Random walks in random environments on trees

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Workshop on Interacting Processes in Random Environments
Toronto 2011

Random walks in random environments (S_n) on $\mathbb{Z}^1$

Let $\omega = \{\omega_x, x \in \mathbb{Z}^1\}$ be a family of i.i.d. random variables (and non constant) taking values in $(0, 1)$. The ω plays the role of random environment. Assume that for some constant $\epsilon > 0$, $\epsilon \leq \omega_x \leq 1 - \epsilon$ almost surely.

Given ω, let $\{S_n, n \geq 0\}$ be a Markov chain taking values in $\mathbb{Z}^1$ starting from 0 with probability transition : ($\mathbb{P}_\omega$ means the probability conditioned on ω)

$$\mathbb{P}_\omega\Big(S_{n+1} = y \big| S_n = x\Big) = \begin{cases} \omega_x, & \text{if } y = x + 1\,; \\ 1 - \omega_x, & \text{if } y = x - 1. \end{cases}$$

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Asymptotic behaviors of (S_n)

References

- P. Révész : *Random walk in random and non-random environments* (1st edition : 1990, 2nd edition : 2005)
- O. Zeitouni : Lecture notes in Mathematics, 2004.

Recurrence/transience criteria : Solomon (1975)

- (S_n) is recurrent if and only if $\mathbb{E}(\log \frac{1-\omega_x}{\omega_x}) = 0$;
- $S_n \to \infty$ if and only if $\mathbb{E}(\log \frac{1-\omega_x}{\omega_x}) < 0$.

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How big is (S_n) ?

Transient case : Kesten, Kozlov and Spiter (1976)

when $S_n \to \infty$, $S_n \approx n^\rho$. The exponent ρ is explicitly determined by the law of ω_x and can vary in $(0, 1]$.

Recurrent case : Sinai (1982)

when (S_n) is recurrent, $\frac{S_n}{\log^2 n}$ converges in law (to some non-degenerated law, explicitly computed by Kesten (1986) and Golosov (1986)).

Question : $\mathbb{Z}^d$?, trees ?, ...

An example of answers on trees :

We may find both subdiffusive and slow movement behaviors in a class of recurrent RWREs on trees.

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RWRE on (regular) trees

Random environments

Let $\mathbb{T}$ be a regular tree rooted at $\varnothing$ and each individual has $\mathfrak{b}$-children [we can also take a Galton-Watson tree $\mathbb{T}$]. Let $\omega = \{(\omega(x,y), y \in \mathbb{T})_{x\in\mathbb{T}}\}$ be a family of independent random vectors such that $\sum_{y\in T: y\sim x}\omega(x,y) = 1$, $\omega(x,y) > 0$ if $x \sim y$ ($x \sim y$ means x and y are adjacent).

Random walk in random environment (X_n)

Conditioned on ω, (X_n) is a Markov chain taking values in $\mathbb{T}$ with probability transition :

$$\mathbb{P}_\omega\Big(X_{n+1} = y \,\big|\, X_n = x\Big) = \omega(x,y).$$

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Notations

For each vertex $x \in \mathbb{T}\backslash\{\varnothing\}$, we denote its parent by $\overleftarrow{x}$, and its children by $(x^{(1)}, \cdots, x^{(\mathtt{b})})$. Write $|x|$ for the generation of x. Instead of looking at $\omega(x, y)$ (for $y \sim x$ and $x \in \mathbb{T}$), it is often more convenient to use the notation

$$A(x) := \frac{\omega(\overleftarrow{x}, x)}{\omega(\overleftarrow{x}, \overleftarrow{\overleftarrow{x}})}, \qquad |x| \geq 2.$$

Recurrence/transience criteria

Lyons and Pemantle (1992)'s theorem :

Assume that all $A(x)$ have the same law as some A and A has good integrability. Let $p := \inf_{0 \le t \le 1} \mathbb{E}(A^t)$.

1. If $p\,\mathtt{b} > 1$, then RWRE (X_n) is a.s. transient.
2. If $p\,\mathtt{b} \le 1$, then RWRE (X_n) is a.s. recurrent ; moreover X is positive recurrent if $p\mathtt{b} < 1$.

Remark : Lyons and Pemantle (1992)'s theorem holds for a very general tree (by replacing $\mathtt{b}$ by the branching number of the tree $\mathbb{T}$).

