf\{\int_\Omega L\Big(x, \nabla u(x), f(x)\Big)\,dx;\, f \in L^q(\Omega; \mathbb{R}^N), -\mathrm{div}(f) = u^*\},$$

is selfdual $W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$.

Variational resolution of the main equation

Let $\beta \in M_{\Omega,p}(\mathbb{R}^N)$ for some $p > 1$, then for every $u^* \in W^{-1,q}(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$, there exist $\bar{u} \in W_0^{1,p}(\Omega)$ and $\bar{f}(x) \in L^q(\Omega;\mathbb{R}^N)$ such that

$$\begin{cases} \bar{f} \in \beta(x, \nabla \bar{u}(x)) & \text{a.e. } x \in \Omega \\ -\mathrm{div}(\bar{f}) = u^*. \end{cases} \tag{6}$$

It is obtained by minimizing the functional

$$I(u) := \inf_{\substack{f \in L^q(\Omega;\mathbb{R}^N) \\ -\mathrm{div}(f) = u^*}} \int_\Omega \Big[L\Big(x, \nabla u(x), f(x)\Big) - \langle u(x), u^*(x) \rangle_{\mathbb{R}^N} \Big] dx$$

on $W^{1,p}(\Omega)$, where L is a state-dependent selfdual Lagrangian on $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$ associated to β in such a way that $\bar{\partial} L(x,\cdot) = \beta(x,\cdot)$ for a.e $x \in \Omega$.

Variational formula for the homogenized field

Given a family β in $M_{\Omega,p}(\mathbb{R}^N)$ that is Q-periodic for an open non-degenerate parallelogram Q in $\mathbb{R}^n$, its homogenization β_{hom} can now be given by a variational formula in terms of a homogenized selfdual Lagrangian L_{hom}.

Theorem: If $\beta \in M_{\Omega,p}(\mathbb{R}^N)$ is Q-periodic and L is a state dependent selfdual Lagrangian on $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$ such that $\beta(x,.) = \bar{\partial} L(x,.)$. Then β_{hom} is given by $\beta_{hom} = \bar{\partial} L_{hom}$ where L_{hom} is the selfdual Lagrangian

$$L_{hom}(\xi,\eta) = \min_{\substack{\varphi \in W^{1,p}_{\#}(Q) \\ g \in L^q_{\#}(Q;\mathbb{R}^N)}} \frac{1}{|Q|} \int_Q L\Big(x, \xi + \nabla\varphi(x), \eta + g(x)\Big)\, dx.$$

$$W^{1,p}_{\#}(Q) = \{u \in W^{1,p}(Q);\ \int_Q u(x)\, dx = 0 \ \text{ and } \ u \text{ is } Q\text{−periodic}\}.$$

$$L^q_{\#}(Q;\mathbb{R}^N) := \Big\{ g \in L^q(Q;\mathbb{R}^N);\ \int_Q \langle g(y), \nabla\varphi(y)\rangle dy = 0, \forall \varphi \in W^{1,p}_{\#}(Q) \Big\}$$

Mosco and Γ-convergence of selfdual functionals

Let F_n and F be functionals on a reflexive Banach space X. The sequence $\{F_n\}$ is said to *Γ-converge* (resp., *Mosco-converge*) to F, if the following two conditions are satisfied:

1. For any sequence $\{u_n\} \subset X$ such that $u_n \to u$ strongly (resp., $u_n \rightharpoonup u$ weakly) in X to some $u \in X$, one has
$$F(u) \leq \liminf_{n\to\infty} F_n(u_n).$$
2. For any $u \in X$, there exists a sequence $\{u_n\} \subset X$ such that $u_n \to u$ strongly in X and
$$\lim_{n\to\infty} F_n(u_n) = F(u).$$

The following is a fundamental property of Mosco-convergence. Let F_n, F be convex lower semi-continuous functionals, then $\{F_n\}$ Mosco-converge to F if and only their Fenchel-Legendre duals $\{F_n^*\}$ Mosco-converge to F^*.

