Asymptotic speed of second class particles in a rarefaction fan

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Joint work, partly with Leandro Pimentel, partly with James Martin

Introduction

- Hammersley-Aldous-Diaconis process
- Busemann functions
- Multiclass process
- Second class particles in a rarefaction fan
- Example

Hammersley-Aldous-Diaconis process

LPP in $\mathbb{R}^2$

Consider a Poisson process $\mathcal{P}$ on $\mathbb{R}^2$. For $\mathbf{x} \leq \mathbf{y} \in \mathbb{R}^2$, define $\Pi(\mathbf{x}, \mathbf{y})$ as the set of up-right paths from $\mathbf{x}$ to $\mathbf{y}$. Define

$$L(\mathbf{x}, \mathbf{y}) = \max_{\varpi \in \Pi(\mathbf{x}, \mathbf{y})} \#(\mathcal{P} \cap \varpi).$$

Call $\varpi(\mathbf{x}, \mathbf{y})$ the lowest path that attains the maximum.

Hammersley-Aldous-Diaconis process

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Particle process

Given an initial configuration ν, which is a counting process on $\mathbb{R}$, that counts negatively when going to the left, we define for $t \geq 0$, $x \in \mathbb{R}$:

$$L_\nu(x, t) = \sup_{z \leq x} \left(\nu(z) + L((z, 0), (x, t))\right).$$

$L_\nu(\cdot, t)$ is the counting process that describes the particle configuration at time t.

Busemann functions

α-rays

A continuous path $\gamma : [0, \infty) \to \mathbb{R}^2$ is called an α-ray, for $\alpha \in (\pi, 3\pi/2)$, if

1. For all $s \geq t \geq 0$, $\gamma(s) \leq \gamma(t)$ and $\gamma([t, s]) = \varpi(\gamma(s), \gamma(t))$.
2. $\|\gamma(t)\| \to \infty$ and $\gamma(t)/\|\gamma(t)\| \to (\cos\alpha, \sin\alpha)$.

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Theorem (Wüttrich)

Fix α. With probability one, there exists a unique α-ray starting at $\mathbf{x}$, for all $\mathbf{x} \in \mathbb{R}^2$; denote it by $\varpi_\alpha(\mathbf{x})$.
For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $\varpi_\alpha(\mathbf{x})$ and $\varpi_\alpha(\mathbf{y})$ will coalesce.

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The proof closely follows ideas by Newman and co-authors for First Passage Percolation.

Busemann functions

Definition

For fixed $\alpha \in (\pi, 3\pi/2)$ we define the Busemann function

$$B_\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} : B_\alpha(\mathbf{x}, \mathbf{y}) = L(\mathbf{y}, \mathbf{c}_\alpha(\mathbf{x}, \mathbf{y})) - L(\mathbf{x}, \mathbf{c}_\alpha(\mathbf{x}, \mathbf{y})),$$

where $c_\alpha(\mathbf{x}, \mathbf{y})$ is a coalescing point of $\pi_\alpha(\mathbf{x})$ and $\pi_\alpha(\mathbf{y})$.

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Properties

- $B_\alpha(\mathbf{x}, \mathbf{z}) = B_\alpha(\mathbf{x}, \mathbf{y}) + B_\alpha(\mathbf{y}, \mathbf{z})$.
- For any $\mathbf{p} \in \mathbb{R}^2$, $B_\alpha(\cdot + \mathbf{p}, \cdot + \mathbf{p}) \stackrel{\mathcal{D}}{=} B_\alpha(\cdot, \cdot)$.
- $B_\alpha(\mathbf{0}, (x, t)) = \sup_{z \le x} \left(B_\alpha(\mathbf{0}, (z, 0)) + L((z, 0), (x, t)) \right)$.

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These properties show that $x \mapsto B_\alpha(\mathbf{0}, (x, 0))$ is a stationary configuration of the Hammersley-Aldous-Diaconis model.

