The weak coupling limit of disordered copolymer models

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Joint work with Giambattista Giacomin (Université Paris Diderot)

Fields Institute, Toronto $\sim$ February 15, 2011

Outline

1. The basic copolymer model
2. The free energy
3. Generalized copolymer models
4. Strategy of the proof

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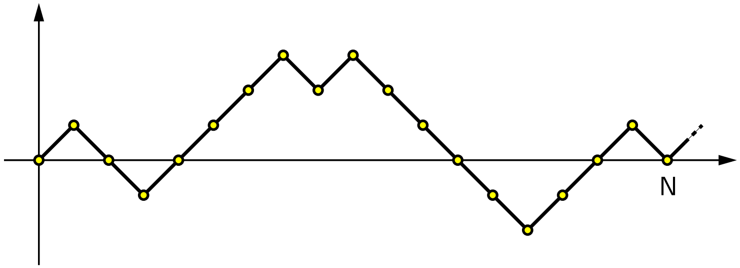
A random walk with a random potential



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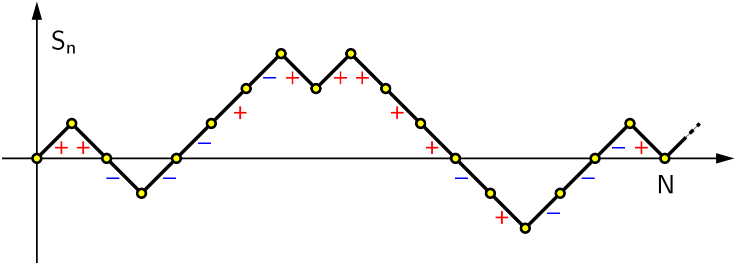

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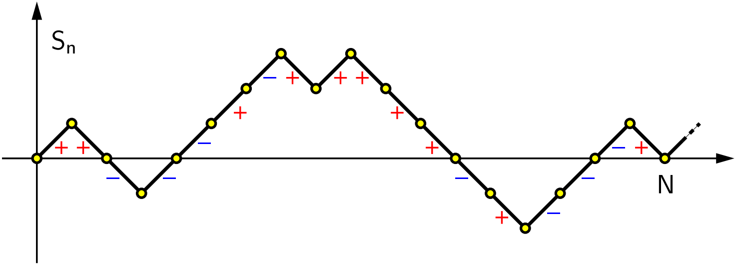

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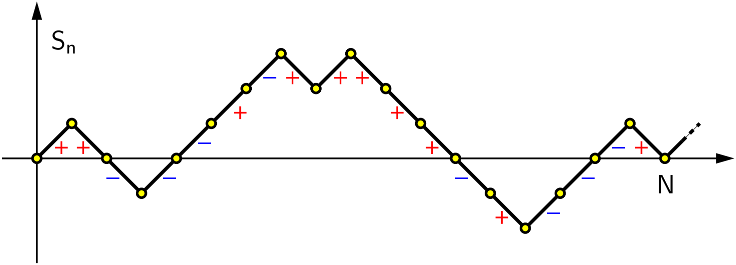

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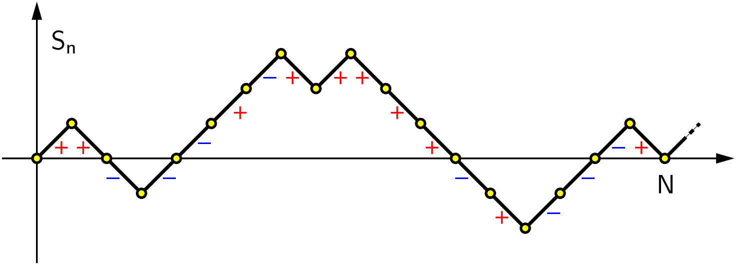

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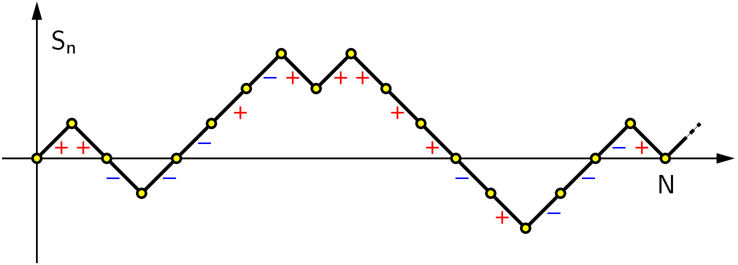

