On the oscillation stability of universal metric spaces

Norbert Sauer

University of Calgary

Thematic Program on Asymptotic Geometric Analysis

Workshop on the Concentration Phenomenon, Transformation Groups and Ramsey Theory,

Oct 12-15, 2010

Oct 14, 9:00-9:50

Outline

A metric space $\mathrm{M} = (M; \mathrm{d})$ is *ultrahomogeneous* if

A metric space $\mathrm{M} = (M; \mathrm{d})$ is *ultrahomogeneous* if

every isometry of a finite subspace of M to M
has an extension to an isometry of M onto M

A metric space $\mathrm{M} = (M; \mathrm{d})$ is *ultrahomogeneous* if

every isometry of a finite subspace of M to M
has an extension to an isometry of M onto M

that is to an *automorphism* of M.

A metric space $\mathrm{M} = (M; \mathrm{d})$ is *ultrahomogeneous* if

every isometry of a finite subspace of M to M
has an extension to an isometry of M onto M

that is to an *automorphism* of M.

A ultrahomogeneous metric space M is *universal* if every finite metric space F with $\mathrm{dist}(\mathrm{F}) \subseteq \mathrm{dist}(\mathrm{M})$ has an isometry into M.

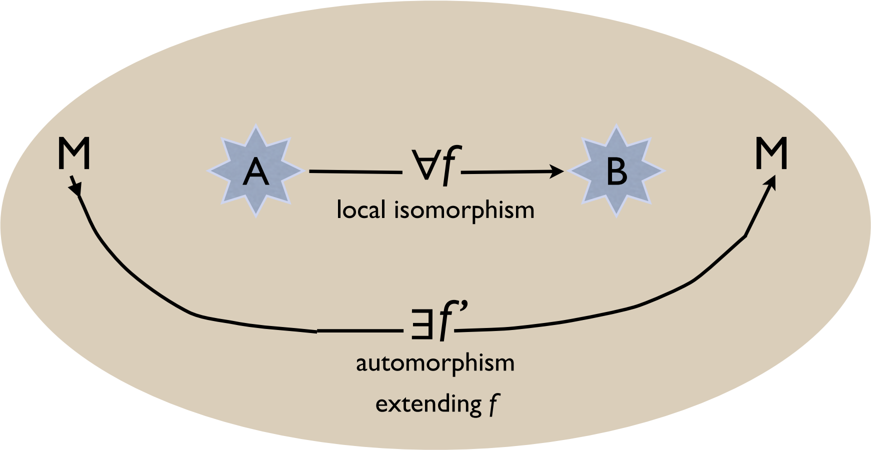
A metric space $\mathrm{M} = (M; \mathrm{d})$ is *ultrahomogeneous* if

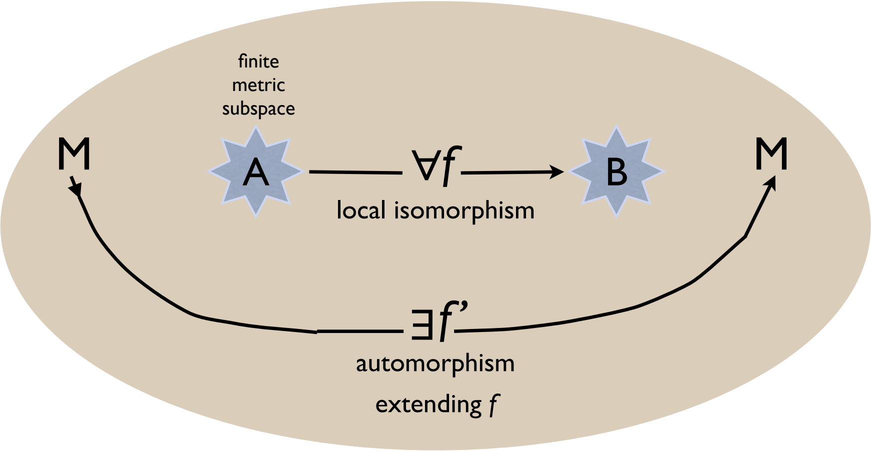
every isometry of a finite subspace of M to M
has an extension to an isometry of M onto M

that is to an *automorphism* of M.

A ultrahomogeneous metric space M is *universal* if every finite metric space F with $\mathrm{dist}(\mathrm{F}) \subseteq \mathrm{dist}(\mathrm{M})$ has an isometry into M.

An isometry of a finite subspace of M to a finite subspace of M is a *local isometry* or *local isomorphism*.

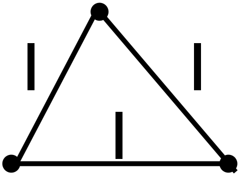

M
A
$\forall f$
B
M
local isomorphism
$\exists f'$
automorphism
extending f


finite
metric
subspace
M
A
$\forall f$
B
M
local isomorphism
$\exists f'$
automorphism
extending f

The rationals $(\mathbb{Q}; \mathrm{d})$ as metric subspace of the line

are ultrahomogeneous but not universal

because the triangle

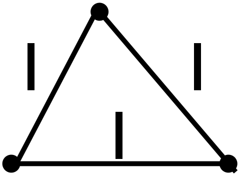


does not isometrically embed into $(\mathbb{Q}; \mathrm{d})$.

The rationals $(\mathbb{Q}; \mathrm{d})$ as metric subspace of the line

are ultrahomogeneous but not universal

because the triangle



does not isometrically embed into $(\mathbb{Q}; \mathrm{d})$.

