Non-ballistic random walks in random environments

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Standard **RWRE** on $\mathbb{Z}^d$: Markov chain with "disordered" transition probabilities.

Disorder: $\omega = \{\omega_x\}_{x\in\mathbb{Z}^d} \in \Omega \stackrel{\text{def}}{=} \mathcal{P}^{\mathbb{Z}^d}$,

$$\omega_x \in \mathcal{P} \stackrel{\text{def}}{=} \{p : \text{prob. distr. on } \{e : |e| = 1\}\}.$$

For a fixed "environment" ω, and starting point $x \in \mathbb{Z}^d$: Markov chain $\{X_n\}_{n\geq 0}$ with "quenched" law

$$\begin{aligned} P_{x,\omega}(X_0 = x) &= 1 \\ P_{x,\omega}(X_{n+1} = y + e |\, X_n = y) &= \omega_y(e). \end{aligned}$$

Assumptions:

(I) The **disorder** is random and i.i.d.: $\mathbb{P}$ on Ω is a **product measure** $\mathbb{P} = \mu^{\mathbb{Z}^d}$, μ a probability distribution on $\mathcal{P}$

(II) **Small disorder assumption**: There exists a (small) $\varepsilon > 0$ with

$$\mu\left(\left\{p : \left|p(e) - \frac{1}{2d}\right| \leq \varepsilon\right\}\right) = 1.$$

(III) μ is invariant under lattice isometries.

Ex: $\mathbb{Z}^2$, choose preferred $\hat{e}$ w.p. $\frac{1}{4}$. Take $p(\hat{e}) = \frac{1}{4} + \varepsilon,\ p(e) = \frac{1}{4} - \frac{\varepsilon}{3},\ e \neq \hat{e}$.
Averaged law: $\mathbb{P} \otimes P_{0,\cdot}$. $\{X_n\}$ is *not* Markovian under this.
Remarks:

1. $P_{x,\omega}$ is *not* reversible for $d \geq 2$ (and not for $d = 1$ dropping nearest neighbor)
2. A lot is known under ballisticity conditions, as Sznitman's condition (T') which is not satisfied under (III). In that case

$$\lim_{n\to\infty} \frac{X_n}{n} \neq 0.$$

$\underline{d = 1:}$ (reversible in the nearest neighbor case).
Solomon, Sinai: subdiffusive behavior: With large $\mathbb{P} \otimes P_{0,\cdot}$-probability $X_n \asymp (\log n)^2$
$\underline{d = 2:}$ The difficult case
$\underline{d \geq 3:}$

- Bricmont & Kupiainen 1990: CLT for $X_n/\sqrt{n}$
- Sznitman & Zeitouni 2007: Continuous time and space: CLT
- B.& Zeitouni 2008: **Exit distributions** from large sets are "the same" as those of ordinary random walks.

Main tool: Perturbation expansion for exit distributions

$$V_L \stackrel{\text{def}}{=} \left\{ x \in \mathbb{Z}^d : |x| \leq L \right\}.$$

$$\Pi_{L,\omega}(x,z) \stackrel{\text{def}}{=} P_{x,\omega}\left(X_{\tau_L} = z\right),\ x \in V_L,\ z \in \partial V_L.$$

∂V_L :outer boundary, τ_L first exit time from V_L.

$$\pi_L(x,z) \stackrel{\text{def}}{=} P_x^{\text{ORW}}\left(X_{\tau_L} = z\right),\ g_L^{\text{ORW}} \stackrel{\text{def}}{=} \text{Green's function in } V_L.$$

$$\begin{aligned}\Pi_{L,\omega}(x,z) = {} & \pi_L(x,z) + \sum_{y \in V_L, e} g_L^{\text{ORW}}(x,y)\left[\omega_y(e) - \frac{1}{2d}\right] \underbrace{\pi_L(y+e,z)}_{\text{can be replaced by } \pi_L(y+e,z)-\pi_L(y,z)} \\ & + \sum_{y,y'_L,e,e'} g_L^{\text{ORW}}(x,y)\left[\omega_y(e) - \frac{1}{2d}\right]\left(g_L^{\text{ORW}}(y+e,y') - g_L^{\text{ORW}}(y,y')\right) \\ & \times \left[\omega_{y'}(e') - \frac{1}{2d}\right]\left(\pi_L(y'+e',z) - \pi_L(y',z)\right) \\ & + \cdots\end{aligned}$$

