Entropy for actions of sofic groups

Hanfeng Li

SUNY at Buffalo

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Fields Institute

Joint work with David Kerr

Dynamical Systems:

G: a countable discrete group.
X: a probability measure space or a compact metrizable space.
α: an action of G on X preserving the structure of X.

Example(Topological Bernoulli shift): For a compact metrizable space Y, consider $X = Y^G$ with the product topology.
X is compact metrizable.
G acts on X by shift: $(sx)_t = x_{s^{-1}t}$.

Example(Measurable Bernoulli shift): For a Borel probability measure ν on Y, consider the product measure ν^G on X:

$$\nu^G(\{x \in X : x_{s_1} \in A_1, \ldots, x_{s_k} \in A_k\}) = \nu(A_1) \cdots \nu(A_k)$$

if $s_1, \ldots, s_k$ in G are pairwise distinct.
The shift action preserves ν^G.

History of measure entropy:

$$G \curvearrowright (X, \mathcal{B}, \mu).$$

For a (measurable countable) partition P of X, define

$$H(P) = -\sum_{p \in P} \mu(p) \log \mu(p).$$

P is *dynamically generating* if the σ-algebra generated by $\bigcup_{s \in G} sP$ is equal to $\mathcal{B}$, modulo μ.

Kolmogorov (1958): $G = \mathbb{Z}$. Defined the dynamical entropy $h_\mu(P)$ for any P. If X has a dynamically generating partition P with $H(P) < +\infty$, then $h_\mu(P)$ does not depend on the choice of such P. Define $h_\mu(X, \mathbb{Z})$ as $h_\mu(P)$.

Example: The trivial action of $\mathbb{Z}$ on the unit interval with Lebesgue measure has no dynamically generating partitions.

Sinai (1959): $G = \mathbb{Z}$. In general, define $h_\mu(X, \mathbb{Z})$ as $\sup_P h_\mu(P)$ for P ranging over partitions with $H(P) < +\infty$.

Moulin Ollagnier (1985), **Ornstein and Weiss** (1987): Extended the theory to countable amenable G.

Lewis Bowen (2008): G countable sofic. Defined the dynamical entropy $h_{\Sigma,\mu}(P)$ for any P. If X has a dynamically generating partition P with $H(P) < +\infty$, then $h_\mu(P)$ does not depend on the choice of such P. Define $h_{\Sigma,\mu}(X, G)$ as $h_{\Sigma,\mu}(P)$.

Question: how to define measure entropy in general for sofic group actions?

Example: For measure Bernoulli shift $G \curvearrowright (Y^G, \nu^G)$,

$$h_{\nu^G}(Y^G, G) = H(\nu),$$

if G is amenable, or sofic with $H(\nu) < +\infty$, where

$$H(\nu) = \sup_Q H(Q),$$

for Q ranging over partitions of Y.

If $Y = \{0, 1\}$, then

$$H(\nu) = -\nu(\{0\}) \log \nu(\{0\}) - \nu(\{1\}) \log \nu(\{1\}).$$

Ornstein-Weiss Example (1987):

Consider $G = \mathbb{F}_2$, the free group generated by 2 elements.

There is a $\mathbb{F}_2$-equivariant measure-preserving map from the Bernoulli shift $((x_1, x_2)^{\mathbb{F}_2}, (1/2, 1/2)^{\mathbb{F}_2})$ to the Bernoulli shift $((x_1, x_2, x_3, x_4)^{\mathbb{F}_2}, (1/4, 1/4, 1/4, 1/4)^{\mathbb{F}_2})$.

$H(1/2, 1/2) = \log 2$ while $H(1/4, 1/4, 1/4, 1/4) = \log 4$.

Thus, Sinai's definition via taking supremum over all partitions does not work.

History of topological entropy:

$$G \curvearrowright X.$$

Adler, Konheim, McAndrew (1965): $G = \mathbb{Z}$. Defined $h_{\mathrm{top}}(X, G)$ using open covers of X.

Rufus Bowen (1971): $G = \mathbb{Z}$. Definition using separated sets.

Moulin Ollagnier (1985): Extended the theory to countable amenable G.

Question: how to define topological entropy for sofic group actions?