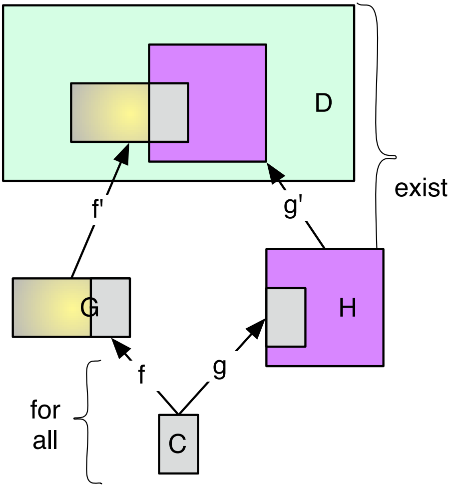
On the other hand, as we will see, the rational Urysohn space

$\mathbf{U}_{\mathbb{Q}}$ is univesal, because it embeds isometrically

every finite metric space with rational distances.

A class $\mathcal{A}$ of finite metric spaces is closed under *amalgamation* if:

A class $\mathcal{A}$ of finite metric spaces is closed under *amalgamation* if:



Theorem (R. Fraïssé)

The finite induced substructures of an ultrahomogeneous metric space are closed under amalgamation.

Theorem (R. Fraïssé)

The finite induced substructures of an ultrahomogeneous metric space are closed under amalgamation.

For every countable class $\mathcal{A}$ of finite metric spaces closed under induced subspaces and amalgamation exists a unique countable ultrahomogeneous metric space $\mathrm{M} = (M; \mathrm{d})$ so that $\mathcal{A}$ is the class of finite metric spaces having an isometry into M.

Theorem (R. Fraïssé)
The finite induced substructures of an ultrahomogeneous metric space are closed under amalgamation.

For every countable class $\mathcal{A}$ of finite metric spaces closed under induced subspaces and amalgamation exists a unique countable ultrahomogeneous metric space $\mathrm{M} = (M; \mathrm{d})$ so that $\mathcal{A}$ is the class of finite metric spaces having an isometry into M.

The space M is the *Fraïssé limit of the age* $\mathcal{A}$.

It is not difficult to see that the class $\mathcal{A}_{\mathbb{Q}}$

of finite metric spaces with rational distances is

countable, closed under subspaces and has amalgamation.

It is not difficult to see that the class $\mathcal{A}_{\mathbb{Q}}$

of finite metric spaces with rational distances is

countable, closed under subspaces and has amalgamation.

The Fraïssé limit of $\mathcal{A}_{\mathbb{Q}}$ is the

rational Urysohn space $\mathbf{U}_{\mathbb{Q}}$

whose completion is the Urysohn space $\mathbf{U}$.

A *copy* of a metric space $\mathrm{M} = (M; \mathrm{d})$

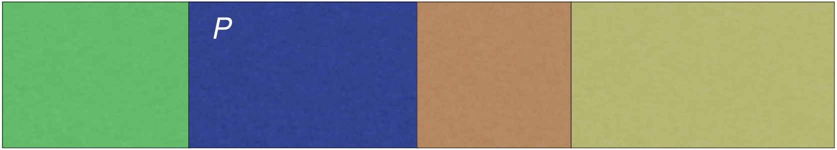
is the image of M under an isometry.

Let $S \subseteq M$. Then

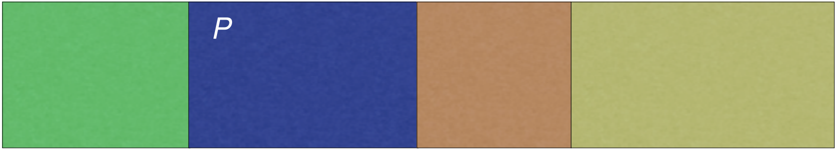
$$(S)_\epsilon = \{p \in M \mid \exists s \in S \; (\mathrm{d}(p, s) < \epsilon)\}.$$

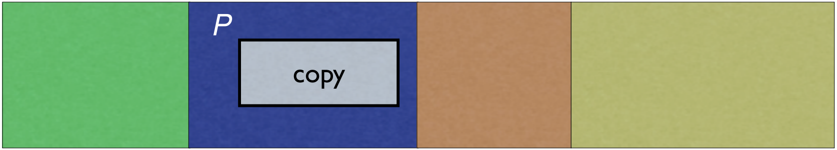
A metric space $\mathrm{M} = (M; \mathrm{d})$ is *indivisible* if for every partition of M into finitely many parts there exists a part P of the partition and an isometry of M into P.

A metric space $\mathrm{M} = (M; \mathrm{d})$ is *indivisible* if for every partition of M into finitely many parts there exists a part P of the partition and an isometry of M into P.



A metric space $\mathrm{M} = (M; \mathrm{d})$ is *indivisible* if for every partition of M into finitely many parts there exists a part P of the partition and an isometry of M into P.





A metric space $\mathrm{M} = (M; \mathrm{d})$ is *approximately indivisible* if

for every partition of M into finitely many parts and every $\epsilon > 0$

there exists

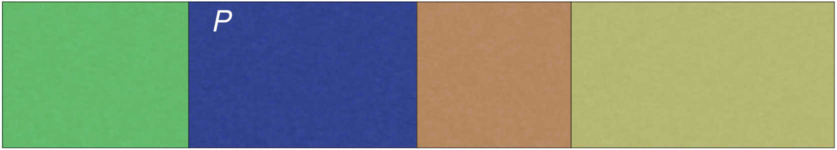
a part P of the partition and an isometry of M into $(P)_\epsilon$.