A slightly more general setting

Hypothesis :

We assume that for all $|x| \geq 2$, $\{A(x^{(1)}), \dots, A(x^{(\mathtt{b})})\}$ has the same law as the vector $\{A_1, \dots, A_{\mathtt{b}}\}$. Define and assume that

$$\phi(t) := \log \mathbb{E}\Big(\sum_{i=1}^{\mathtt{b}} A_i^t \Big), \quad \text{is finite on } (-\delta, 1+\delta),$$

for some $\delta > 0$ (If $A_i \stackrel{\text{law}}{=} A$, then $\phi(t) = \log(b\mathbb{E}(A^t))$).

Lyons and Pemantle (1992)'s theorem says :

1. if $\inf_{0\leq t\leq 1}\phi(t) > 0$, then RWRE (X_n) is a.s. transient.
2. If $\inf_{0\leq t\leq 1}\phi(t) = 0$ (critical case), then RWRE (X_n) is a.s. recurrent.
3. If $\inf_{0\leq t\leq 1}\phi(t) < 0$, then (X_n) is a.s. positive recurrent.

Critical case : $\inf_{0\le t\le 1} \phi(t) = 0$

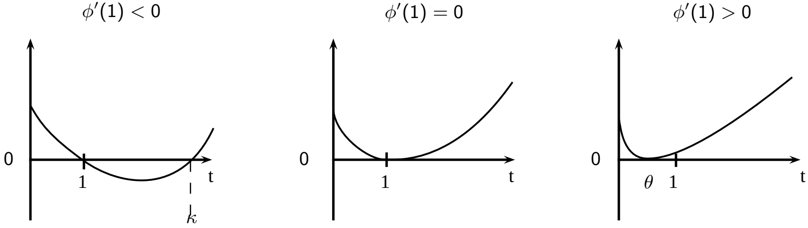


FIGURE: Three different shapes of ϕ in the critical case [and $\phi'(\theta) = 0$].

Critical case : $\inf_{0\le t\le 1}\phi(t)=0$. Subdiffusive case

Theorem 1 (Hu and Shi 2007)

[GW without leaves]. If $\inf_{0\le t\le 1}\phi(t)=0$ and $\phi'(1)<0$, then almost surely,

$$\max_{0\le i\le n}|X_i| = n^{\nu+o(1)},$$

where

$$\nu := 1-\max\left(\frac{1}{2},\frac{1}{\kappa}\right),$$

and

$$\kappa := \inf\{t>1 : \phi(t)=0\} \in (1,\infty].$$

References

- Ben Arous and Hammond (2011, arXiv), Hammond (2011, arXiv), Gantert, Muller, Popov and Vachkovskaia (2011, arXiv), Ben Arous and Cerny (2008), Comets and Simenhaus (2008).
- E. Aidékon (2008a, 2008b) for rate of convergence and large deviations (transient case).
- Case $\kappa > 2$: is there an invariance principle to (reflected) Brownian motion ? Faraud (2008+) confirms it for $\kappa > 5$.
- If the ω are non random and $\mathbb{T}$ is a Galton-Watson tree, the model corresponds to the so-called biased random walk on (Galton-Watson) trees (see Peres and Zeitouni (2008) for a CLT in the recurrent case and Dembo and Sun (2010+) for the multi-type GW).

Critical case : $\inf_{0\le t\le 1}\phi(t)=0$. Slow movement :

Theorem 2 (Faraud, Hu and Shi (2010+))

If $\inf_{0\le t\le 1}\phi(t)=0$ and $\phi'(1)\ge 0$, then almost surely,

$$\lim_{n\to\infty}\frac{1}{\log^3 n}\max_{0\le i\le n}|X_i|=c,$$

where

$$c:=\begin{cases}\frac{8}{3\pi^2\phi''(1)}, & \text{if } \phi'(1)=0\,;\\ \frac{2\theta}{3\pi^2\phi''(\theta)}, & \text{if } \phi'(1)>0,\end{cases},$$

where $\theta\in(0,1]$ denotes the unique zero :$\phi'(\theta)=0$.

Remark : Discontinuity of c when $\theta\to 1$!

