Limiting distributions for a one-dimensional random walk in a random environment

Jonathon Peterson

Cornell University
Department of Mathematics

Joint work with Gennady Samorodnitsky

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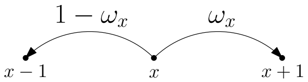
RWRE in $\mathbb{Z}$ with i.i.d. environment

An *environment* $\omega = \{\omega_x\}_{x \in \mathbb{Z}} \in \Omega = [0,1]^{\mathbb{Z}}$.
P an i.i.d. product measure on Ω.



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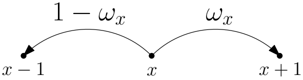

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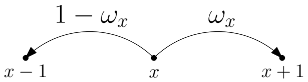

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Averaged law $\mathbb{P}$: average over environments.

$$\mathbb{P}(G) := \int_\Omega P_\omega(G)dP(\omega)$$

Recurrence/Transience and LLN

$$\rho_x = \frac{1-\omega_x}{\omega_x}$$

Theorem (Solomon '75)

- $E[\log \rho_0] < 0 \Rightarrow X_n \to +\infty$
- $E[\log \rho_0] > 0 \Rightarrow X_n \to -\infty$
- $E[\log \rho_0] = 0 \Rightarrow X_n$ *is recurrent.*

Theorem (Solomon '75)

If $E[\log \rho_0] < 0$ *then*

$$\lim_{n\to\infty} \frac{X_n}{n} = v_P = \begin{cases} \frac{1-E[\rho_0]}{1+E[\rho_0]} & E[\rho_0] < 1 \\ 0 & E[\rho_0] \geq 1. \end{cases}$$

Averaged Limiting Distributions

$\kappa > 0$ solves $E\rho^\kappa = 1$.

Theorem (Kesten, Kozlov, Spitzer '75)

There exists a constant b such that

$$(a) \quad \kappa \in (0,1) \Rightarrow \qquad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n}{n^\kappa} \le x \right) = 1 - L_{\kappa,b}(x^{-1/\kappa})$$

$$(b) \quad \kappa \in (1,2) \Rightarrow \qquad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - nv_P}{n^{1/\kappa}} \le x \right) = 1 - L_{\kappa,b}(-x)$$

$$(c) \quad \kappa > 2 \Rightarrow \qquad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - nv_P}{b\sqrt{n}} \le x \right) = \Phi(x)$$

where $L_{\kappa,b}$ is a κ-stable distribution function.

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Limits for X_n: $\mathbb{P}(T_n > t) \approx \mathbb{P}(X_t < n)$.

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$T_n = \sum_{i=1}^n (T_i - T_{i-1})$

Quenched Limiting Distribution: Gaussian Regime

Theorem (Goldsheid '06, P. '06)

If $\kappa > 2$ then

$$\lim_{n\to\infty} P_\omega\left(\frac{T_n - E_\omega T_n}{\sigma\sqrt{n}} \leq x\right) = \Phi(x), \quad P-a.s.$$

where $\sigma^2 = E(Var_\omega T_1)$, and

$$\lim_{n\to\infty} P_\omega\left(\frac{X_n - nv_P + Z_n(\omega)}{v_P^{3/2}\sigma\sqrt{n}} \leq x\right) = \Phi(x), \quad P-a.s.$$

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- Requires a *random* centering.
- Scaling constant is different from averaged CLT.

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$$\frac{T_n - n/v_P}{\sqrt{n}} = \frac{T_n - E_\omega T_n}{\sqrt{n}} + \frac{E_\omega T_n - n/v_P}{\sqrt{n}}$$

- Terms on right are asymptotically independent.
- $(E_\omega T_n - n/v_P)/\sqrt{n}$ is approximately mean zero Gaussian.

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Question

What happens when $\kappa < 2$?
Do we get quenched stable laws?

Quenched Limits: $\kappa < 2$

Theorem (P. '07)

If $\kappa < 2$ then $P-$ a.s. there exist random subsequences $n_k = n_k(\omega)$, and $m_k = m_k(\omega)$ such that

$$(a) \quad \lim_{k\to\infty} P_\omega\left(\frac{T_{n_k} - E_\omega T_{n_k}}{\sqrt{Var_\omega T_{n_k}}} \leq x \right) = \Phi(x)$$

$$(b) \quad \lim_{k\to\infty} P_\omega\left(\frac{T_{m_k} - E_\omega T_{m_k}}{\sqrt{Var_\omega T_{m_k}}} \leq x \right) = \begin{cases} 0 & \text{if } x < -1 \\ 1 - e^{-x-1} & \text{if } x \geq -1 \end{cases}$$

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- Which limit is more likely?