A metric space $\mathrm{M} = (M; \mathrm{d})$ is *approximately indivisible* if

for every partition of M into finitely many parts and every $\epsilon > 0$

there exists

a part P of the partition and an isometry of M into $(P)_\epsilon$.

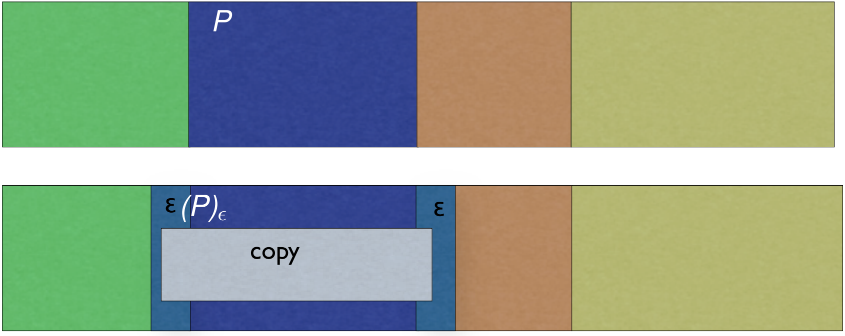


A metric space $\mathrm{M} = (M; \mathrm{d})$ is *approximately indivisible* if

for every partition of M into finitely many parts and every $\epsilon > 0$

there exists

a part P of the partition and an isometry of M into $(P)_\epsilon$.



Problem
Characterize the indivisible metric spaces.

Problem

Characterize the indivisible metric spaces.

Theorem (Delhomme, Laflamme, Pouzet, Sauer)

Divisible metric spaces are bounded.

Problem
Characterize the indivisible metric spaces.

Theorem (Delhomme, Laflamme, Pouzet, Sauer)
Divisible metric spaces are bounded.

Approximately indivisible metric spaces are bounded.

Problem
Characterize the indivisible metric spaces.

Theorem (Delhomme, Laflamme, Pouzet, Sauer)
Divisible metric spaces are bounded.

Approximately indivisible metric spaces are bounded.

Theorem (Delhomme, Laflamme, Pouzet, Sauer)
A countable metric space which is indivisible is totally Cantor disconnected.

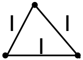
Problem
Characterize the indivisible metric spaces.

Theorem (Delhomme, Laflamme, Pouzet, Sauer)
Divisible metric spaces are bounded.

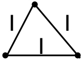
Approximately indivisible metric spaces are bounded.

Theorem (Delhomme, Laflamme, Pouzet, Sauer)
A countable metric space which is indivisible is totally Cantor disconnected.

Problem
Characterize the approximately indivisible countable universal metric spaces.

Let $\mathcal{A}$ be the class of finite metric spaces with distance set $\{0, 1, 2\}$ which do not contain a 

triangle.

Let $\mathcal{A}$ be the class of finite metric spaces with distance set $\{0, 1, 2\}$ which do not contain a 

triangle.

It is not difficult to see that $\mathcal{A}$ is closed under substructures and amalgamation. Let $F(\mathcal{A})$ be the Fraïssé limit of $\mathcal{A}$, then

Theorem (P. Komjath, V. Rödl)
The metric space $F(\mathcal{A})$ is indivisible.

The motivation to look at approximately indivisible metric spaces has its origin in Dvoretzki's theorem which then was sharpened by Milman and then restated by Milman and again reformulated by V. Pestov to:

The motivation to look at approximately indivisible metric spaces has its origin in Dvoretzki's theorem which then was sharpened by Milman and then restated by Milman and again reformulated by V. Pestov to:

Theorem

Let $\mathbb{S}^\infty$ be the unit sphere in ℓ_2. Let f $: \mathbb{S}^\infty \to \Re$ be a uniformly continuous function. Then for every $\epsilon > 0$ and every natural number n there exists an n-dimensional linear subspace V such that: $\operatorname{osc}(f|(V \cap \mathbb{S}^\infty)) < \epsilon$.

The motivation to look at approximately indivisible metric spaces has its origin in Dvoretzki's theorem which then was sharpened by Milman and then restated by Milman and again reformulated by V. Pestov to:

Theorem

Let $\mathbb{S}^\infty$ be the unit sphere in ℓ_2. Let $f : \mathbb{S}^\infty \to \Re$ be a uniformly continuous function. Then for every $\epsilon > 0$ and every natural number n there exists an n-dimensional linear subspace V such that: $\mathrm{osc}(f|(V \cap \mathbb{S}^\infty)) < \epsilon$.

A metric space M is *oscillation stable* if for every bounded 1-Lipschitz function of M into the reals and every $\epsilon > 0$ the space contains an isometric copy on which the oscillation of f is less than ϵ.

The motivation to look at approximately indivisible metric spaces has its origin in Dvoretzki's theorem which then was sharpened by Milman and then restated by Milman and again reformulated by V. Pestov to:

Theorem

Let $\mathbb{S}^\infty$ be the unit sphere in ℓ_2. Let $f : \mathbb{S}^\infty \to \Re$ be a uniformly continuous function. Then for every $\epsilon > 0$ and every natural number n there exists an n-dimensional linear subspace V such that: $\operatorname{osc}(f|(V \cap \mathbb{S}^\infty)) < \epsilon$.