Multiclass process

Theorem

Suppose $\alpha_1 \leq \alpha_2$. Define B_{α_1} and B_{α_2} simultaneously. Then for every $x_1 \leq x_2$ and $t \in \mathbb{R}$

$$B_{\alpha_1}((x_1, t), (x_2, t)) \leq B_{\alpha_2}((x_1, t), (x_2, t)).$$

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Proof

Set $\mathbf{x}_1 = (x_1, t)$ and $\mathbf{x}_2 = (x_2, t)$. Define $\mathbf{c}_1$ as coalescing point of $\varpi_{\alpha_1}(\mathbf{x}_1)$ and $\varpi_{\alpha_1}(\mathbf{x}_2)$, and likewise $\mathbf{c}_2$. Define $\mathbf{m}$ as the crossing of $\varpi_{\alpha_1}(\mathbf{x}_2)$ and $\varpi_{\alpha_2}(\mathbf{x}_1)$.

$$\begin{aligned}
B_{\alpha_2}(\mathbf{x}_1, \mathbf{x}_2) - B_{\alpha_1}(\mathbf{x}_1, \mathbf{x}_2) &= L(\mathbf{c}_2, \mathbf{x}_2) - L(\mathbf{c}_2, \mathbf{x}_1) - (L(\mathbf{c}_1, \mathbf{x}_2) - L(\mathbf{c}_1, \mathbf{x}_1)) \\
&= L(\mathbf{c}_1, \mathbf{x}_1) + L(\mathbf{c}_2, \mathbf{x}_2) - (L(\mathbf{c}_1, \mathbf{m}) + L(\mathbf{m}, \mathbf{x}_2)) \\
&\quad -(L(\mathbf{c}_2, \mathbf{m}) + L(\mathbf{m}, \mathbf{x}_1)) \\
&= L(\mathbf{c}_1, \mathbf{x}_1) - (L(\mathbf{c}_1, \mathbf{m}) + L(\mathbf{m}, \mathbf{x}_1)) \\
&\quad +L(\mathbf{c}_2, \mathbf{x}_2) - (L(\mathbf{c}_2, \mathbf{m}) + L(\mathbf{m}, \mathbf{x}_2)) \\
&\ge 0.
\end{aligned}$$

Multiclass process

Uniqueness

Ferrari and Martin ('09) proved that there is a unique ergodic multiclass system that is invariant in the Hammersley process. Our construction with the Busemann functions must therefore be the same! This allows explicit calculations for several Busemann functions at the same time.

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Queueing construction

Consider a Poisson counting process S_1 of services on $\mathbb{R}$, intensity μ_1, and an independent arrival process A_1, intensity $\mu_2 < \mu_1$. Construct the corresponding stationary queue Q_1, and define the departure process $D_1(z) = \int_0^z 1_{\{Q(z)>0\}}\, dS_1(z)$.

Multiclass process

Queueing construction

Define S_1 as the first and second class particles and D_1 as the second class particles. Then (S_1, D_1) is invariant under the Hammersley evolution. Note that D_1 is a Poisson process with intensity μ_2.
Now define recursively for $i \geq 2$, $S_i = D_{i-1}$ as the i^{th} service process, define an independent arrival process A_i of intensity $\mu_{i+1} < \mu_i$, and the corresponding departure process D_i. The sequence $(S_1, S_2, \ldots, S_n)$ is stationary for the Hammersley evolution.

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Busemann functions

Take angles $\alpha_1 \geq \ldots \geq \alpha_n$. The vector valued process $z \mapsto (B_{\alpha_1}(\mathbf{0}, (z, 0)), \ldots, B_{\alpha_n}(\mathbf{0}, (z, 0)))$ is stationary for the Hammersley evolution, with intensity $\mu_i = \sqrt{\tan \alpha_i}$. Therefore,

$$(B_{\alpha_1}(\mathbf{0}, (\cdot, 0)), \ldots, B_{\alpha_n}(\mathbf{0}, (\cdot, 0))) \stackrel{\mathcal{D}}{=} (S_1, S_2, \ldots, S_n).$$

Second class particles in a rarefaction fan

Second class particle as a competition interface

Consider an initial condition ν. Add a second class particle at x_0, define $X(t)$ as its position at time t. Then $X(t)$ is the position of a competition interface: for $x > x_0$,

$$\{X(t) \le x\} = \{\sup_{z \le x_0} [\nu(z) + L((z,0),(x,t))] \le \sup_{z > x_0} [\nu(z) + L((z,0),(x,t))]\}.$$

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Asymptotic direction

We know that second class particles almost surely have an asymptotic direction, or angle; denote it by $A \in [0, \pi/2)$. It follows that

$$\mathbb{P}(A \geq \alpha) = \lim_{t \to \infty} \mathbb{P}(X(t) \leq t / \tan \alpha).$$