Example: G amenable. For topological Bernoulli shift $G \curvearrowright Y^G$,

$$h_{\mathrm{top}}(Y^G, G) = \log |Y|.$$

Variational Principle: For $G \curvearrowright X$, G amenable,

$$h_{\mathrm{top}}(X, G) = \sup_{\mu} h_{\mu}(X, G),$$

for μ ranging over G-invariant Borel probability measures on X.

Sofic group:

For $d \in \mathbb{N}$, denote by $\mathrm{Sym}(d)$ the permutation group of $\{1, \ldots, d\}$.

A natural metric on $\mathrm{Sym}(d)$:

$$\rho(\varphi, \psi) = \frac{|\{a \in \{1, \ldots, d\} : \varphi(a) \neq \psi(a)\}|}{d}.$$

Gromov (1999):
G is *sofic* if for any finite $F \subseteq G$ and $\varepsilon > 0$, there are some $d \in \mathbb{N}$ and $\sigma : G \to \mathrm{Sym}(d)$ with

$$\rho(\sigma_{st}, \sigma_s \sigma_t) < \varepsilon \text{ for all } s, t \in F,$$

and

$$\rho(\sigma_s, \sigma_t) > 1 - \varepsilon \text{ for all distinct } s, t \in F.$$

Sofic groups $\supseteq$ amenable groups, residually finite groups ($\supseteq$ free groups), subgroups of $GL(n, k)$ for any field k, subgroups of compact groups.

Currently it is open whether every group is sofic.

Main idea for sofic topological entropy:

G: sofic group.
Fix $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}_{i\in\mathbb{N}}$ such that:
(1). For any $s, t \in G$,

$$\lim_{i\to\infty} \rho(\sigma_{i,st}, \sigma_{i,s}\sigma_{i,t}) \to 0;$$

(2). For any distinct $s, t \in G$,

$$\lim_{i\to\infty} \rho(\sigma_{i,s}, \sigma_{i,t}) \to 1;$$

(3). $\lim_{i\to\infty} d_i = \infty$. (This is a consequence of (2) if G is infinite.)

$$G \curvearrowright X.$$

Idea: Think of σ_i as an approximate action of G on $\{1, \ldots, d_i\}$. Use it as a model. Count the number of copies of this "action" in $G \curvearrowright X$, i.e., maps $\{1, \ldots, d_i\} \to X$ which are approximately equivariant.

Motivation:

1. The origin of $H(P)$ for finite partition P comes from counting number of partitions of $\{1, \ldots, d\}$ under the uniform probability measure approximating P, as $d \to \infty$.

2. In Lewis Bowen's work, count number of partitions of $\{1, \ldots, d_i\}$ under the uniform probability measure dynamically approximating P.

3. In Rufus Bowen's definition,

T : homeomorphism of X. ρ: a compatible metric on X.

For $n \in \mathbb{N}$, define a metric ρ_n on X:

$$\rho_F(x, y) = \max_{0 \leq k \leq n-1} \rho(T^k x, T^k y).$$

For $\varepsilon > 0$, $Z \subseteq X$ is (ρ_n, ε)-*separated* if $\rho_n(x, y) \geq \varepsilon$ for all distinct $x, y \in Z$.

Use $N_\varepsilon(X, \rho_n)$: the maximal cardinality of (ρ_n, ε)-separated subsets of X.

For $\sigma_n : \mathbb{Z} \to \mathrm{Sym}(n) = \mathrm{Sym}(\mathbb{Z}/n\mathbb{Z})$ by $\sigma_n(a)(b) = a + b \mod n$, the partial orbit $x, Tx, \ldots, T^{n-1}x$ of each point x of X can be counted as a "copy" of $\{1, \ldots, n\}$ in X.

Sofic topological entropy:

G: sofic group. Fix $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}_{i \in \mathbb{N}}$. $G \curvearrowright X$.

Let $d \in \mathbb{N}$ and $\sigma : G \to \mathrm{Sym}(d)$. Let ζ be the uniform probability measure on $\{1, \dots, d\}$.
ρ: a compatible metric on X.