Back of an envelope computation: Disorder only in the first coordinate.

$$\xi\left(x\right) \stackrel{\text{def}}{=} \omega_x\left(e_1\right) - \omega_x\left(-e_1\right) = E_{x,\omega}\left(\left(X_1 - x\right)_1\right),$$

$$\xi_L \stackrel{\text{def}}{=} \frac{E_{0,\omega}\left(\left(X_{\tau_L}\right)_1\right)}{L}.$$

$$\xi_L = \frac{1}{L}\sum_x g_L\left(0,x\right)\xi\left(x\right) + \frac{1}{L}\sum_{x,y} g_L\left(0,x\right)\xi\left(x\right)\nabla_1 g_L\left(x,y\right)\xi\left(y\right) + \cdots$$

$$\nabla_1 g_L\left(x,y\right) \stackrel{\text{def}}{=} \frac{1}{2}\left[g_L\left(x+e_1,y\right) - g_L\left(x-e_1,y\right)\right].$$

Remark: For fixed L, $\varepsilon \to 0$, this leads to a valid expansion of $\mathcal{L}\left(\xi_L\right)$.

In first order: $\delta \overset{\text{def}}{=} \operatorname{var}_{\mathbb{P}}(\xi(x))$

$$\operatorname{var}_{\mathbb{P}}(\xi_L) \approx \left[L^{-2} \sum\nolimits_{x \in V_L} g_L(0,x)^2\right] \delta.$$

$\underline{d = 1:}$ $g_L(0,x) \approx L$. Therefore

$$\operatorname{var}(\xi_L) \approx \text{const} \times L\delta.$$

In fact

$$\lim_{L \to \infty} \mathcal{L}(\Pi_L(0,L)) = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_0, \ \forall \delta > 0.$$

$\underline{d = 2:}$ $g_L(0,x) \approx 1$

$$\lim_{L \to \infty} \lim_{\delta \to 0} \frac{1}{\delta} \operatorname{var}(\xi_L) = \frac{2}{\pi}.$$

$\underline{d \geq 3:}$

$$\operatorname{var}(\xi_L) \approx \begin{cases} L^{-1}\delta & \text{for } d = 3 \\ (\log L) L^{-2}\delta & \text{for } d = 4 \\ L^{-2}\delta & \text{for } d \geq 5 \end{cases}.$$

Problems with this computation:

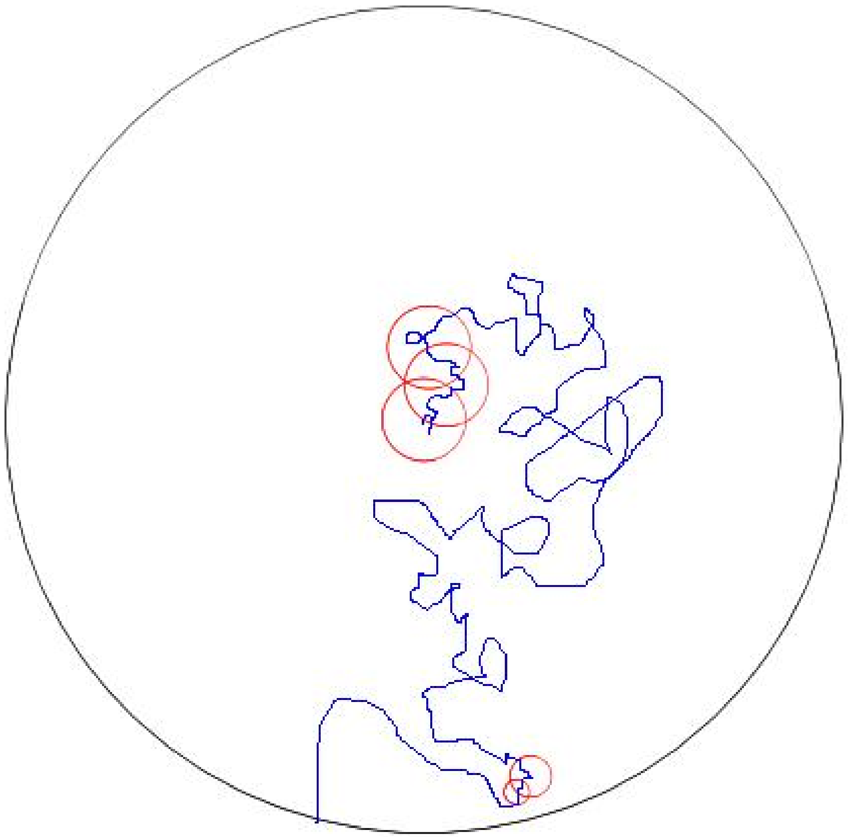
- We want δ (or ε) > 0 fixed (small), and $L \to \infty$, and *not* L fixed (large), $\delta \to 0$.
  Way out: Multiscale procedure: Take a sequence of scales $1 = L_1 < L_2 < \cdots$ such that from scale L_k to L_{k+1} one can work with the expansion.
- There is the question what exactly one wants to propagate. In the 2008 paper with Ofer Zeitouni, we did propagate estimates about the distance between the RWRE exit distribution and the ORW exit distribution, essentially in total variation.
- $\Pi_{L,\omega}(0,\cdot) - \pi_L(0,\cdot)$ cannot go to 0 in total variation if $L \to \infty$: If $z \in \partial V_L$, $\Pi_{L,\omega}(0,z)$ "feels" the disorder near z.
  Way out: Do some smoothing after exit.

Theorem $(d \geq 3)$. *There exists* $\varepsilon > 0$ *such that if (I)-(III) are satisfied, and* $m_L \to \infty$ *(arbitrary slowly)*

$$\left\| \sum_{z \in \partial V_L} \left[\Pi_{L,\omega}(0,z) - \pi_L(0,z) \right] \varphi_{m_L}(z,\cdot) \right\|_{\mathrm{var}} \to 0 \text{ in prob.}$$

where φ_m *is a smoothing kernel on scale* m. *Everything with explicit (but certainly bad) estimates.*

Key recursion: Represent $\Pi_{L_{k+1}}$ through centered exits distributions from $V_{L_k} + x,\ x \in V_{L_{k+1}}$:



Key technical steps and problems:

1. Scheme needs modifications near the boundary: We took $L_{k+1} = L_k (\log L_k)^3$, and started to shrink at distance $\approx \mathrm{const} \times L_k$ from the boundary.
2. For the induction step, one uses the (schematic) expansion

$$\begin{aligned} \Pi_{L_{k+1}} &= \pi_{L_{k+1}} + g_{L_{k+1}}^{(L_k)} \Delta^{(L_k)} \pi_{L_{k+1}} \\ &\quad + g_{L_{k+1}}^{(L_k)} \Delta^{(L_k)} g_{L_{k+1}}^{(L_k)} \Delta^{(L_k)} \pi_{L_{k+1}} + \cdots , \end{aligned}$$

   where

$$\Delta^{(L_k)} (x, \cdot) \stackrel{\mathrm{def}}{=} \Pi_{V_{L_k}+x} (x, \cdot) - \pi_{V_{L_k}+x} (x, \cdot) ,$$

   $g_{L_{k+1}}^{(L_k)}$ the Green's function of L_k-coarse grained ORW, killed when exiting $V_{L_{k+1}}$. The choice of the sequence is made in such a way that one uses sophisticated estimates only for the first term, and can estimates the others very crudely.
3. One needs $g^{(L_k)}$ or $\pi_{L_{k+1}}$ as smoothing operations. There is the problem that g is not a very good smoother.

4. We propagated two properties: Estimates for the globally smoothed differences, and for the non-smoothed differences. The latter is needed for the part of $g^{(L_k)}$ which is not properly smoothing.