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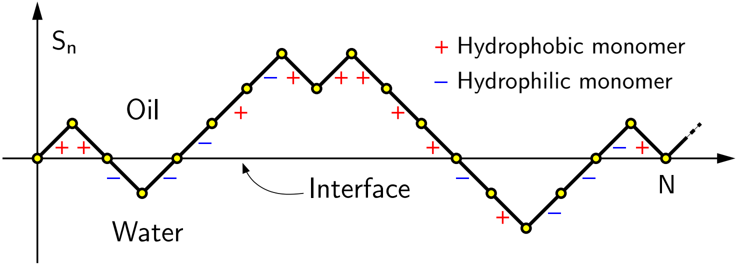
Localization or **Delocalization** ?

A polymer model interpretation



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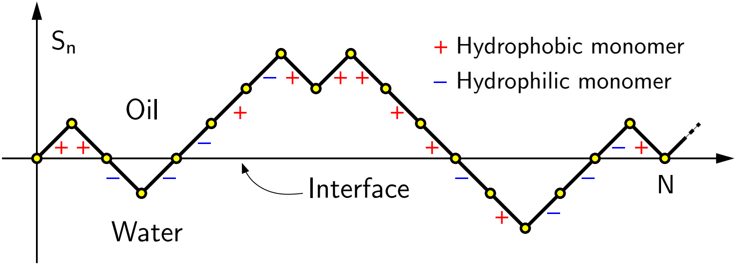

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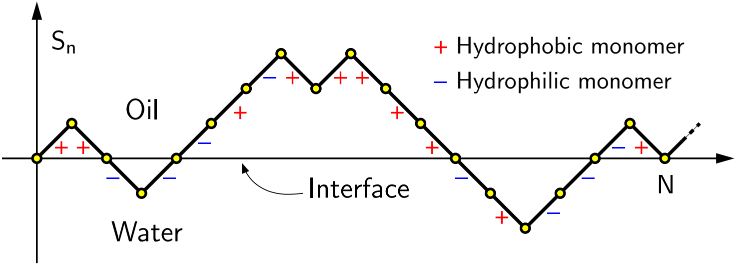

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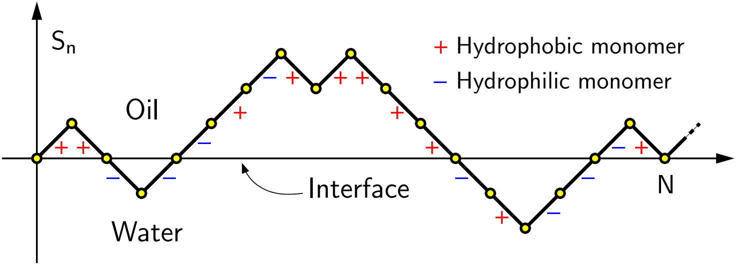

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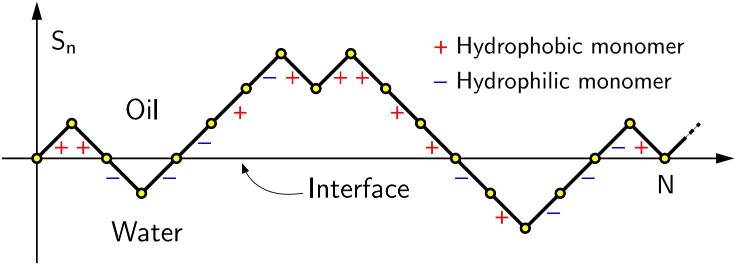

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- Several contributions in the physics literature too.

