Applications of continuous logic to the theory of Polish groups

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Workshop on Concentration Phenomenon, Transformation Groups and Ramsey Theory

Fields Institute, Toronto

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- The Urysohn space of diameter 1 $(\mathbb{U}_1, d)$, i.e the unique Polish metric space of diameter 1 which is both *universal* and approximately ultrahomogeneous.
- The group of measure-preserving automorphisms of $[0, 1]$, seen as $Aut(MALG_\mu, d, d(., \emptyset))$, with

$$d(A, B) = \mu(A \Delta B) \ .$$

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Theorem

The automorphism group of a separable metric sructure is a Polish group.
Conversely, any Polish group is (isomorphic to) the automorphism group of some approximately ultrahomogeneous separable metric structure.

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d_u is always complete and usually nonseparable. For instance, the uniform metric on $\mathcal{U}(\ell_2)$ is given by the operator norm.

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If (G, τ, ∂) is a Polish topometric group and $A \subseteq G$, we let, for any $\varepsilon > 0$,

$$(A)_\varepsilon = \{g \in G \colon \exists a \in A, d(a, g) < \varepsilon\}$$

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The three groups above have meagre conjugacy classes.

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Theorem (Kechris-Rosendal)

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Assume (G, τ, ∂) is a Polish topometric group with ample generics, H is a topological group with uniform Suslin number $< 2^{\aleph_0}$ and $\varphi\colon G \to H$ is a homomorphism such that $\varphi\colon (G, \partial) \to H$ is continuous. Then φ is continuous from (G, τ) to H.

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Assume (G, τ, ∂) is a Polish topometric group with ample generics, H is a topological group with uniform Suslin number $< 2^{\aleph_0}$ and $\varphi\colon G \to H$ is a homomorphism such that $\varphi\colon (G, \partial) \to H$ is continuous. Then φ is continuous from (G, τ) to H.

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For any seminorm l on G which is ∂-lower semicontinuous and has a density character $< 2^{\aleph_0}$, l is τ-continuous.

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Extreme amenability and finite oscillation stability (I)

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Definition (Milman)

Let a group G act on a set X, and consider a function $f: X \to [0,1]$. f is *finitely oscillation stable* if for any finite $F \subseteq X$ and any $\varepsilon > 0$ there is $g \in G$ such that the oscillation of f on gF is $< \varepsilon$.

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Proposition (Pestov)

Let G be a Polish group. Then the following are equivalent:

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- There is a directed collection of bounded left-invariant continuous pseudometrics $\{d_i\}$, determining the topology of G and such that each metric space G/d_i is finitely oscillation stable.

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