Large Deviations for partition functions.

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Layout

1. Inroduction and Results.
   - Quadrant directed polymers
   - The Results
   - Boundary *log*-gamma model
   - Burke Property

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   - Quadrant directed polymers
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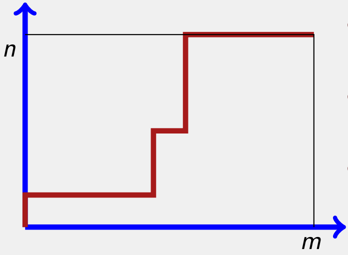
2. The proofs.
   - Decomposition and Burke
   - Estimation
   - Inversion
   - Proofs for unconstrained endpoint model

Directed polymer in space-time random environment



- Nearest neighbor, up-right path $(x(u))$, $u \in \mathbb{N}^2$.
- "Space-time" environment $\{\omega(u) : u \in \mathbb{N}^2\}$.
- $\Pi(m,n) =$ Up-right paths $(x_{m,n})$ from $(1,1)$ to (m,n).
- Quenched probability measure on the paths

$$Q_{m,n}\{x_{m,n}(\cdot)\} = \frac{1}{Z_{m,n}} \exp\Big\{\beta \sum_{u \in x_{m,n}(\cdot)} \omega(u)\Big\}$$

Inverse temperature $\beta > 0$ (set to be 1 for the majority of the talk) .
The normalizing constant $Z_{m,n}$ is the *partition function*, given by

$$Z_{m,n} = \sum_{x \in \Pi(m,n)} \exp\Big\{\beta \sum_{u \in x(\cdot)} \omega(u)\Big\}$$

$\mathbb{P}$ is the probability distribution on the environment ω, $\{\omega(u)\}$ i.i.d.

Questions and previous results

Question: Large deviations or Concentration Inequalities for the partition function.

- Concentration Inequalities:
  1. Carmona - Hu: Order n concentration inequality (Gaussian environment)
  2. Comets - Shiga - Yoshida: Order $n^{1/3}$ concentration inequality.
  3. Liu - Watbled: Order n concentration inequality.

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- Large deviations.
  1. Carmona - Hu: Upper and lower tails normalizations (Gaussian environment)
  2. Ben-Ari: Lower tail large deviation regimes.
  3. Cranston - Gauthier - Mountford: Upper tail normalizations, lower tail regimes (Parabolic Anderson model).

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Explicit rate functions for any model?

Existence of Rate functions

Assumptions:

- d-dimensional rectangle (only steps parallel to the positive axes are allowed).
- General i.i.d. weights, so that a $\xi > 0$ that depends on the distributions of the weights ω exists, such that

$$\mathbb{E}\left(e^{\xi|\omega(u)|}\right) < \infty.$$

- $\beta < \xi$.

Theorem

For $t > 0$, $u \in \mathbb{R}_+^d$ and $r \in \mathbb{R}$ there exists a nonnegative function that satisfies

$$J_u^\beta(r) = -\lim_{n\to\infty} n^{-1} \log \mathbb{P}\{\log Z_{\lfloor nu \rfloor}^\beta \geq nr\}.$$

J is convex in the variable (u, r). The rate function is continuous in (u, r) where it is finite.

Log-Gamma Model

- Dimension $1+1$
- I.i.d. weights, with distributions

$$\omega(i,j) \sim \log Y_{i,j}, \quad \text{where } Y_{i,j}^{-1} \sim \mathit{Gamma}(\mu)$$

  Gamma density: $\Gamma(\mu)^{-1} x^{\mu-1} e^{-x}$
- The partition function satisfies a law of large numbers:

$$\lim_{n\to\infty} n^{-1} \log Z_{\lfloor ns \rfloor, \lfloor nt \rfloor} = f_{s,t}(\mu).$$

- $J_{s,t}(r) = -\lim_{n\to\infty} n^{-1} \log \mathbb{P}\{\log Z_{\lfloor ns \rfloor, \lfloor nt \rfloor} \geq nr\}.$

Results - The log-gamma rate function (fixed endpoint)

Definitions:

For $0 < \xi < (\mu + \xi)/2 \leq \theta < \mu$, define

- $h_\xi(\theta) = \log \Gamma(\mu - \theta) - \log \Gamma(\mu - \theta + \xi)$.
- $d_\xi(\theta) = \log \Gamma(\theta - \xi) - \log \Gamma(\theta)$.