Discontinuity in the limits

Let $\phi'(1) = 0$. Let $\beta > 1$ and consider a new random environment $\omega^{(\beta)}$ which corresponds to $(A_i^\beta, 1 \le i \le \mathtt{b})$. Let $(X_n^{(\beta)})$ be the RWRE in the environment $\omega^{(\beta)}$. Our result says : almost surely,

$$\lim_{n\to\infty} \frac{1}{\log^3 n} \max_{0\le i\le n} |X_i^{(\beta)}| = \begin{cases} \frac{8}{3\pi^2\phi''(1)}, & \text{if } \beta = 1\,; \\ \\ \frac{2}{3\pi^2\beta\phi''(1/\beta)}, & \text{if } \beta > 1. \end{cases}$$

We see the discontinuity of the limit at $\beta = 1$.

An associated branching random walk

The potential process associated with the random environment is defined by $V(\varnothing) := 0$ and

$$V(x) := - \sum_{y \in \rrbracket \varnothing, x \rrbracket} \log A(y), \qquad x \in \mathbb{T} \backslash \{\varnothing\},$$

so that $(V(x),\, x \in \mathbb{T})$ is a branching random walk.

A relationship between (X_n) and V

In the recurrent case, for any $k \geq 0$, let

$$\tau_k := \inf\{j \geq 1 : |X_j| = k\}, \qquad \inf \varnothing := \infty.$$

So τ_0 is the first *return* time to the root if the walk starts from $\varnothing$. Let $\varrho_n := P_\omega\{\tau_n < \tau_0\}$. Then almost surely, if for some positive constant c,

$$\varrho_n = e^{-(c+o(1))n^{1/3}},$$

then

$$\lim_{n\to\infty} \frac{1}{(\log n)^3} \max_{0\leq k\leq n} |X_k| = c^{-3}.$$

A lower bound of ϱ_n in terms of V

There exists some $0 < c(\omega) < \infty$ such that for any $n \geq 1$,

$$\varrho_n := P_\omega\{\tau_n < \tau_0\} \geq \max_{|x|=n} P_\omega\{T_x < \tau_0\} \geq \frac{c(\omega)}{n} \exp\Big(-\min_{|x|=n} \overline{V}(x)\Big),$$

where, for any vertex x, $T_x := \inf\{j \geq 0 : X_j = x\}$ and

$$\overline{V}(x) := \max_{y \in [\![\varnothing, x]\!]} V(y).$$

Rate of $\overline{V}$

Theorem 3 (Independently obtained by Fang and Zeitouni (2010))

Assume $\inf_{t\in[0,1]} \phi(t) = 0$ and let $\theta \in (0, 1]$ be such that $\phi'(\theta) = 0$. We have

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \min_{|x|=n} \overline{V}(x) = \Big(\frac{3\pi^2\sigma_\theta^2}{2}\Big)^{1/3}, \qquad \mathbb{P}\text{-a.s.},$$

where

$$\sigma_\theta^2 := \frac{1}{\theta}\mathbb{E}\Big\{ \sum_{|x|=1} V(x)^2 \mathrm{e}^{-\theta V(x)} \Big\}.$$

Behaviors of ϱ_n

Recall that

$$\varrho_n := P_\omega\{\tau_n < \tau_0\} \geq \frac{c(\omega)}{n} \exp\Big(- \min_{|x|=n} \overline{V}(x)\Big).$$

There are 2 cases :

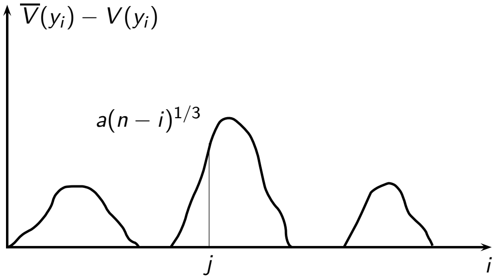
1. if $\phi'(1) > 0$, then the above lower bound for ϱ_n is sharp.
2. if $\phi'(1) = 0$, $\varrho_n \gg \exp\Big(- \min_{|x|=n} \overline{V}(x)\Big)$.