A metric space M is *oscillation stable* if for every bounded 1-Lipschitz function of M into the reals and every $\epsilon > 0$ the space contains an isometric copy on which the oscillation of f is less than ϵ.
A complete ultrahomogeneous metric space is approximately indivisible if and only if it is oscillation stable.

A set N of reals is *universal* if there exists

a universal metric space $\mathrm{M} = (M; \mathrm{d})$ with $\mathrm{dist}(\mathrm{M}) = N$.

Problem
Characterize the bounded sets N of reals for which

the corresponding universal metric space is

approximately indivisible

A set N of reals is *universal* if there exists

a universal metric space $\mathrm{M} = (M; \mathrm{d})$ with $\mathrm{dist}(\mathrm{M}) = N$.

Problem

Characterize the bounded sets N of reals for which

the corresponding universal metric space is

approximately indivisible

$N = [0, 1) \cup \{2, 3\}$

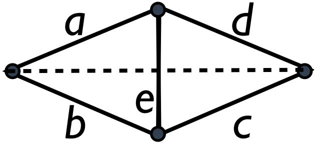
Given a countable set N of positive reals and

the complete graph on four vertices with edges labeled

$\{a, b, c, d, e\} \subseteq N$ so that the

two labeled triangles

are metric.



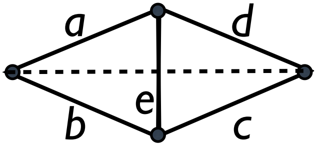
Given a countable set N of positive reals and

the complete graph on four vertices with edges labeled

$\{a, b, c, d, e\} \subseteq N$ so that the

two labeled triangles

are metric.



FACT: If there is always a number

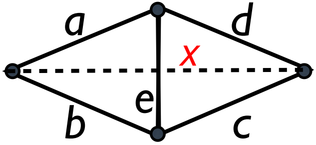
Given a countable set N of positive reals and

the complete graph on four vertices with edges labeled

$\{a, b, c, d, e\} \subseteq N$ so that the

two labeled triangles

are metric.



FACT: If there is always a number x in N so that the resulting labeled graph is metric, then N is universal. Conversely, if M is a universal metric space and $N = \mathrm{dist}(\mathrm{M})$, there always is such a number x.

In the Abad, Nguyen paper:

Theorem

Every complete separable ultrahomogeneous metric space Y *includes a countable ultrahomogeneous dense metric subspace.*

In the Abad, Nguyen paper:

Theorem
Every complete separable ultrahomogeneous metric space Y *includes a countable ultrahomogeneous dense metric subspace.*

Theorem (J. Lopez-Abad and L. Nguyen Van Thé)
The following are equivalent:

1. $\boldsymbol{S}$ *is oscillation stable (equiv., approximately indivisible).*
2. *For every strictly positive* $m \in \omega$, $\boldsymbol{U}_m$ *is indivisible.*

In the Abad, Nguyen paper:

Theorem
Every complete separable ultrahomogeneous metric space Y *includes a countable ultrahomogeneous dense metric subspace.*

Theorem (J. Lopez-Abad and L. Nguyen Van Thé)
The following are equivalent:

1. $\boldsymbol{S}$ *is oscillation stable (equiv., approximately indivisible).*
2. *For every strictly positive* $m \in \omega$, $\boldsymbol{U}_m$ *is indivisible.*

Theorem (L. Nguyen Van Thé and N. Sauer)
For every strictly positive $m \in \omega$, $\boldsymbol{U}_m$ *is indivisible.*

In the Abad, Nguyen paper:

Theorem
Every complete separable ultrahomogeneous metric space Y *includes a countable ultrahomogeneous dense metric subspace.*

Theorem (J. Lopez-Abad and L. Nguyen Van Thé)
The following are equivalent:

1. $\boldsymbol{S}$ *is oscillation stable (equiv., approximately indivisible).*
2. *For every strictly positive* $m \in \omega$, $\boldsymbol{U}_m$ *is indivisible.*

Theorem (L. Nguyen Van Thé and N. Sauer)
For every strictly positive $m \in \omega$, $\boldsymbol{U}_m$ *is indivisible.*

Theorem (N. Sauer)
Every countable universal metric space with a finite set of distances is indivisible.

Let N, a subset of the non negative reals be given.

A subset S of reals has property $\mathfrak{e}$ if

for every $\epsilon > 0$ and every $s \in S$

$N \cap (s - \epsilon, s] \neq \emptyset$.

Let N, a subset of the non negative reals be given.

A subset S of reals has property $\mathfrak{e}$ if

for every $\epsilon > 0$ and every $s \in S$

$N \cap (s - \epsilon, s] \neq \emptyset$.

For two reals a and b let:

$$a \oplus b = \sup\{z \in N \mid z < a + b\},$$
$$a \boxplus b = \sup\{z \in N \mid z \leq a + b\}.$$

Given $\epsilon > 0$.

Given $\epsilon > 0$.

1. **1.** The set $S = \{\frac{i}{n} \mid 0 \leq i \leq n\}$ for sufficiently large n has the property that for all $x \in [0, 1] \cap \mathbb{Q}$ exists $s \in S$ so that $x \leq s$ and $s - x$ is "small".

Given $\epsilon > 0$.