Theorem

Let $r \in \mathbb{R}$, $0 \leq s \leq t$. Then

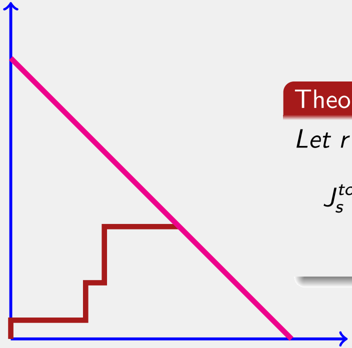
$$J_{s,t}(r) = \sup_{\xi \in [0,\mu)} \sup_{h_\xi((\mu+\xi)/2) \leq v} \Big\{(r, -s) \cdot (\xi, v) - t(d_\xi \circ h_\xi^{-1})(v)\Big\},$$
$$= \sup_{\xi \in [0,\mu)} \sup_{\theta \in [(\mu+\xi)/2, \mu)} \Big\{r\xi - sh_\xi(\theta) - td_\xi(\theta)\Big\},$$

The function $J_{s,t}(r)$ is strictly positive for $r > r_0 = f_{s,t}(\mu)$.

Results - The log-gamma rate function (free endpoint)

Let $s > 0$. The free-endpoint directed polymer model has partition function

$$Z^{tot}_{\lfloor ns \rfloor} = \sum_{x:\lfloor ns \rfloor\text{-paths } x} \exp\Big\{ \sum_{u \in x(\cdot)} \omega(u) \Big\}$$



Theorem

Let $r \in \mathbb{R}$, $s > 0$. Then

$$J^{tot}_s(r) = - \lim_{n \to \infty} n^{-1} \log \mathbb{P}\{\log Z^{tot}_{\lfloor ns \rfloor} \geq nr\}$$
$$= J_{s/2,s/2}(r).$$

Results - Asymptotic Behavior close to the zero

Theorem

Let $s = t = 1$, $r_0 = -2\Psi_0(\mu/2)$ and $r = r_0 + \varepsilon$ for some $\varepsilon > 0$. Then the maximizing ξ (from the previous theorem) satisfies $\xi = \mathcal{O}(\sqrt{\varepsilon})$ and there exist positive constants C_1, C_2

$$C_1(r - r_0)^{3/2} + \mathcal{O}\big((r - r_0)^{5/2}\big) < J_{1,1}(r) < C_2(r - r_0)^{3/2} + \mathcal{O}\big((r - r_0)^{5/2}\big).$$

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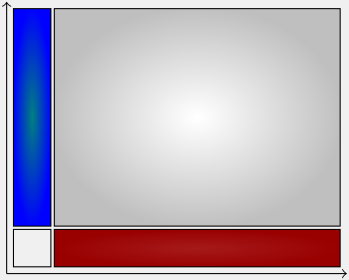
Remark: This is consistent with the order $n^{2/3}$ fluctuations for the partition function. Bounding the extreme tail we get

$$\mathbb{E}(Z_{n,n} - nr_0)_+^2 \leq Cn^{2/3}.$$

Computational tool: Log-gamma polymer with boundary

Define multiplicative weights $Y_{i,j} = e^{\omega(i,j)}$ independent.



Environment (distribution $\mathbb{P}$)

Horizontal Weights U

$Y_{i,0}^{-1} = U_{i,0}^{-1} \sim Gamma(\theta)$

Vertical Weights V

$Y_{0,j}^{-1} = V_{0,j}^{-1} \sim Gamma(\mu - \theta)$

Bulk Weights Y

$Y_{i,j}^{-1} \sim Gamma(\mu)$

- Gamma(μ) density: $\Gamma(\mu)^{-1} x^{\mu-1} e^{-x}$.

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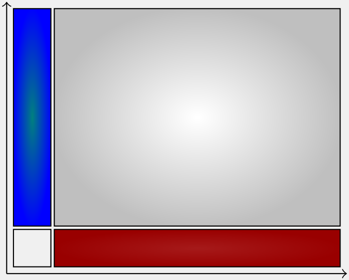

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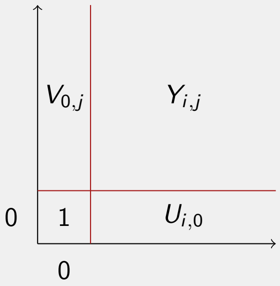
This model allows specific calculations. Reason: Burke property.

Burke Property for the boundary model.