The basic copolymer model

Definition of the model: $\dfrac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) := \dfrac{1}{Z_{N,\omega}} \exp\big(- H_{N,\omega}(S)\big)$

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The law $\mathbf{P}_{N,\omega}$ remains the same. New Hamiltonian (and $Z_{N,\omega}$):

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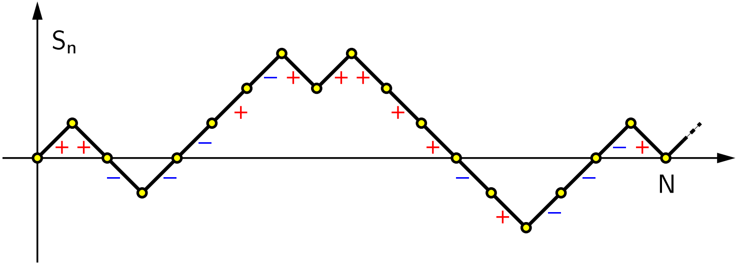

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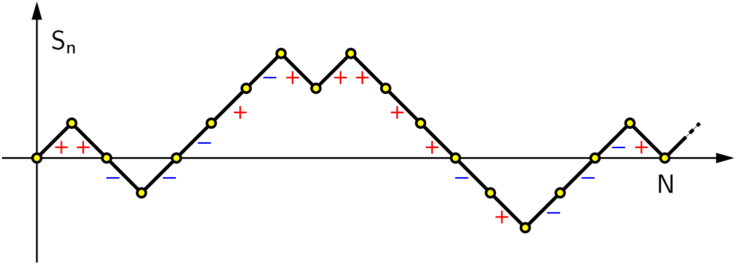

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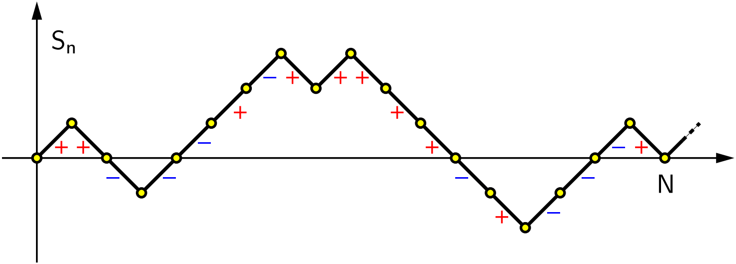

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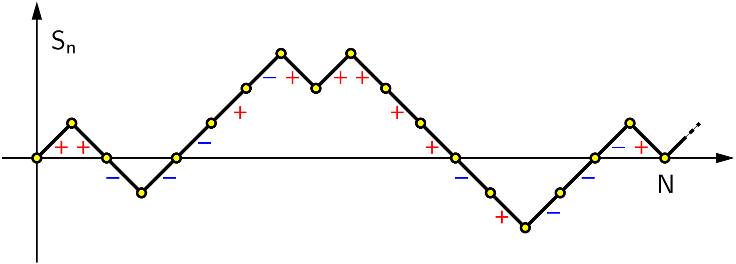

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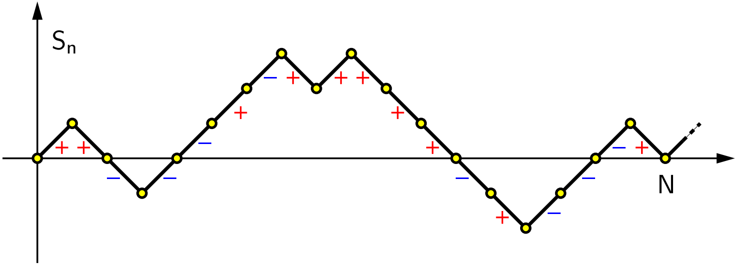

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(This definition does correspond to sharply different path behaviors)

The phase diagram: discrete model

Theorem

The regions $\mathcal{L}$ and $\mathcal{D}$ are separated by a strictly increasing, continuous critical line $h_c(\cdot)$:

$$\mathcal{L} = \{(\lambda, h) : \; h < h_c(\lambda)\} \qquad \mathcal{D} = \{(\lambda, h) : \; h \geq h_c(\lambda)\} .$$

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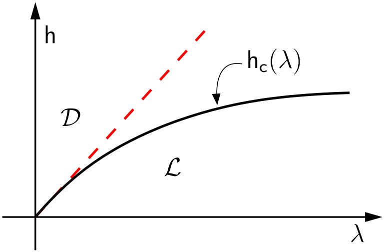

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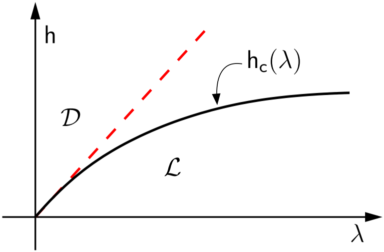