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FAQ

- Which limit is more likely?
- What other subsequential limits are possible?
- Where do the κ-stable distributions come from.

Weak Quenched Limits: $\kappa < 2$

For any $x \in \mathbb{R}$,

$$P_\omega\left(\frac{T_n - E_\omega T_n}{n^{1/\kappa}} \le x \right)$$

is a random variable.

- Doesn't converge almost surely to a deterministic constant.
- Maybe converges in some weaker sense (in distribution).

Random distributions

$\mathcal{M}_1 =$ Probability distributions on $\mathbb{R}$ (with Prohorov metric ρ).

$$\mu_{n,\omega}(\cdot) = \begin{cases} P_\omega\left(\frac{T_n - E_\omega T_n}{n^{1/\kappa}} \in \cdot \right) & \kappa < 2 \\ P_\omega\left(\frac{T_n - E_\omega T_n}{\sqrt{n}} \in \cdot \right) & \kappa > 2. \end{cases}$$

$\mu_{n,\omega}$ is an $\mathcal{M}_1$-valued random variable.

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$\mu_{n,\omega}$ converges weakly (in distribution) on $\mathcal{M}_1$.

Weak Quenched Limiting Distributions: $\kappa < 2$

Theorem (P. & Samorodnitsky '10)

If $\kappa < 2$, then there exists a $\lambda > 0$ such that

$$\mu_{n,\omega} \underset{n\to\infty}{\Longrightarrow} \mu_{\lambda,\kappa},$$

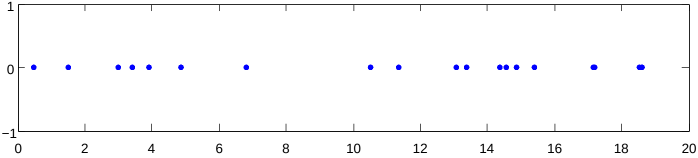
where $\mu_{\lambda,\kappa}$ is a random probability distribution defined by

$$\mu_{\lambda,\kappa}(A) = \mathbf{P}\left(\sum_{i=1}^{\infty} a_i(\tau_i - 1) \in A \,\middle|\, a_i,\ i \geq 1 \right)$$

where $\{a_i\}_{i\geq 1}$ are the points of a non-homogeneous Poisson point process with intensity $\lambda\kappa x^{-\kappa-1}$.

Non-homogeneous Poisson point processes

Let $\{\Gamma_i\}_{i\geq 1}$ be a PPP with constant intesity 1.



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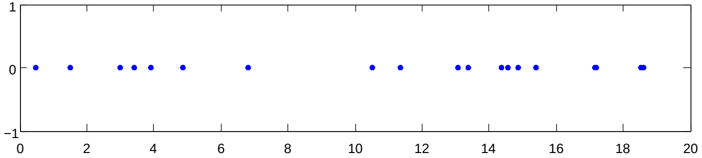
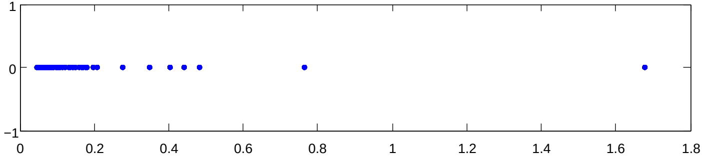


Let $a_i = \lambda^{1/\kappa}\Gamma_i^{-1/\kappa}$. Then, $\{a_i\}_{i\geq 1}$ is a PPP with intensity $\lambda\kappa x^{-\kappa-1}$.



Weak Quenched Limits: Averaged Centering ($\kappa < 1$)

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Note: Implies the averaged stable limit laws.

$$\mathbb{P}\left(\frac{T_n}{n^{1/\kappa}} \leq x\right) = E\left[\tilde{\mu}_{n,\omega}((-\infty, x])\right] \underset{n\to\infty}{\longrightarrow} \mathbf{E}\left[\tilde{\mu}_{\lambda,\kappa}((-\infty, x])\right]$$

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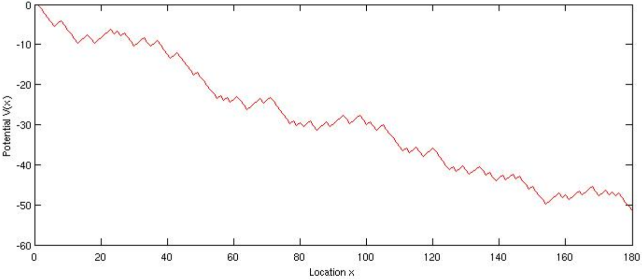
- $a_i \tau_i$ are atoms of PPP with intensity $\lambda\kappa\Gamma(\kappa+1)x^{-\kappa-1}\,dx$.
- $\sum_{i\geq 1} a_i \tau_i$ is a κ-stable random variable.