1. The set $S = \{\frac{i}{n} \mid 0 \leq i \leq n\}$ for sufficiently large n has the property that for all $x \in [0,1] \cap \mathbb{Q}$ exists $s \in S$ so that $x \leq s$ and $s - x$ is “small".
2. S has property $\mathfrak{e}$ with respect to the set $[0,1] \cap \mathbb{Q}$.

Given $\epsilon > 0$.

1. The set $S = \{\frac{i}{n} \mid 0 \leq i \leq n\}$ for sufficiently large n has the property that for all $x \in [0, 1] \cap \mathbb{Q}$ exists $s \in S$ so that $x \leq s$ and $s - x$ is “small".
2. S has property $\mathfrak{e}$ with respect to the set $[0, 1] \cap \mathbb{Q}$.
3. There exists a finite set $T \supseteq S$ of numbers having property $\mathfrak{e}$ which is closed under $\oplus$ and $\boxplus$, namely $T = S$.

Given $\epsilon > 0$.

1. The set $S = \{\frac{i}{n} \mid 0 \leq i \leq n\}$ for sufficiently large n has the property that for all $x \in [0, 1] \cap \mathbb{Q}$ exists $s \in S$ so that $x \leq s$ and $s - x$ is "small".
2. S has property $\mathfrak{e}$ with respect to the set $[0, 1] \cap \mathbb{Q}$.
3. There exists a finite set $T \supseteq S$ of numbers having property $\mathfrak{e}$ which is closed under $\oplus$ and $\boxplus$, namely $T = S$.
4. There exists a finite set $R \subseteq [0, 1]$ so that for all $t \in T$ there is a $r \in R$ for which $|r - t|$ is "small", take $R = T = S$, having property $\mathfrak{e}$ and which is universal.

Given $\epsilon > 0$.

1. The set $S = \{\frac{i}{n} \mid 0 \leq i \leq n\}$ for sufficiently large n has the property that for all $x \in [0,1] \cap \mathbb{Q}$ exists $s \in S$ so that $x \leq s$ and $s - x$ is “small".
2. S has property $\mathfrak{e}$ with respect to the set $[0,1] \cap \mathbb{Q}$.
3. There exists a finite set $T \supseteq S$ of numbers having property $\mathfrak{e}$ which is closed under $\oplus$ and $\boxplus$, namely $T = S$.
4. There exists a finite set $R \subseteq [0,1]$ so that for all $t \in T$ there is a $r \in R$ for which $|r - t|$ is “small", take $R = T = S$, having property $\mathfrak{e}$ and which is universal.
5. The universal structure U with $\mathrm{dist}(\mathrm{U}) = R = S$ is indivisible according to the Nguyen, Sauer theorem.

Given $\epsilon > 0$.

1. The set $S = \{\frac{i}{n} \mid 0 \leq i \leq n\}$ for sufficiently large n has the property that for all $x \in [0,1] \cap \mathbb{Q}$ exists $s \in S$ so that $x \leq s$ and $s - x$ is “small".
2. S has property $\mathfrak{e}$ with respect to the set $[0,1] \cap \mathbb{Q}$.
3. There exists a finite set $T \supseteq S$ of numbers having property $\mathfrak{e}$ which is closed under $\oplus$ and $\boxplus$, namely $T = S$.
4. There exists a finite set $R \subseteq [0,1]$ so that for all $t \in T$ there is a $r \in R$ for which $|r - t|$ is “small", take $R = T = S$, having property $\mathfrak{e}$ and which is universal.
5. The universal structure U with $\mathrm{dist}(\mathrm{U}) = R = S$ is indivisible according to the Nguyen, Sauer theorem.
6. The universal structure U has an isometric copy U^* in $\boldsymbol{S}_{\mathbb{Q}}$ so that $(\mathrm{U}^*)_\epsilon = \boldsymbol{S}_{\mathbb{Q}}$.

Given $\epsilon > 0$.

1. The set $S = \{\frac{i}{n} \mid 0 \leq i \leq n\}$ for sufficiently large n has the property that for all $x \in [0,1] \cap \mathbb{Q}$ exists $s \in S$ so that $x \leq s$ and $s - x$ is “small".
2. S has property $\mathfrak{e}$ with respect to the set $[0,1] \cap \mathbb{Q}$.
3. There exists a finite set $T \supseteq S$ of numbers having property $\mathfrak{e}$ which is closed under $\oplus$ and $\boxplus$, namely $T = S$.
4. There exists a finite set $R \subseteq [0,1]$ so that for all $t \in T$ there is a $r \in R$ for which $|r - t|$ is “small", take $R = T = S$, having property $\mathfrak{e}$ and which is universal.
5. The universal structure U with $\mathrm{dist}(\mathrm{U}) = R = S$ is indivisible according to the Nguyen, Sauer theorem.
6. The universal structure U has an isometric copy U^* in $\boldsymbol{S}_{\mathbb{Q}}$ so that $(\mathrm{U}^*)_\epsilon = \boldsymbol{S}_{\mathbb{Q}}$.
7. The copy U^* can be chosen in such a way that the indivisibility of U implies the approximate indivisibility of $\boldsymbol{S}_{\mathbb{Q}}$, the difficult part of the Lopez-Abad, Nguyen theorem.

Given $\epsilon > 0$ and a bounded non negative universal set N of reals with the property that for every $\delta > 0$ the set $N \cap (0, \delta) \neq \emptyset$. Let M be the corresponding universal metric space.