Upper bound of ϱ_n : case $\phi'(1) = 0$

For each $|x| = n$, by considering the first $j \in [1, n]$ such that $\overline{V}(x_j) - V(x_j) \geq a(n-j)^{1/3}$, we get that

$$\tau_n = \inf_{|x|=n} T_x \geq \min_{1 \leq j \leq n} \inf\{T_y : |y| = j \text{ and } E(y) \text{ holds}\},$$

with $E(y)$ given as follows :



Upper bound of ϱ_n : case $\phi'(1) = 0$

Hence

$$\varrho_n = P_\omega\{\tau_n > \tau_0\} \le \sum_{j=1}^{n} \sum_{|y|=j} \mathbf{1}_{E(y)} e^{V(y_1) - \overline{V}(y)}.$$

An important formula (many-to-one, change of measure...) see Biggins and Kyprianou (1997) :

For any $n \ge 1$ and any measurable function $F : \mathbb{R}^n \to [0, \infty)$, we have

$$\mathbb{E}\Big\{ \sum_{|x|=n} e^{-V(x)} F(V(x_i), \, 1 \le i \le n) \Big\} = \mathbb{E}\Big\{ F(S_i, \, 1 \le i \le n) \Big\},$$

where (S_n) is a centered random walk.

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Upper bound of ϱ_n : case $\phi'(1) = 0$

Then,

$$\begin{aligned}
\mathbb{E}(\varrho_n) &\le \sum_{j=1}^{n} \mathbb{E}\Big\{ e^{S_j} \mathbf{1}_{\{\overline{S}_j - S_j > a(n-j)^{1/3},\, \overline{S}_i - S_i \le a(n-i)^{1/3},\, \forall i<j\}} e^{-\overline{S}_j} \Big\} \\
&\le \sum_{j=1}^{n} \mathrm{e}^{-a(n-j)^{1/3}} \, \mathbb{P}\Big\{ \overline{S}_i - S_i \le a(n-i)^{1/3},\ \forall i < j \Big\} \\
&= e^{-\min(a, \frac{3\pi^2\sigma^2}{8a^2})(1+o(1)) n^{1/3}},
\end{aligned}$$

by an application of Mogul'skii (1974).

Upper bound on ϱ_n : case $\phi'(1) = 0$

Then

$$\mathbb{E}(\varrho_n) \leq e^{-\min(a, \frac{3\pi^2\sigma^2}{8a^2})(1+o(1))n^{1/3}}.$$

Taking $a = \frac{3\pi^2\sigma^2}{8a^2}$, namely $a = (\frac{3\pi^2\sigma^2}{8})^{1/3} := a_*$, gives the upper bound for ϱ_n :

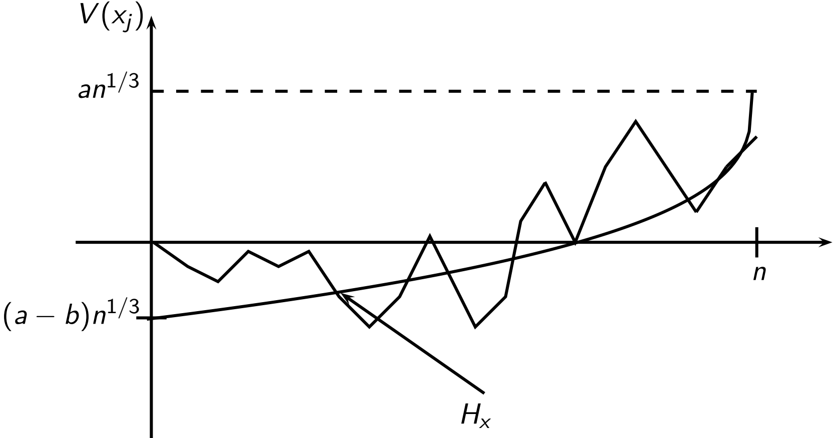
$$\varrho_n \leq e^{-(a_*+o(1))n^{1/3}}.$$

The lower bound of ϱ_n : difficult part...

Proof of Theorem 3 (lower bound)

Let $a^* := (\frac{3\pi^2\sigma^2}{2})^{1/3}$ and $0 < a < a^* < b < (\frac{3\pi^2\sigma^2}{2a^2})^{1/3}$. We are going to bound $\mathbb{P}(\min_{|x|=n} \overline{V}(x) \leq an^{1/3})$. For all $|x| = n$, let

$$H_x := \inf\{j \in [1, n] : V(x_j) \leq an^{1/3} - b(n-j)^{1/3}\}.$$



Proof of Theorem 3 (lower bound)

By considering $H_x = j$, we get

$$
\begin{aligned}
& \mathbb{P}\Big(\min_{|x|=n} \overline{V}(x) \le a n^{1/3} \Big) \\
\le \ & \sum_{j=1}^{n} \mathbb{E}\Big(\sum_{|y|=j} \mathbf{1}_{\{V(y) \le a n^{1/3} - b(n-j)^{1/3},\ a n^{1/3} \ge V(y_i) > a n^{1/3} - b(n-i)^{1/3}, \forall i \le j\}} \Big) \\
= \ & \sum_{j=1}^{n} \mathbb{E} e^{S_j} \mathbf{1}_{\{S_j \le a n^{1/3} - b(n-j)^{1/3}, a n^{1/3} \ge S_i > a n^{1/3} - b(n-i)^{1/3}, \forall i \le j\}},
\end{aligned}
$$

by the change of probability formula.