5. Presence of bad L_k-balls inside $V_{L_{k+1}}$. The induction of course should tell us that the probability that a box is bad is decaying. “Badness” is measured in terms of total variations of (smoothed) exit distribution differences..

6. An advantage of working with exit distributions: The “degree of badness” (in total variation) is limited. If a subbox in scale L_k is bad, and is in the bulk, the L_{k+1}-box already gets improved with high probability, provided it is the only bad subbox. Technically we work with three levels of badness.

An improved and shorter version of the approach was recently worked out by **Erich Baur**: Better induction ansatz on the coarse grained RWRE Green’s functions.

Dimension 2 (work in progress with Ofer Zeitouni):
Back to the "back of envelope computation": $\xi(x) = \omega_x(e_1) - \omega_x(-e_1)\,,\ \omega_x(e) = 1/2d$ for $e \neq \pm e_1$.

$$\begin{aligned}\xi_L &= \frac{1}{L}\sum_x g_L(0,x)\,\xi(x) + \frac{1}{L}\sum_{x,y} g_L(0,x)\,\xi(x)\,\nabla_1 g_L(x,y)\,\xi(y) + \cdots \\ &= \Xi_{1,L} + \Xi_{2,L} + \cdots\ ,\ \text{say,}\end{aligned}$$

where

$$\nabla_1 g_L(x,y) \stackrel{\text{def}}{=} \frac{1}{2}\left[g_L(x+e_1) - g_L(x-e_1)\right].$$

$$\mathrm{var}_{\mathbb{P}}(\Xi_{1,L}) \stackrel{L\to\infty}{\to} \frac{2}{\pi}\mathrm{var}_{\mathbb{P}}(\xi_1) = \frac{2}{\pi}\delta.$$

First (important) observation: $2/\pi$ has "morally" to be 1.
For any (arbitrary large) L, $\mathrm{var}_{\mathbb{P}}(\Xi_{1,L})$ dominates $\mathrm{var}_{\mathbb{P}}(\xi_L)$ for $\delta \to 0$.
Problem: From scale L to $L' > L$, there is no simple way to present $\xi_{L'}$ in terms of ξ_L as above.

Take $1 \ll L \ll L'$. Then on one hand

$$\operatorname{var}_{\mathbb{P}}(\xi_{L'}) \approx \frac{2}{\pi}\delta,\ \operatorname{var}_{\mathbb{P}}(\xi_L) \approx \frac{2}{\pi}\delta,\ \delta \text{ small},$$

but the multiplication should also appear when going from scale L to L', i.e.

$$\operatorname{var}_{\mathbb{P}}(\xi_{L'}) \approx \frac{2}{\pi}\operatorname{var}_{\mathbb{P}}(\xi_L),\ \text{i.e. } 2/\pi = (2/\pi)^2 .$$

The solution of the paradox is that when going from scale L to L', one has to take into account that regions do overlap, and therefore the L-scale random transitions are not independent.
Taking that into account, one in fact gets in first order

$$\operatorname{var}_{\mathbb{P}}(\xi_{L'}) \approx \operatorname{var}_{\mathbb{P}}(\xi_L),\ \text{as } \delta \to 0.$$

$\Longrightarrow$*No contraction of disorder in leading order*.
One has to take the next term in the expansion. Bad news: $\Xi_{1,L}$ and $\Xi_{2,L}$ are uncorrelated $\Longrightarrow$

$$\operatorname{var}_{\mathbb{P}}(\Xi_{1,L} + \Xi_{2,L}) = \operatorname{var}_{\mathbb{P}}(\Xi_{1,L}) + \operatorname{var}_{\mathbb{P}}(\Xi_{2,L}) .$$

so the second order term pushes the disorder in the wrong direction. Disregarding the dependencies:

$$\Xi_{2,L} = \frac{1}{L}\sum_{x,y} g(0,x)\,\xi(x)\,\nabla_1 g(x,y)\,\xi(y)\,,$$

$$\begin{aligned}\mathrm{var}_{\mathbb{P}}(\Xi_{2,L}) &= \frac{\delta^2}{L^2}\sum_{x,y} g(0,x)^2\left(\nabla_1 g(x,y)\right)^2 + \frac{\delta^2}{L^2}\sum_{x,y} g(0,x)\,g(0,y)\,\nabla_1 g(x,y)\,\nabla_1 g(y,x)\\ &\quad + \frac{\mathbb{E}\left(\xi^4\right)}{L^2}\sum_{x} g(0,x)^2\left(\nabla_1 g(x,x)\right)^2.\end{aligned}$$