Given $\epsilon > 0$ and a bounded non negative universal set N of reals with the property that for every $\delta > 0$ the set $N \cap (0, \delta) \neq \emptyset$. Let M be the corresponding universal metric space.

1. There exists a finite set S of numbers so that for all $x \in N$ exists $s \in S$ so that $x \leq s$ and $s - x$ is “small".

Given $\epsilon > 0$ and a bounded non negative universal set *N* of reals with the property that for every $\delta > 0$ the set $N \cap (0, \delta) \neq \emptyset$. Let M be the corresponding universal metric space.

1. There exists a finite set *S* of numbers so that for all $x \in N$ exists $s \in S$ so that $x \leq s$ and $s - x$ is “small".
2. *S* has property $\mathfrak{e}$ with respect to the set *N*.

Given $\epsilon > 0$ and a bounded non negative universal set *N* of reals with the property that for every $\delta > 0$ the set $N \cap (0, \delta) \neq \emptyset$. Let M be the corresponding universal metric space.

1. There exists a finite set *S* of numbers so that for all $x \in N$ exists $s \in S$ so that $x \leq s$ and $s - x$ is “small".
2. *S* has property $\mathfrak{e}$ with respect to the set *N*.
3. There exists a finite set $T \supseteq S$ of numbers having property $\mathfrak{e}$ which is closed under $\oplus$ and $\boxplus$.

Given $\epsilon > 0$ and a bounded non negative universal set N of reals with the property that for every $\delta > 0$ the set $N \cap (0, \delta) \neq \emptyset$. Let M be the corresponding universal metric space.

1. There exists a finite set S of numbers so that for all $x \in N$ exists $s \in S$ so that $x \leq s$ and $s - x$ is “small".
2. S has property $\mathfrak{e}$ with respect to the set N.
3. There exists a finite set $T \supseteq S$ of numbers having property $\mathfrak{e}$ which is closed under $\oplus$ and $\boxplus$.
4. There exists a finite set $R \subseteq N$ so that for all $t \in T$ there is an $r \in R$ for which $|r - t|$ is “small" and which is universal.

Given $\epsilon > 0$ and a bounded non negative universal set N of reals with the property that for every $\delta > 0$ the set $N \cap (0, \delta) \neq \emptyset$. Let M be the corresponding universal metric space.

1. There exists a finite set S of numbers so that for all $x \in N$ exists $s \in S$ so that $x \leq s$ and $s - x$ is “small".
2. S has property $\mathfrak{e}$ with respect to the set N.
3. There exists a finite set $T \supseteq S$ of numbers having property $\mathfrak{e}$ which is closed under $\oplus$ and $\boxplus$.
4. There exists a finite set $R \subseteq N$ so that for all $t \in T$ there is an $r \in R$ for which $|r - t|$ is “small" and which is universal.
5. The universal structure U with $\mathrm{dist}(\mathrm{U}) = R$ is indivisible according to the Sauer theorem.

Given $\epsilon > 0$ and a bounded non negative universal set N of reals with the property that for every $\delta > 0$ the set $N \cap (0, \delta) \neq \emptyset$. Let M be the corresponding universal metric space.

1. There exists a finite set S of numbers so that for all $x \in N$ exists $s \in S$ so that $x \leq s$ and $s - x$ is “small".
2. S has property $\mathfrak{e}$ with respect to the set N.
3. There exists a finite set $T \supseteq S$ of numbers having property $\mathfrak{e}$ which is closed under $\oplus$ and $\boxplus$.
4. There exists a finite set $R \subseteq N$ so that for all $t \in T$ there is an $r \in R$ for which $|r - t|$ is “small" and which is universal.
5. The universal structure U with $\mathrm{dist}(\mathrm{U}) = R$ is indivisible according to the Sauer theorem.

6. **Theorem (N. Sauer)**

Every countable universal metric space M has for every $\epsilon > 0$ a universal subspace U with a finite set of distances and with $\left(U\right)_{\epsilon} = M$.

Let $N \subseteq \Re_{\geq 0}$ with an accumulation point at 0 be universal.

Let $N \subseteq \Re_{\geq 0}$ with an accumulation point at 0 be universal.

Then $N \setminus (0, \delta] := N'$ for some sufficiently small δ is again universal. Let $\delta \in N$.

Let $N \subseteq \Re_{\geq 0}$ with an accumulation point at 0 be universal.

Then $N \setminus (0, \delta] := N'$ for some sufficiently small δ is again universal. Let $\delta \in N$.

Let U be the universal metric space with $\mathrm{dist}(\mathrm{U}) = N$.

Let $N \subseteq \Re_{\geq 0}$ with an accumulation point at 0 be universal.

Then $N \setminus (0, \delta] := N'$ for some sufficiently small δ is again universal. Let $\delta \in N$.

Let U be the universal metric space with $\mathrm{dist}(\mathrm{U}) = N$.

Let V be the universal metric space with $\mathrm{dist}(\mathrm{V}) = N'$

Let $N \subseteq \Re_{\geq 0}$ with an accumulation point at 0 be universal.

Then $N \setminus (0, \delta] := N'$ for some sufficiently small δ is again universal. Let $\delta \in N$.

Let U be the universal metric space with $\mathrm{dist}(\mathrm{U}) = N$.

Let V be the universal metric space with $\mathrm{dist}(\mathrm{V}) = N'$

There exists a copy V^* of V in U with $\left(\mathrm{V}^*\right)_\delta = \mathrm{U}$.