Proofs

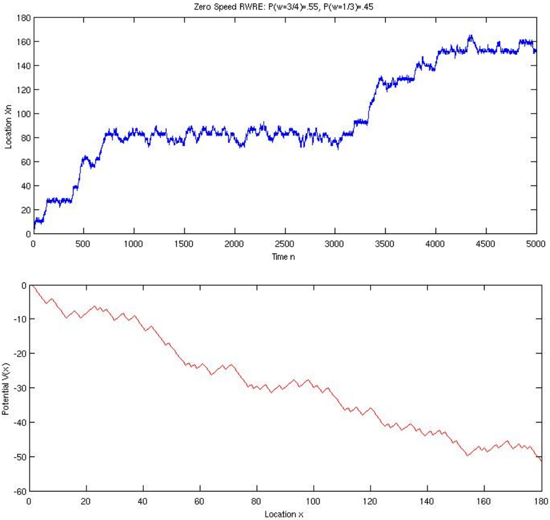
Potential:

$$V(i) := \begin{cases} \sum_{k=0}^{i-1} \log \rho_k, & i > 0 \\ 0, & i = 0 \\ \sum_{k=i}^{-1} -\log \rho_k, & i < 0 \end{cases}$$

Trap: Atypical section where the potential is increasing.



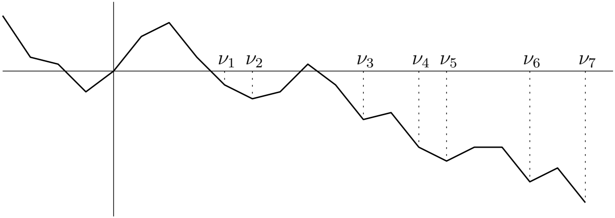
Trapping Effects



Blocks of the environment

Ladder locations $\{\nu_n\}$ defined by $\nu_0 = 0$,

$$\nu_n := \inf\{i > \nu_{n-1} : V(i) < V(\nu_{n-1})\}$$



Reduction to T_{ν_n}

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Enough to study

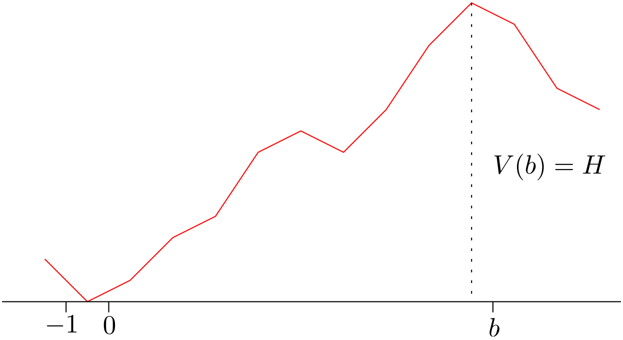
$$\phi_{n,\omega}(\cdot) = P_\omega \left(\frac{T_{\nu_n} - E_\omega T_{\nu_n}}{n^{1/\kappa}} \in \cdot \right)$$

Want to show $\phi_{n,\omega} \Longrightarrow \mu_{\lambda,\kappa}$ for some λ.

Crossing Hills

Probability of escaping a trap of Height H.

$$P_\omega(T_b < T_{-1}) = \frac{1}{1 + \sum_{j=1}^{b} e^{V(j)}}$$



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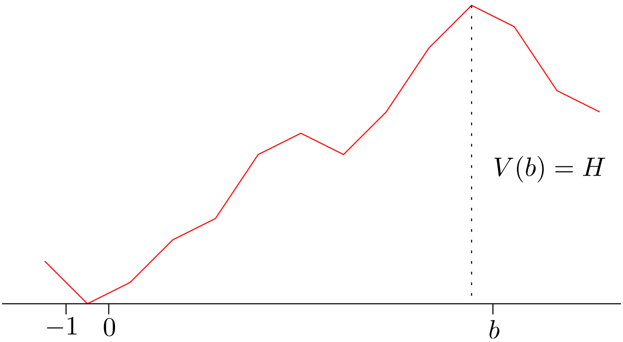