Given initial weights $(i, j \in \mathbb{N})$

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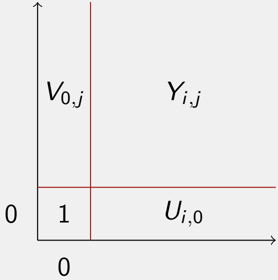

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Compute $Z_{m,n}$ for all $(m,n) \in \mathbb{Z}_+^2$ and then define

$$U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}} \qquad V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}} \qquad X_{m,n} = \left(\frac{Z_{m,n}}{Z_{m+1,n}} + \frac{Z_{m,n}}{Z_{m,n+1}}\right)^{-1}$$

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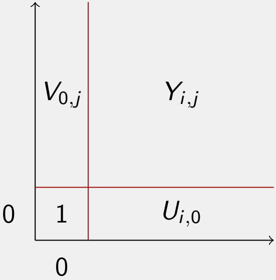

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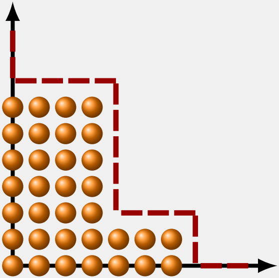
For an undirected edge f: $\quad T_f = \begin{cases} U_x & f = \{x - e_1, x\} \\ V_x & f = \{x - e_2, x\} \end{cases}$

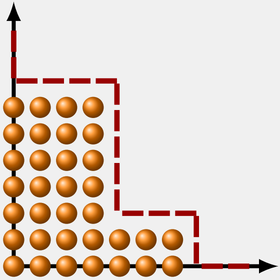


down-right path (z_k) with edges

$$f_k = \{z_{k-1}, z_k\}, k \in \mathbb{Z}$$

interior points $\mathcal{I}$ of path (z_k)

Burke Property



down-right path (z_k) with edges

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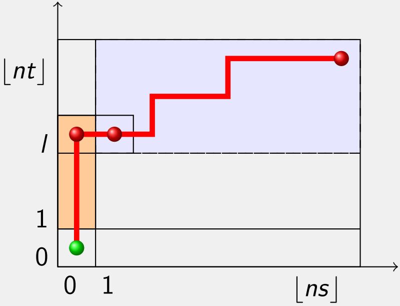
Theorem

Variables $\{\, T_{f_k}, X_z : k \in \mathbb{Z},\ z \in \mathcal{I} \,\}$ *are independent with marginals* $U^{-1} \sim Gamma(\theta)$, $V^{-1} \sim Gamma(\mu - \theta)$, *and* $X^{-1} \sim Gamma(\mu)$.

Idea of Proof - Decomposition

We start by decomposing $Z_{\lfloor ns \rfloor, \lfloor nt \rfloor}$ according to the exit point of the polymer path from the boundary:



The shaded part represents the partition function conditioned on $(0, l)$ being the exit point of the polymer path from the boundary.

$$\left(\prod_{j=1}^{l} V_{0,j} \right) Z^{\square}_{(1,l)}(\lfloor ns \rfloor, \lfloor nt \rfloor)$$

Proof - Decomposition

$$Z_{\lfloor ns \rfloor, \lfloor nt \rfloor} = \sum_{x_\cdot} \prod_{j=1}^{\lfloor ns \rfloor + \lfloor nt \rfloor} Y_{x_j}$$

$$= \sum_{l=1}^{\lfloor nt \rfloor} \left\{ \left(\prod_{j=1}^{l} V_{0,j} \right) Z^{\square}_{(1,l)}(\lfloor ns \rfloor, \lfloor nt \rfloor) \right\} + \sum_{k=1}^{\lfloor ns \rfloor} \left\{ \left(\prod_{i=1}^{k} U_{i,0} \right) Z^{\square}_{(k,1)}(\lfloor ns \rfloor, \lfloor nt \rfloor) \right\}.$$

Consequence of the Burke Property:

$$Z_{\lfloor ns \rfloor, \lfloor nt \rfloor} = \prod_{j=1}^{\lfloor nt \rfloor} V_{0,j} \prod_{i=1}^{\lfloor ns \rfloor} U_{i, \lfloor nt \rfloor}$$

Idea of Proof - Burke Trick

Divide both sides by $\prod_{j=1}^{\lfloor nt \rfloor} V_{0,j}$:

$$\prod_{i=1}^{\lfloor ns \rfloor} U_{i,\lfloor nt \rfloor} = \sum_{l=1}^{\lfloor nt \rfloor} \left\{ \left(\prod_{j=l+1}^{\lfloor nt \rfloor} V_{0,j}^{-1} \right) Z^{\square}_{(1,l)}(\lfloor ns \rfloor, \lfloor nt \rfloor) \right\}$$

$$+ \sum_{k=1}^{\lfloor ns \rfloor} \left\{ \left(\prod_{j=1}^{\lfloor nt \rfloor} V_{0,j}^{-1} \prod_{i=1}^{k} U_{i,0} \right) Z^{\square}_{(k,1)}(\lfloor ns \rfloor, \lfloor nt \rfloor) \right\}.$$

$$= \sum_{\substack{k=-\lfloor nt \rfloor \\ k \neq 0}}^{\lfloor ns \rfloor} \eta_k Z^{\square}_{\mathbf{k}}(\lfloor ns \rfloor, \lfloor nt \rfloor).$$