Let $N \subseteq \Re_{\geq 0}$ with an accumulation point at 0 be universal.

Then $N \setminus (0, \delta] := N'$ for some sufficiently small δ is again universal. Let $\delta \in N$.

Let U be the universal metric space with $\mathrm{dist}(\mathrm{U}) = N$.

Let V be the universal metric space with $\mathrm{dist}(\mathrm{V}) = N'$

There exists a copy V^* of V in U with $\left(\mathrm{V}^*\right)_\delta = \mathrm{U}$.

It follows that V is $\delta + \epsilon$-approximately indivisible for every $\epsilon > 0$.

Let $N \subseteq \Re_{\geq 0}$ with an accumulation point at 0 be universal.

Then $N \setminus (0, \delta] := N'$ for some sufficiently small δ is again universal. Let $\delta \in N$.

Let U be the universal metric space with $\mathrm{dist}(\mathrm{U}) = N$.

Let V be the universal metric space with $\mathrm{dist}(\mathrm{V}) = N'$

There exists a copy V^* of V in U with $\left(\mathrm{V}^*\right)_\delta = \mathrm{U}$.

It follows that V is $\delta + \epsilon$-approximately indivisible for every $\epsilon > 0$.

If $\frac{\delta}{2} \in N$ then there exists a copy V^* of V in U with $\left(\mathrm{V}^*\right)_{\frac{\delta}{2}} = \mathrm{U}$.

Let $N \subseteq \Re_{\geq 0}$ with an accumulation point at 0 be universal.

Then $N \setminus (0, \delta] := N'$ for some sufficiently small δ is again universal. Let $\delta \in N$.

Let U be the universal metric space with $\mathrm{dist}(\mathrm{U}) = N$.

Let V be the universal metric space with $\mathrm{dist}(\mathrm{V}) = N'$

There exists a copy V^* of V in U with $\left(\mathrm{V}^*\right)_\delta = \mathrm{U}$.

It follows that V is $\delta + \epsilon$-approximately indivisible for every $\epsilon > 0$.

If $\frac{\delta}{2} \in N$ then there exists a copy V^* of V in U with $\left(\mathrm{V}^*\right)_{\frac{\delta}{2}} = \mathrm{U}$.

It follows that V is $\frac{\delta}{2} + \epsilon$-approximately indivisible for every $\epsilon > 0$ and hence indivisible.

Let $N \subseteq \Re_{\geq 0}$ with an accumulation point at 0 be universal.

Then $N \setminus (0, \delta] := N'$ for some sufficiently small δ is again universal. Let $\delta \in N$.

Let U be the universal metric space with $\mathrm{dist}(\mathrm{U}) = N$.

Let V be the universal metric space with $\mathrm{dist}(\mathrm{V}) = N'$

There exists a copy V^* of V in U with $\left(\mathrm{V}^*\right)_\delta = \mathrm{U}$.

It follows that V is $\delta + \epsilon$-approximately indivisible for every $\epsilon > 0$.

If $\frac{\delta}{2} \in N$ then there exists a copy V^* of V in U with $\left(\mathrm{V}^*\right)_{\frac{\delta}{2}} = \mathrm{U}$.

It follows that V is $\frac{\delta}{2} + \epsilon$-approximately indivisible for every $\epsilon > 0$ and hence indivisible.

(Actually any number in $[\frac{\delta}{2}, \delta)$ would do to give this result.)

Let $N \subseteq \Re_{\geq 0}$ be universal and 0 is not an accumulation point.

W

Let N be a finite universal set of numbers.

Let $\mathrm{M} = (M; \mathrm{d})$ the corresponding countable universal metric space with $\mathrm{dist}(\mathrm{M}) = N$.

Let N be a finite universal set of numbers.

Let $\mathrm{M} = (M; \mathrm{d})$ the corresponding countable universal metric space with $\mathrm{dist}(\mathrm{M}) = N$.

For r a positive real let $r^{\langle - \rangle} = \max\big([0, r) \cap N\big)$.

For $r < \max N$ let $r^{\langle + \rangle} = \min\big((r, \max N] \cap N\big)$. a

For $r = \max N$ let $r^{\langle + \rangle} = r$.

Let N be a finite universal set of numbers.

Let $\mathrm{M} = (M; \mathrm{d})$ the corresponding countable universal metric space with $\mathrm{dist}(\mathrm{M}) = N$.

For r a positive real let $r^{\langle - \rangle} = \max\big([0, r) \cap N\big)$.

For $r < \max N$ let $r^{\langle + \rangle} = \min\big((r, \max N] \cap N\big)$. a

For $r = \max N$ let $r^{\langle + \rangle} = r$.

The number $r \in N$ is a *jump number* if $r^{\langle + \rangle} > 2 \cdot r$.

The set $\mathcal{B} \subseteq N$ is a *block* of N if there exists an enumeration $(b_i : i \in n+1)$ of $\mathcal{B}$ so that:

1. $b_i < b_{i+1}$ for all $i \in n$.
2. $b_0 > b_0^{\langle - \rangle} + b_0^{\langle - \rangle}$.
3. $b_{i+1} = b_i^{\langle + \rangle}$ for all $i \in n$.
4. $b_i + b_0 \geq b_{i+1}$ for all $i \in n$.