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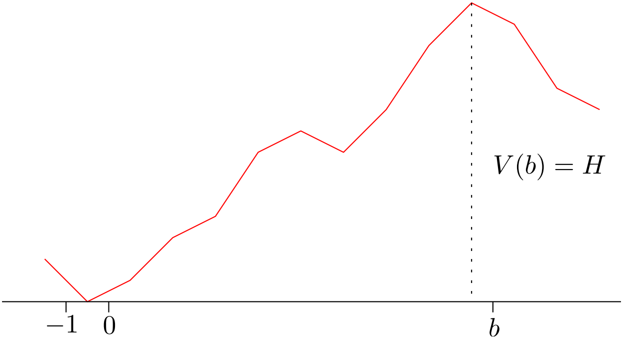
Time to escape trap of height $H \approx Exp(e^{-H})$.



Comparison with exponentials

Want: $T_\nu \stackrel{Law}{\approx} (E_\omega T_\nu)\tau$.

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Decompose T_ν into trials

$$T_\nu = S + \sum_{i=1}^{G} F_i$$

Where

$$\begin{aligned} S &\sim (T_\nu \mid T_\nu < T_0^+) \\ F_i &\sim (T_0^+ \mid T_0^+ < T_\nu) \\ G &\sim \mathrm{Geo}(p_\omega), \qquad \text{where } p_\omega = P_\omega(T_\nu < T_0^+) \end{aligned}$$

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Couple G with an exponential τ

$$G = \left\lfloor \frac{\tau}{\log(1/(1-p_\omega))} \right\rfloor.$$

Comparison with mixture of exponentials

Let

$$\beta_i = \beta_i(\omega) = E_\omega(T_{\nu_i} - T_{\nu_{i-1}}).$$

The coupling gives

$$\frac{T_{\nu_n} - E_\omega T_{\nu_n}}{n^{1/\kappa}} \approx \frac{1}{n^{1/\kappa}} \sum_{i=1}^{n} \beta_i(\tau_i - 1)$$

where τ_i are i.i.d. Exp(1).

Comparison with mixture of exponentials

Let

$$\beta_i = \beta_i(\omega) = E_\omega(T_{\nu_i} - T_{\nu_{i-1}}).$$

The coupling gives

$$\frac{T_{\nu_n} - E_\omega T_{\nu_n}}{n^{1/\kappa}} \approx \frac{1}{n^{1/\kappa}} \sum_{i=1}^{n} \beta_i(\tau_i - 1)$$

where τ_i are i.i.d. Exp(1).

$$\lim_{n\to\infty} n^{-2/\kappa}\, \mathrm{Var}_\omega\left(T_{\nu_n} - \sum_{i=1}^{n} \beta_i \tau_i \right) = 0, \quad \text{in } P\text{-probability.}$$

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$$\phi_{n,\omega}(\cdot) = P_\omega\left(\frac{T_{\nu_n} - E_\omega T_{\nu_n}}{n^{1/\kappa}} \in \cdot \right)$$

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Need to understand the distribution of $\{\beta_i\}_{i=1}^n$

Point Process Convergence

$\mathcal{M}_p =$ Point processes $\sum_{i\geq 1} \delta_{x_i}$ on $(0,\infty)$.

$$N_n = N_n(\omega) = \sum_{i=1}^{n} \delta_{\beta_i / n^{1/\kappa}}$$

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Proposition (P. & Samorodnitsky '10)

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1. $\{\beta_i\}$ is (almost) stationary.
2. The β_i have heavy tails: $\beta_1 = E_\omega T_\nu \approx e^H$

$$P(e^H > x) \sim Cx^{-\kappa} \qquad \text{as } x \to \infty.$$

3. $\{\beta_i\}$ has good mixing properties.

Define $H : \mathcal{M}_p \longrightarrow \mathcal{M}_1$ by

$$H(\zeta)(\cdot) = \mathbf{P}\left(\sum_{i \geq 1} x_i(\tau_i - 1) \in \cdot \right), \qquad \text{for } \zeta = \sum_{i \geq 1} \delta_{x_i}.$$

Remark: Only defined if $\sum x_i^2 < \infty$.

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H is NOT continuous.