Proof of Theorem 3 (lower bound)

It follows that

$$\begin{aligned}
&\mathbb{P}\Big(\min_{|x|=n} \overline{V}(x) \le an^{1/3}\Big) \\
\le\ & \sum_{j=1}^{n} e^{an^{1/3}-b(n-j)^{1/3}}\mathbb{P}\Big(an^{1/3} \ge S_i > an^{1/3} - b(n-i)^{1/3}, \forall i \le j\Big) \\
=\ & e^{(a-\min(b,\frac{3\pi^2\sigma^2}{2b^2})+o(1))n^{1/3}}.
\end{aligned}$$

Hence by letting $b \to a^*$ and $\epsilon \to 0$, we obtain that for any $a < a^*$,

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbb{P}\Big(\min_{|x|=n} \overline{V}(x) \le an^{1/3}\Big) \le a - a^*,$$

implying the lower bound.

Proof of Theorem 1

Recall that $\tau_n := \inf\{i \geq 0 : |X_i| = n\}$ be the first hitting time at nth generation of the tree by the walk. We are mostly interested in

$$\varrho_n(x) := \mathbb{P}_{x,\omega}\Big(\tau_n < T_{\overleftarrow{x}}\Big), \qquad |x| \leq n,$$

where $T_{\overleftarrow{x}}$ means the first hitting time on $\overleftarrow{x}$. In particular, for $x = \varnothing$ the root, $\varrho_n := \varrho(\varnothing)$.

Main technical estimate

Assume $\phi'(1) < 0$

1. If $\kappa \in (2, \infty]$, then

$$\varrho_n \approx \mathbb{E}(\varrho_n) \approx \frac{1}{n}.$$

2. If $\kappa \in (1, 2]$, then

$$\varrho_n \approx \mathbb{E}(\varrho_n) \approx n^{-1/(\kappa-1)}.$$

Recurrence equation

Recurrence equation

For $|x| = n$, $\varrho_n(x) = 1$ and

$$\varrho_n(x) = \frac{\sum_{i=1}^{b} A(x^{(i)})\varrho_n(x^{(i)})}{1 + \sum_{i=1}^{b} A(x^{(i)})\varrho_n(x^{(i)})}, \qquad |x| < n.$$

Rough upper bound of ϱ_n

Since $\varrho_n(x) \leq \sum_{i=1}^{b} A(x^{(i)})\varrho_n(x^{(i)})$, by iterating these inequalities we get $\varrho_n(\varnothing) \leq M_n$, where

$$M_n := \sum_{|x|=n} \prod_{y \in (\varnothing, x]} A(y).$$

It is easy to check that (M_n) is a positive martingale, called Mandelbrot's multiplicative cascade.

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Where does come from κ ?

It is known (Liu 2001) that $M_n \to M_\infty \in (0, \infty)$, if $\kappa < \infty$,

$$\mathbf{P}\Big(M_\infty > x\Big) \approx x^{-\kappa}.$$

We have

$$\frac{\varrho_n}{\mathbb{E}(\varrho_n)} \xrightarrow{(d)} M_\infty.$$

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An elementary inequality

Let $\xi \geq 0$ be a random variable. Assume that $\mathbb{E}(\xi^a) < \infty$ for some $a > 1$.

$$\mathbb{E}[(\frac{\xi}{1+\xi})^a] \leq \mathbb{E}(\xi^a),$$

and

$$[\mathbb{E}(\frac{\xi}{1+\xi})]^a \leq [\mathbb{E}\xi]^a.$$

Then (!)

$$\frac{\mathbb{E}[(\frac{\xi}{1+\xi})^a]}{[\mathbb{E}(\frac{\xi}{1+\xi})]^a} \leq \frac{\mathbb{E}(\xi^a)}{[\mathbb{E}\xi]^a}.$$

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THANK YOU !