Theorem (N. Sauer)

The distance set N of a universal metric space is the union of disjoint blocks $\mathcal{B}_i$ so that:

1. $x < y$ *for all* $x \in \mathcal{B}_i$ *and* $y \in \mathcal{B}_{i+1}$.
2. $2 \cdot \max \mathcal{B}_i < \min \mathcal{B}_{i+1}$.
3. $x + \min(\mathcal{B}_i) \geq x^{\langle + \rangle}$ *for all* $x \in \mathcal{B}_i$.

Let $\mathrm{M} = (M; \mathrm{d})$ be a metric space.

The set $A \subseteq M$ is *age complete* if every finite subset $F \subseteq M$ has an isometry into A.

(Negation, *age incomplete*.)

WB

The metric space $\mathrm{M} = (M; \mathrm{d})$ is *weakly indivisible* if $A \cup B = M$ and A age incomplete implies that there is an isometric embedding of M into B.

The metric space $\mathrm{M} = (M; \mathrm{d})$ is *weakly indivisible* if $A \cup B = M$ and A age incomplete implies that there is an isometric embedding of M into B.

The metric space M is *age indivisible* if $A \cup B = M$ and A age incomplete implies that B is age complete.

The metric space $\mathrm{M} = (M; \mathrm{d})$ is *weakly indivisible* if $A \cup B = M$ and A age incomplete implies that there is an isometric embedding of M into B.

The metric space M is *age indivisible* if $A \cup B = M$ and A age incomplete implies that B is age complete.

So far there are only two and to some extent unsatisfactory examples of age indivisible but not weakly indivisible ultrahomogenous structures known.

The metric space $\mathrm{M} = (M; \mathrm{d})$ is *weakly indivisible* if $A \cup B = M$ and A age incomplete implies that there is an isometric embedding of M into B.

The metric space M is *age indivisible* if $A \cup B = M$ and A age incomplete implies that B is age complete.

So far there are only two and to some extent unsatisfactory examples of age indivisible but not weakly indivisible ultrahomogenous structures known.

There are no age indivisible but not weakly indivisible universal metric spaces known.

Let $\mathbf{U}$ be the Urysohn space and $\mathbf{U}_{\mathbb{Q}}$ the universal metric space with the non negative rationals as distances and $\mathbf{U}_{\mathbb{N}}$ the universal metric space with the non negative integers as distance set.

Using the Hales-Jewett theorem it is easy to see that universal metric spaces are age indivisible. Hence $\mathbf{U}_{\mathbb{N}}$ and $\mathbf{U}_{\mathbb{Q}}$ are age indivisible.

Because $\mathbf{U}_{\mathbb{Q}}$ is not totally Cantor disconnected it is not indivisible and because $\mathbf{U}_{\mathbb{N}}$ is not bounded it is not indivisible according to the Delhomme, Laflamme, Pouzet, Sauer theorems.

Theorem (L. Nguyen Van Thé and N. Sauer)
The space $\mathbf{U}_{\mathbb{N}}$ is weakly indivisible.

Theorem (L. Nguyen Van Thé and N. Sauer)
Let $\mathbf{U}_{\mathbb{Q}} = A \cup B$ and $\epsilon > 0$. If A is age incomplete then $\mathbf{U}_{\mathbb{Q}}$ embeds into $(B)_\epsilon$.

Theorem (L. Nguyen Van Thé and N. Sauer)
Let $\mathbf{U}_{\mathbb{Q}} = A \cup B$ and $\epsilon > 0$. If a compact metric subspace $\mathbf{K}$ of $\mathbf{U}$ does not embed into $(A)_\epsilon$ then $\mathbf{U}$ embeds into $(B)_\epsilon$.

Let S be the set of rationals in $[0,2]$ and $\mathfrak{C}_S$ the class of all finite metric spaces X with distances in S which embed isometrically into the unit sphere $\mathbb{S}^\infty$ of ℓ_2 with the property that $\{0_{\ell_2}\} \cup X$ is affinely independent.

Theorem (L. Nguyen Van Thé)

There is a unique countable ultrahomogeneous metric space $\mathbb{S}^\infty_S$ whose class of finite metric subspaces is exactly $\mathfrak{C}_S$. Moreover, the metric completion of $\mathbb{S}^\infty_S$ is $\mathbb{S}^\infty$.

Theorem (L. Nguyen Van Thé and N. Sauer)

The space $\mathbb{S}^\infty_S$ is age indivisible but not weakly indivisible.

MR

Theorem (C. Laflamme, L. Nguyen Van Thé, M. Pouzet, N. Sauer)

Let V be a vector space of countable dimension over $\mathbb{Q}$ and M_V be the midpoint structure associated with the vector space V. Then:

1. M_V *is age indivisible.*
2. M_V *is not weakly indivisible.*
3. M_V *is universal fir its age: Every countable structure with the same age as* M_V *is embeddable into* M_V.

Theorem

Let $\epsilon > 0$ and k a positive integer. There exists $N = N(k, \epsilon)$ so that: Every normed $n \geq N$ dimensional space X contains a k-dimensional subspace E_k with $\mathrm{d}(E_k, \ell_2^k) \leq 1 + \epsilon$.

V. Milman $k \geq c\epsilon^2 \log n$.

DVB

Theorem (J. Matoušek, V. Rödl)

Let X be an affinely independent finite metric subspace of $\mathbb{S}^\infty$ with circumradius r, and let $\alpha > 0$. Then there exists a finite metric subspace Z of $\mathbb{S}^\infty$ with circumradius $r + \alpha$ such that for every partition $Z = B \cup R$, the space X embeds in B or R.

MRB