The third summand is negligible. The second summand is

$$\begin{aligned}&\frac{\delta^2}{4L^2}\sum_{x,y} g(0,x)\,g(0,y)\left[g(x+e_1,y)-g(x-e_1,y)\right]\underbrace{\left[g(y+e_1,x)-g(y-e_1,x)\right]}_{\approx g(y,x-e_1)-g(y,x+e_1)}\\ &\approx -\frac{\delta^2}{L^2}\sum_{x,y} g(0,x)\,g(0,y)\left[\nabla_1 g(x,y)\right]^2.\end{aligned}$$

Both, the first and the second summand are of order $\delta^2 \log L$, but the second summand is negative, and there is a cancellation of the $\log L$-divergency! So

$$\operatorname{var}_{\mathbb{P}}\left(\Xi_{2,L}\right) \approx \text{const} \times \delta^2, \ \text{const} > 0.$$

However, $\delta + \text{const} \times \delta^2$ wouldn't propagate properly in a multi-scale picture.
One has to develop up to third order with

$$\Xi_{3,L} = \frac{1}{L} \sum_{x,y,z} g\left(0,x\right) \xi\left(x\right) \nabla_1 g\left(x,y\right) \xi\left(y\right) \nabla_1 g\left(y,z\right) \xi\left(z\right).$$

Now, $\operatorname{var}_{\mathbb{P}}\left(\Xi_{3,L}\right)$ is of order δ^3, and $\Xi_{3,L}$ and $\Xi_{2,L}$ are uncorrelated, but $\Xi_{3,L}$ and $\Xi_{1,L}$ are correlated with a value of order δ^2!

$$\mathbb{E}\left(\Xi_{1,L}\Xi_{3,L}\right) = \frac{1}{L^2}\mathbb{E} \sum_{x'} g_L\left(0,x'\right) \xi\left(x'\right) \sum_{x,y,z} g\left(0,x\right) \xi\left(x\right) \nabla_1 g\left(x,y\right) \xi\left(y\right) \nabla_1 g\left(y,z\right) \xi\left(z\right).$$

The dominating term comes only when matching x' with y, and x with z, leading to

$$\mathbb{E}\left(\Xi_{1,L}\Xi_{3,L}\right) \approx -\frac{\delta^2}{L^2} \sum_{x,y} g\left(0,x\right) g\left(0,y\right) \left[\nabla_1 g\left(x,y\right)\right]^2 \approx -\text{const} \times \delta^2 \log L.$$

Therefore

$$\mathrm{var}_{\mathbb{P}}(\xi_L) \approx \delta - \alpha\delta^2 \log L.$$

This propagates properly: If $1 \ll L \ll L'$

$$\left(\delta - \alpha\delta^2 \log L\right) - \alpha \left(\delta - \alpha\delta^2 \log L\right)^2 \log \frac{L'}{L} = \delta - \alpha\delta^2 \log L' + \cdots$$

and should lead to

$$\mathrm{var}(\xi_L) \approx \frac{1}{C + \alpha \log L}.$$

There are many problems when implementing this for $L_k \to L_{k+1}$. The $\xi_{L_{k+1}}$ are not directly representable in terms of ξ_{L_k}. We in fact do a propagation of (L_{k-1}, L_k) to (L_k, L_{k+1}) : Key properties in the expansion when expressing $\xi_{L_{k+1}}$ can be estimated by going one scale down.

Intended **Theorem:** If $\varepsilon > 0$ is small enough, then

$$\lim_{L\to\infty} \mathrm{var}_{\mathbb{P}}(\xi_L) = 0.$$