Define a metric ρ_2 on the set of maps $\{1, \dots, d\} \to X$:

$$\begin{aligned}\rho_2(\varphi, \psi) &= \|\{1, \dots, d\} \ni a \mapsto \rho(\varphi(a), \psi(a))\|_2 \\ &= (\frac{1}{d}\sum_{a=1}^{d}(\rho(\varphi(a), \psi(a)))^2)^{1/2}.\end{aligned}$$

F: a nonempty finite subset of G. $\delta > 0$.
Consider the space of (F, δ)-approximately equivariant maps from $\{1, \dots, d\}$ to X:

$$\mathrm{Map}(\rho, F, \delta, \sigma) := \\ \{\varphi : \{1, \dots, d\} \to X : \rho_2(\alpha_s \circ \varphi, \varphi \circ \sigma_s) < \delta \text{ for all } s \in F\}.$$

Define

$$h_\Sigma(\rho) = \sup_{\varepsilon>0} \inf_{F} \inf_{\delta>0} \limsup_{i\to\infty} \frac{\log N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma_i), \rho_2)}{d_i}.$$

Theorem(Kerr, L. 2010): The number $h_\Sigma(\rho)$ does not depend on the choice of ρ. Define $h_\Sigma(X, G)$ to be this number.

Main idea for sofic measure entropy:

Consider $G \curvearrowright X$ and μ is a G-invariant Borel probability measure on X.

The approximate action σ_i preserves the uniform probability measure ζ on $\{1, \ldots, d_i\}$, thus can be used as a model for measure-presserving actions.

Just count the number of approximately equivariant maps $\{1, \ldots, d_i\} \to X$ which are also approximately measure-preserving.

Sofic measure entropy:

G: sofic group. Fix Σ. $G \curvearrowright (X', \mu')$.

(X', μ'): a standard probability space.

Then there exists a *topological model*:
a compact metrizable X, a continuous action $G \curvearrowright X$, a G-invariant Borel probability measure μ on X such that $G \curvearrowright (X', \mu')$ is isomorphic to $G \curvearrowright (X, \mu)$.

F: a nonempty finite subset of G. $\delta > 0$.
L: a finite subset of $C(X)$.
Consider the space of approximately equivariant and approximately measure-preserving maps from $\{1, \ldots, d\}$ to X:

$$\mathrm{Map}_\mu(\rho, F, L, \delta, \sigma) = \{\varphi \in \mathrm{Map}(\rho, F, \delta, \sigma) : \max_{f \in L} |\varphi_* \zeta(f) - \mu(f))| < \delta\}.$$

Define

$$h_{\Sigma,\mu}(\rho) = \sup_{\varepsilon>0} \inf_F \inf_L \inf_{\delta>0} \limsup_{i\to\infty} \frac{\log N_\varepsilon(\mathrm{Map}_\mu(\rho, F, L, \delta, \sigma_i), \rho_2)}{d_i}.$$

Theorem(Kerr, L. 2010): The number $h_{\Sigma,\mu}(\rho)$ does not depend on the choice of the topological model and ρ. Define $h_{\Sigma,\mu'}(X', G)$ to be this number.

Remark: In both topological and measurable cases, it is enough to require ρ to be a *dynamically generating* continuous pseudometric on X:
for any distinct $x, y \in X$, there exists $s \in G$ with $\rho(sx, sy) > 0$.

Outline of the proof for independence of the choice of topological model:

Take a *dynamically generating* sequence $\mathcal{S}$ in the unit ball of $L^\infty_{\mathbb{R}}(X', \mu')$: $\bigcup_{s \in G} s\mathcal{S}$ generates $L^\infty_{\mathbb{C}}(X', \mu')$ as a von Neumann algebra. Such $\mathcal{S}$ always exists.

Definition 1: Count the number of unital positive linear maps $L^\infty(X', \mu') \to \mathbb{C}^{d_i} = L^\infty_{\mathbb{C}}(\{1, \ldots, d_i\}, \zeta)$ which are approximately multiplicative, approximately equivariant, and approximately measure-preserving. (The definition uses $\mathcal{S}$.)

Definition 2: Fix $\lambda > 1$. As in Definition 1, but require the map to have L^2-norm at most λ.