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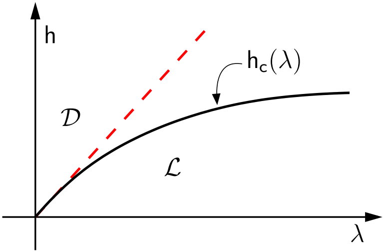

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$$\underline{h}'(0+) = \frac{2}{3} , \quad \overline{h}'(0+) = 1 - \epsilon .$$

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The continuum free energy $\tilde{\mathtt{F}}(\lambda, h)$ is defined analogously:

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Again $\tilde{\mathrm{F}}(\lambda, h) \geq 0$. We then define $\widetilde{\mathcal{L}}$ocalization and $\widetilde{\mathcal{D}}$elocalization:

$$\widetilde{\mathcal{L}} = \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) > 0\} \qquad \widetilde{\mathcal{D}} = \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) = 0\}$$

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Theorem ([BdH 97])

For all $\lambda, h \geq 0$

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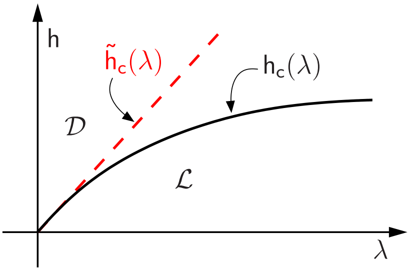

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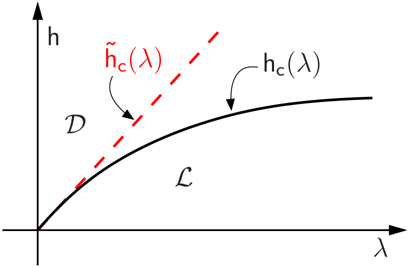

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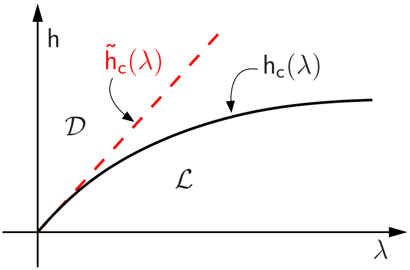

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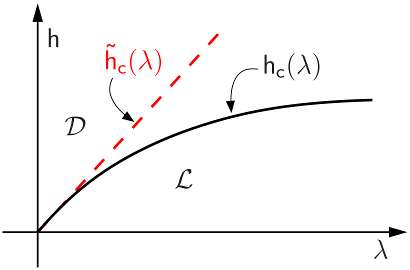

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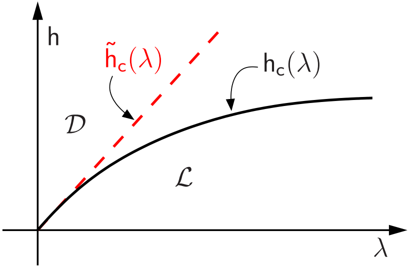

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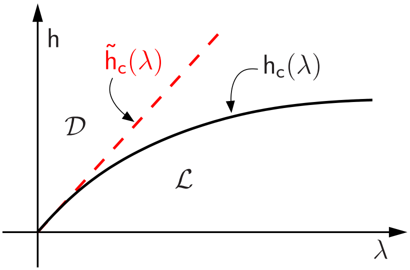

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  ...unfortunately for just one discrete model. Generalization?

Outline

1. The basic copolymer model
2. The free energy
3. Generalized copolymer models
4. Strategy of the proof

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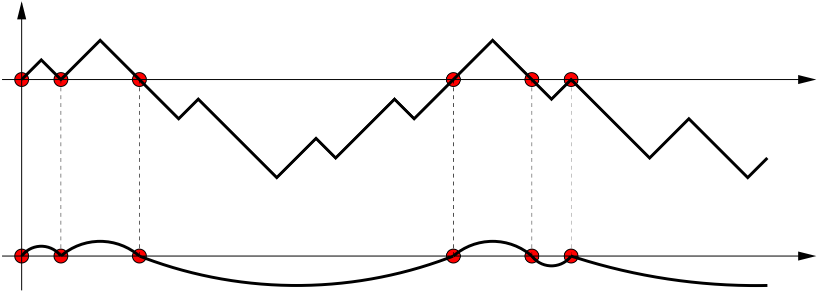