This implies the agreable fact that Mosco and Γ-convergence are actually equivalent for a sequence of selfdual Lagrangians $\{L_n\}$, as long as the limiting Lagrangian L is itself selfdual.

Theorem

Let $\{L_n\}$ be a family of selfdual Lagrangians on $X \times X^*$, where X is a reflexive Banach space, and let L be a Lagrangian on $X \times X^*$. The following statements are then equivalent:

1. *$\{L_n\}$ Mosco-converges to L.*
2. *L is selfdual and $\{L_n\}$ Γ-converges to F.*
3. *L is selfdual and for any $(u, u^*) \in X \times X^*$, there exists a sequence (u_n, u_n^*) converging strongly to (u, u^*) in $X \times X^*$ such that*

$$\limsup_n L_n(u_n, u_n^*) \leq L(u, u^*).$$

Graph Convergence

Considering a sequence of sets $\{A_n\}$ in X, the corresponding sequential lower and upper limit sets are respectively given by

$$Li_X\big(A_n\big) = \{u \in X \ : \ \exists u_n \to u,\, u_n \in A_n\},$$

$$Ls_X\big(A_n\big) = \{u \in X \ : \ \exists k(n) \to \infty,\, \exists u_{n(k)} \to u,\, u_{n(k)} \in A_k\}.$$

In other words, Limit vs. cluster points.
Clearly, $Li_X(A_n) \subseteq Ls_X(A_n)$.
A sequence of subsets $\{A_n\}$ of X is said to converge to $A \subset X$, in the sense of *Kuratowski-Painlevé*, if

$$Ls_X(A_n) = A = Li_X(A_n).$$

This definition, when X is replaced by the phase space $X \times X^*$ and when the subsets A_n are graphs of maps from X to X^*, is also refered to as *graph*-convergence.

Continuity of $L \to \bar{\partial} L$ for Γ to Graph convergence

One of the most attractive properties of Mosco convergence is the fact that for convex functions it implies the graph convergence (or *Kuratowski-Painlevé convergence*) of their corresponding subdifferentials.
A similar result holds for self-dual Lagrangian (and Γ-convergence).

Theorem

Let X be a reflexive Banach space and suppose $\{L_n\}$ is a family of selfdual Lagrangians on $X \times X^*$.
If $L : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ is a selfdual Lagrangian that is a Γ-limit of $\{L_n\}$, then the graph of $\bar{\partial} L_n$ converge to the graph of $\bar{\partial} L$ in the sense of Kuratowski-Painlevé.

Variational approach to gradient flows (Brezis-Ekeland,1976)

$$\begin{cases} -\dot{v}(t) & \in & \partial\varphi(v(t)) \quad \text{a.e.} \quad \text{on} \quad [0,T], \\ v(0) & = & v_0. \end{cases}$$

φ convex l.s.c on Hilbert space H. (e.g., $\varphi(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx$)
Minimize

$$I(u) = \int_0^T \left[\varphi(u(t)) + \varphi^*(-\dot{u}(t))\right] dt + \frac{1}{2}|u(0)|^2 - 2\langle u(0), v_0\rangle + |v_0|^2 + \frac{1}{2}|u(T)|^2$$

$$I(u) = \int_0^T L(t, u(t), \dot{u}(t))dt + \ell(u(0), u(T)) \quad \text{Selfdual} \quad \text{form!}$$

Using that $\int_0^T \langle u(t), \dot{u}(t)\rangle dt = \frac{1}{2}|u(T)|^2 - \frac{1}{2}|u(0)|^2$,

$$I(u) = \int_0^T \Big[\varphi(u(t)) + \varphi^*(-\dot{u}(t)) + \langle u(t), \dot{u}(t)\rangle\Big] dt + |u(0) - v_0|^2 \geq 0,$$

The important factor is that with selfduality we can prove
$\inf I(u) = I(\bar{u}) = 0$, then we are done by Legendre duality.

Analogue on Wasserstein space!!!!!????