Definition 3: Let $G \curvearrowright (X, \mu)$ be a topological model. Take a sequence $\mathcal{S}$ in the unit ball of $C_{\mathbb{R}}(X)$ dynamically generating $C_{\mathbb{R}}(X)$ as a unital C^*-algebra. Count the number of unital algebra homomorphisms $C_{\mathbb{R}}(X) \to \mathbb{C}^{d_i}$ which are approximately equivariant, and approximately measure-preserving.

Step 1: Show Definition 1 $\Leftrightarrow$ Definition 2.

Step 2: Use step 1 to show Definition 1 does not depend on the choice of $\mathcal{S}$.

Step 3: Show Definition 1 $\Leftrightarrow$ Definition 3.

Step 4: Show Definition 3 $\Leftrightarrow$ the topological model definition.

Relation with Lewis Bowen's work:

Range: In general, $h_\Sigma(X,G), h_{\Sigma,\mu}(X,G) \in \{-\infty\} \cup [0,+\infty]$.

Theorem(Kerr, L. 2010):
For $G \curvearrowright (X,\mu)$, when X has a generating partition P with $H(P) < +\infty$, our entropy coincides with Lewis Bowen's.

Amenable group case:

Suppose G is amenable.

Theorem(Lewis Bowen, 2010):
For $G \curvearrowright (X, \mu)$, when X has a finite generating partition, the sofic measure entropy $=$ the classical entropy.

Theorem(Kerr, L. 2010):
For $G \curvearrowright (X, \mu)$, the sofic measure entropy $=$ the classical entropy.

Theorem(Kerr, L. 2010):
For $G \curvearrowright X$, the sofic topological entropy $=$ the classical entropy.

Sofic entropy for Bernoulli shifts:

Theorem(Lewis Bowen, 2008):
If $H(\nu) < \infty$, then $h_{\Sigma,\nu^G}(Y^G, G) = H(\nu)$.

Theorem(Kerr, L. 2010):
If $H(\nu) = +\infty$, then $h_{\Sigma,\nu^G}(Y^G, G) = H(\nu)$.

Corollary(Kerr, L., 2010): If $H(\nu) = +\infty$ and G is sofic, then Y^G has no dynamically generating partition P with $H(P) < +\infty$.
(Lewis Bowen showed it in the case G is sofic and contains F_2.)

Theorem(Kerr, L. 2010):
$h_\Sigma(Y^G, G) = \log|Y|$. If $|Y| < +\infty$ and Z is a proper closed G-invariant subset of Y^G, then $h_\Sigma(Z, G) < \log|Y|$.

Corollary(Gromov, 1999): Gottschalk's question (1973) has positive answer when G is sofic: $\{1, \ldots, k\}^G$ has no proper subshift isomorphic to itself.

Variation principle for sofic entropy:

Theorem(Kerr, L. 2010): The variational principle holds for sofic entropy. In particular, if $h_{\Sigma}(X, G) \geq 0$, then X has a G-invariant Borel probability measure.

Principal algebraic actions:

For $f \in \mathbb{Z}G$, one may consider the shift action of G on the Pontryagin dual X_f of $\mathbb{Z}G/\mathbb{Z}Gf$. Denote by μ the normalized Haar measure on X_f.

Theorem (L. 2010): When G is amenable and f is invertible in $C^*(G)$,

$$h(X_f, G) = h_\mu(X_f, G) = \log \mathrm{det}_{\mathcal{L}G} f.$$

Theorem (Bowen, 2009): When G is residually finite, Σ comes from finite quotients of G, f is invertible in $\ell^1(G)$ (this is stronger than f being invertible in $C^*(G)$),

$$h_{\Sigma,\mu}(X_f, G) = \log \mathrm{det}_{\mathcal{L}G} f.$$

Theorem (Kerr, L. 2010): When G is residually finite, Σ comes from finite quotients of G, f is invertible in $C^*(G)$,

$$h_\Sigma(X_f, G) = \log \mathrm{det}_{\mathcal{L}G} f.$$

Open question:

Consider an action of G on a compact metrizable group X by automorphisms. When G is amenable, Deninger (2006) showed that the topological entropy is equal to the measure entropy for the normalized Haar measure. For sofic G, what conditions on G, Σ, or the action would guarantee this?