Second class particles in a rarefaction fan

Asymptotic direction

Define $\mathbf{x}_\alpha(t) = (t/\tan\alpha, t)$ and $\mathbf{z} = (z, 0)$. Then

$$\begin{aligned}
\mathbb{P}(A \geq \alpha) &= \mathbb{P}\Big(\sup_{z \leq x_0} [\nu(z) + L(\mathbf{z}, \mathbf{x}_\alpha(t))] \leq \sup_{z > x_0} [\nu(z) + L(\mathbf{z}, \mathbf{x}_\alpha(t))] \Big) \\
&= \mathbb{P}\Big(\sup_{z \leq x_0} [\nu(z) + (L(\mathbf{z}, \mathbf{x}_\alpha(t)) - L(\mathbf{0}, \mathbf{x}_\alpha(t)))] \leq \\
&\qquad \sup_{z > x_0} [\nu(z) + (L(\mathbf{z}, \mathbf{x}_\alpha(t)) - L(\mathbf{0}, \mathbf{x}_\alpha(t)))] \Big)
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\end{aligned}$$

We know that uniformly on compacta,

$$L(\mathbf{z}, \mathbf{x}_\alpha(t)) - L(\mathbf{0}, \mathbf{x}_\alpha(t)) \longrightarrow B_\alpha(\mathbf{0}, \mathbf{z}).$$

So if we can prove that the exit point is $O_p(1)$, then

$$\mathbb{P}(A \geq \alpha) = \mathbb{P}\left(\sup_{z \leq x_0} [\nu(z) + B_\alpha(\mathbf{0}, \mathbf{z})] \leq \sup_{z > x_0} [\nu(z) + B_\alpha(\mathbf{0}, \mathbf{z})] \right).$$

Second class particles in a rarefaction fan

The condition we need on the initial condition ν is:

$$a_\nu := \limsup_{z\to\infty} \frac{\nu(z)}{z} < \liminf_{z\to-\infty} \frac{\nu(z)}{z} =: b_\nu.$$

Each α corresponds to an intensity μ_α of the stationary process induced by B_α: $\mu_\alpha = \sqrt{\tan\alpha}$.

Theorem

For $\alpha \in (\pi, 3\pi/2)$ such that $a_\nu < \mu_\alpha < b_\nu$, we have that

$$\mathbb{P}(A \geq \alpha - \pi) = \mathbb{P}\left(\sup_{z\leq x_0}[\nu(z) - B_\alpha(\mathbf{0},\mathbf{z})] \leq \sup_{z>x_0}[\nu(z) - B_\alpha(\mathbf{0},\mathbf{z})]\right).$$

Second class particles in a rarefaction fan

Higher class particles

Suppose we have a second class particle (index 0) at x_0. Now we add a third class particle (index 1) at x_1. We have seen that almost surely,

$$\{A_0 \geq \alpha_0 - \pi\} = \left\{ \sup_{z \leq x_0} [\nu(z) - B_{\alpha_0}(\mathbf{0}, \mathbf{z})] \leq \sup_{z > x_0} [\nu(z) - B_{\alpha_0}(\mathbf{0}, \mathbf{z})] \right\}.$$

Remark that the third class particle does not see the difference between first and second class particles. So define $\nu_1 = \nu + \delta_{x_0}$, then almost surely.

$$\{A_1 \geq \alpha_1 - \pi\} = \left\{ \sup_{z \leq x_0} [\nu_1(z) - B_{\alpha_1}(\mathbf{0}, \mathbf{z})] \leq \sup_{z > x_0} [\nu_1(z) - B_{\alpha_1}(\mathbf{0}, \mathbf{z})] \right\}.$$