Here we defined

$$\eta_k = \begin{cases} \prod_{j=-k}^{\lfloor nt \rfloor} V_{0,j}^{-1}, & \text{for } -\lfloor nt \rfloor \leq k \leq -1, \\ \eta_{-1}, & k = 0 \\ \eta_0 \prod_{i=1}^{k} U_{i,0}, & \text{for } 0 < k \leq \lfloor ns \rfloor, \end{cases}$$

Idea of Proof - Estimation

$$
\begin{aligned}
R_s(r) &= -\lim_{n\to\infty} n^{-1} \log \mathbb{P}\left\{ \log \prod_{i=1}^{\lfloor ns \rfloor} U_{i,\lfloor nt \rfloor} \geq nr \right\} \\
&\sim -n^{-1} \log \mathbb{P}\left\{ \log \sum_{\substack{k=-\lfloor nt \rfloor \\ k\neq 0}}^{\lfloor ns \rfloor} \eta_k Z^{\square}_{\mathbf{k}}(\lfloor ns \rfloor, \lfloor nt \rfloor) \geq nr \right\} \\
&\sim -n^{-1} \log \mathbb{P}\{\max_k \log \eta_k Z^{\square}_{\mathbf{k}}(\lfloor ns \rfloor, \lfloor nt \rfloor) \geq nr\} \\
&\sim -n^{-1} \max_k \log \mathbb{P}\{\log \eta_k + \log Z^{\square}_{\mathbf{k}}(\lfloor ns \rfloor, \lfloor nt \rfloor) \geq nr\} \\
&\sim \inf_{-t\leq a \leq s} \inf_{x\in\mathbb{R}} \{\kappa_a(x) + J^a_{s,t}(r-x)\}
\end{aligned}
$$

Idea of Proof - Inversion

$$R_s^*(\xi) = \sup_{-t\le a\le s}\left\{\kappa_a^*(\xi) + (J_{s,t}^a)^*(\xi)\right\}$$

$$F^s(\theta,\xi) = \sup_{-t\le a\le s}\left\{au(\theta) - (-(J_{s,t}^a)^*(\xi))\right\}$$

$$F^t(u^{-1}(v),\xi) \stackrel{\text{magic}}{=} \sup_{0\le a\le t}\left\{av - (-(J_{t,t}^a)^*(\xi))\right\}$$

$$F^t(u^{-1}(v),\xi) = \sup_{0\le a\le t}\left\{av - G_\xi(a)\right\}$$

$$F^t(u^{-1}(v),\xi) = G_\xi^*(v)$$

Then,

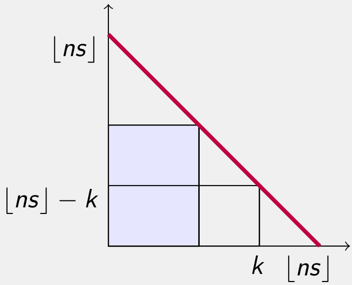
$$\begin{aligned} J_{t,t}^a(r) &= \sup_{\xi\in[0,\mu)}\{r\xi - (J_{t,t}^a)^*(\xi)\} \\ &= \sup_{\xi\in[0,\mu)}\{r\xi + G_\xi(a)\} \\ &= \sup_{\xi\in[0,\mu)}\sup_{v\in\mathbb{R}}\{r\xi + av - G_\xi^*(v)\}. \quad \square \end{aligned}$$

Proof for unconstrained endpoint model

$$\log\left(Z_{\lfloor n\frac{s}{2}\rfloor,\lfloor ns\rfloor-\lfloor n\frac{s}{2}\rfloor}\right) \leq \log Z^{tot}_{\lfloor ns\rfloor} \leq$$
$$\leq \log(ns+1) + \log\left(\max_k Z_{k,\lfloor ns\rfloor-k}\right).$$

After some estimates, this translates to the rate functions:

$$\inf_{0\leq a\leq s} J_{a,s-a}(r) \leq J^{tot}_s(r) \leq J_{s/2,s/2}(r).$$



Use convexity of the point-to-point rate functions to get that

$$J_{s/2,s/2}(r) \leq \inf_{0\leq a\leq s} J_{a,s-a}(r).$$

Thank You...

Questions & Comments ?

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