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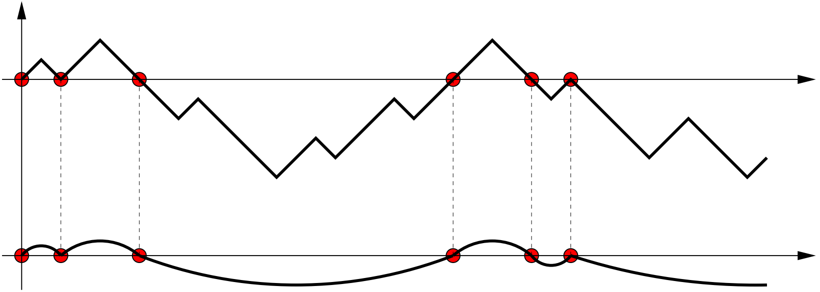

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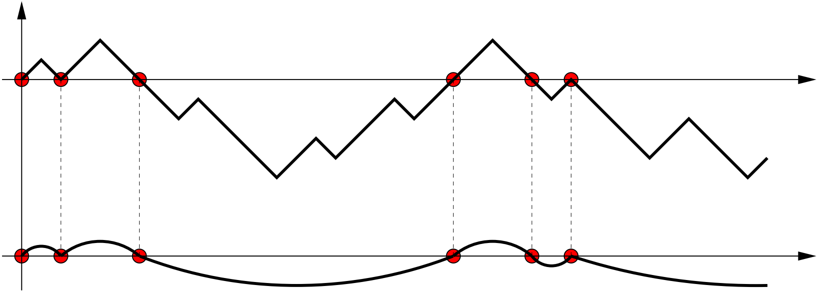

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Discrete Bessel-like process $(c_\alpha = 1/2 - \alpha)$

$$\mathbf{P}(S_{n+1} = x \pm 1 | S_n = x) = \frac{1}{2}\left(1 \pm \frac{c_\alpha}{x} + o\left(\frac{1}{x}\right)\right) \text{ yields } (\star) \text{ asymp.}$$

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Note that $\mathtt{F}(\cdot,\cdot)$ and $h_c(\cdot)$ do depend on the choice of $\mathbf{P}$ and $\mathbb{P}$

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From $\tilde{\tau}^\alpha$ we obtain $(\tilde{\Delta}^\alpha = \{\tilde{\Delta}_t^\alpha\}_{t \geq 0}, \tilde{\mathbf{P}})$ (For $\alpha = \frac{1}{2}$ we recover BM)

The continuum α-copolymer model: free energy

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Theorem [CG]

$\tilde{\mathrm{F}}^\alpha(\lambda, h)$ exists and is self-averaging

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- Continuity theory of Gaussian processes plays essential role

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Theorem

For all $\lambda, h > 0$ and $\epsilon \in (0,1)$ there exists $a_0 > 0$ s.t. for all $a < a_0$

$$\tilde{\mathrm{F}}^\alpha\left(\frac{\lambda}{1+\epsilon}, \frac{h}{1-\epsilon}\right) \leq \frac{\mathrm{F}(a\lambda, ah)}{a^2} \leq \tilde{\mathrm{F}}^\alpha\left((1+\epsilon)\lambda, (1-\epsilon)h\right)$$

Outline

1. The basic copolymer model
2. The free energy
3. Generalized copolymer models
4. Strategy of the proof

Strategy of the proof

Goal: $\forall\ \lambda, h > 0$, $\epsilon \in (0,1)$ one has for $a \ll 1$

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Heuristics

Recall that $\Delta_n = \mathbf{1}_{\{(S_{n-1}, S_n) < 0\}}$ and

$$-H_N^0(a\lambda, ah) = -2a\lambda \sum_{n=1}^{N} (\omega_n + ah)\Delta_n$$

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We need to show that $\approx$ can be made $\asymp$.

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Showing that $H^0_N \asymp H^1_N$ is delicate and very technical.

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Step 2: From discrete charges to the white noise.

H_N^2 is obtained from H_N^1 by replacing the charges ω_n by i.i.d. $N(0,1)$ (discrete white noise).

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This step is analogous to step 1.

Thank you.