Second class particles in a rarefaction fan

Theorem

Let ν satisfy the rarefaction condition. Suppose we have a second class particle at x_0, a third class particle at x_1, up to a $n+2$ class particle at x_n. Define $A_0, \ldots, A_n$ as the asymptotic angles of these particles. Define the i^{th} initial condition as

$$\nu_i = \nu + \sum_{j=0}^{i-1} \delta_{x_i}.$$

For any angles $\alpha_0, \ldots, \alpha_n \in (\pi, 3\pi/2)$ such that $a_\nu < \mu_{\alpha_0}, \ldots, \mu_{\alpha_n} < b_\nu$, we have

$$\begin{aligned} \mathbb{P}(\; & A_0 \geq \alpha_0 - \pi, \ldots, A_n \geq \alpha_n - \pi) = \\ & \mathbb{P}\big(\sup_{z \leq x_0} [\nu(z) - B_{\alpha_0}(\mathbf{0}, \mathbf{z})] \leq \sup_{z > x_0} [\nu(z) - B_{\alpha_0}(\mathbf{0}, \mathbf{z})], \ldots \\ & \ldots \sup_{z \leq x_0} [\nu_n(z) - B_{\alpha_n}(\mathbf{0}, \mathbf{z})] \leq \sup_{z > x_0} [\nu_n(z) - B_{\alpha_n}(\mathbf{0}, \mathbf{z})] \big) \end{aligned}$$

Example

Two second class particle in the empty halfline

$$\nu(z) = \begin{cases} -\infty & \text{if } z \le 0, \\ 0 & \text{if } z > 0. \end{cases}$$

Put a second class particle at $x_0 > 0$ and a third class at $x_1 > x_0$. Then for any $\alpha_0, \alpha_1 \in (0, \pi/2)$ we get

$$\mathbb{P}(A_0 \ge \alpha_0, A_1 \ge \alpha_1) = \int_{\mu_0}^{\infty} \int_{\mu_1}^{\infty} f(\lambda, \rho)\, d\rho d\lambda + \int_{\max(\mu_0,\mu_1)}^{\infty} f_d(\lambda)\, d\lambda,$$

$$f(\lambda, \rho) = \begin{cases} \frac{\rho x_1 - 1}{\lambda^2}\, e^{-\rho x_1} + \frac{\lambda^2 x_0 x_1 - (\lambda x_0 + 1)(\rho x_1 - 1)}{\lambda^2}\, e^{-\lambda x_0 - \rho x_1} & \text{if } \lambda < \rho, \\ x_0 x_1 \left(e^{-\lambda x_0 - \rho x_1} - e^{-\lambda x_1 - \rho x_0} \right) & \text{if } \lambda > \rho, \end{cases}$$

and

$$f_{\mathrm{d}}(\lambda) = x_1 e^{-\lambda x_1} - (x_1 - x_0) e^{-\lambda(x_0 + x_1)} - \frac{e^{-\lambda x_1} - e^{-\lambda(x_0 + x_1)}}{\lambda}.$$

Example

Corollary

$$\mathbb{P}(A_0 > A_1) = \frac{x_1 - x_0}{x_1 + x_0}$$

$$\mathbb{P}(A_0 < A_1) = \log\left(\frac{x_0 + x_1}{x_1}\right)$$

$$\mathbb{P}(A_0 = A_1) = 1 - \frac{x_1 - x_0}{x_1 + x_0} - \log\left(\frac{x_0 + x_1}{x_1}\right)$$

Example

Calculation

Suppose $\alpha_0 > \alpha_1$, so $\mu_0 > \mu_1$.

$$
\begin{aligned}
\mathbb{P}(\ A_0 \geq \alpha_0 - \pi, A_n \geq \alpha_1 - \pi) = \\
&= \mathbb{P}\big(0 \leq -B_{\alpha_0}(\mathbf{0}, (x_0, 0)) \ \text{ and} \\
&\qquad (1 - B_{\alpha_1}(\mathbf{0}, (x_0, 0)))_+ \leq 1 - B_{\alpha_1}(\mathbf{0}, (x_1, 0))\big) \\
&= \mathbb{P}\left(B_{\alpha_0}(\mathbf{0}, (x_0, 0)) = 0, B_{\alpha_1}(\mathbf{0}, (x_1, 0)) = 0\right) \\
&= \mathbb{P}(S([0, x_0]) = 0, D([0, x_1]) = 0).
\end{aligned}
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&= \mathbb{P}(S([0,x_0]) = 0, D([0,x_1]) = 0).
\end{aligned}
$$

Departures depend on $Q(0) \sim \mathrm{Geo}(\mu_1/\mu_0)$. If $Q(0) > 0$, then no services in $[0, x_1]$. Suppose $Q(0) = 0$ and the first arrival time is t, then there are no services in $[0, x_0] \cup [t, x_1]$, where $[t, x_1] = \emptyset$ if $t > x_1$. These are elementary probabilities.

THANK YOU!!