Embeddings and Hypergraph of Copies of Countable Homogeneous Structures

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General setting

Let $\mathcal{R} = (E, ..)$ be a structure and consider:

- $Aut(\mathcal{R})$ = automorphisms of $\mathcal{R}$
- $Emb(\mathcal{R})$ = embeddings of $\mathcal{R}$
- $\mathcal{I}so(\mathcal{R}) = \{range(f) : f \in Emb(\mathcal{R})\}$ = set of copies of $\mathcal{R}$

Basic facts

- $|Emb(\mathcal{R})| = |\mathcal{I}so(\mathcal{R})||Aut(\mathcal{R})|$
- $\overline{Aut(\mathcal{R})} \subseteq Emb(\mathcal{R})$
- If E countable, then $Aut(\mathcal{R})$ is a G_δ subset of $Emb(\mathcal{R})$
- If E countable, then $\mathcal{I}so(\mathcal{R})$ is an analytic subset of 2^E

Example

One can have two countable structures $\mathcal{R}$ and $\mathcal{R}'$, each embeddable in the other, but

$$\begin{array}{lll} |Aut(\mathcal{R})| = 1 & |\mathcal{I}so(\mathcal{R})| = \aleph_0 & |Emb(\mathcal{R})| = \aleph_0 \\ |Aut(\mathcal{R}')| = 2^{\aleph_0} & |\mathcal{I}so(\mathcal{R}')| = 2^{\aleph_0} & |Emb(\mathcal{R}')| = 2^{\aleph_0} \end{array}$$

Hypergraph of copies

What can be said about:

- $\mathcal{H}_{\mathcal{R}}$, the hypergraph of copies of $\mathcal{R}$?
- $Aut(\mathcal{H}_{\mathcal{R}})$, its automorphism group?

In particular

- How are $Aut(\mathcal{R})$ and $Aut(\mathcal{H}_{\mathcal{R}})$ related if at all?

Homogeneous structures

A structure is *(ultra)homogeneous* if every isomorphism between finite substructures extends to an automorphism of the entire structure.

Notice that in this case if $\overline{Aut(\mathcal{R})} = Emb(\mathcal{R})$

Examples:

Fraïssé classes	**Fraïssé limits**
Finite linear orders	Rationals
Finite graphs	Rado graph
Finite graphs omitting K_n	K_n-free graph
Finite metric spaces with rational dist.	Rational Urysohn space
And many more....	

Rationals

Observation

If $f : \mathbb{Q} \to \mathbb{Q}$ is a bijection preserving copies (and conversely), then f is order preserving or reverse order preserving.

So $Aut(\mathcal{H}_{\mathbb{Q}})$ "is" $Aut(\mathbb{Q}, <)$

Rationals

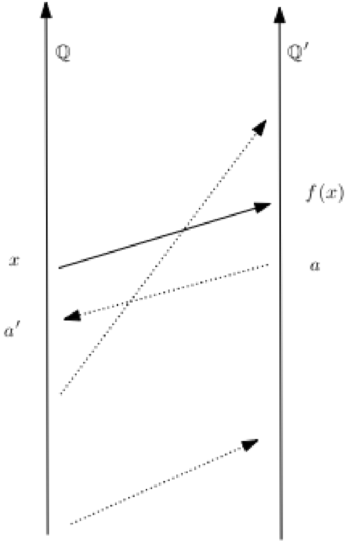
Proof.

Else fix x such that f sends $L = (-\infty, x)$ both above and below $f(x)$.

Now L is copy of $\mathbb{Q}$, but $L \cup \{x\} = (\infty, x]$ is not, so the same holds for their image.

Thus $f(x)$ either there is a largest element a in $f(L)$ below $f(x)$, or a least element in $f(L)$ above $f(x)$.

Let $a' = f^{-1}(a)$. But now $L \cup \{x\} \setminus \{a'\}$ is not copy, while its image is. $\square$



K_n-Free

Theorem

If $\Gamma = (V, E)$ is the K_n-free graph, then $Aut(\mathcal{H}_\Gamma)$ is $Aut(\Gamma)$

That is no bijection, other than graph isomorphisms of Γ, which preserves copies of Γ (and conversely).

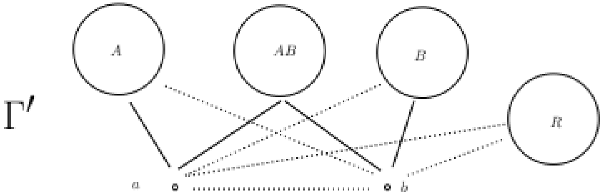
Proof.

(Triangle-Free) Let $f : V \to V$ preserve copies, and suppose some edge (a, b) is mapped to a non edge.
For simplicity define a new graph $\Gamma' = (V, E')$ by $(x, y) \in E' \leftrightarrow (f(x), f(y)) \in E$. So $X \subseteq V$ is a copy in Γ iff it is a copy in Γ'.

In Γ':

1. $A \cup B \cup R \cup \{a, b\}$ is NOT a copy.
2. $A \cup B \cup R \cup \{a\}$ IS a copy.
3. $A \cup B \cup R \cup \{b\}$ IS a copy.

So the same is true in Γ. But in Γ, 2&3 $\implies$ 1. Contradiction.



Rado Graph

Definition

For $\mathcal{R} = (E, ..)$ a relational structures, then $X \subseteq E$ is called **thin** *is does not contain a copy of $\mathcal{R}$.*

Theorem

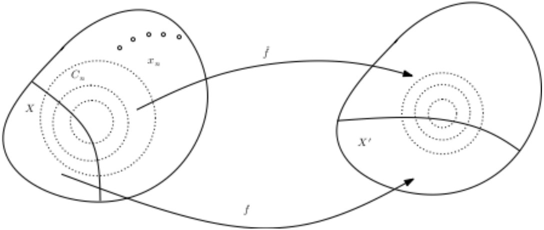
Let $\Gamma = (V, E)$ be the Rado graph, and X, X' two thin sets.

Then any bijection $f : X \to X'$ extends to an automorphism of $\mathcal{H}_\Gamma$.

Proof.

Let $f : X \to X'$ be a bijection between thin sets.
Write $V = \bigcup_n C_n$, and list $V \setminus X = < x_n : n \in \omega >$.
Extend f as $\hat{f} = \bigcup_n f_n$ such that for each n:

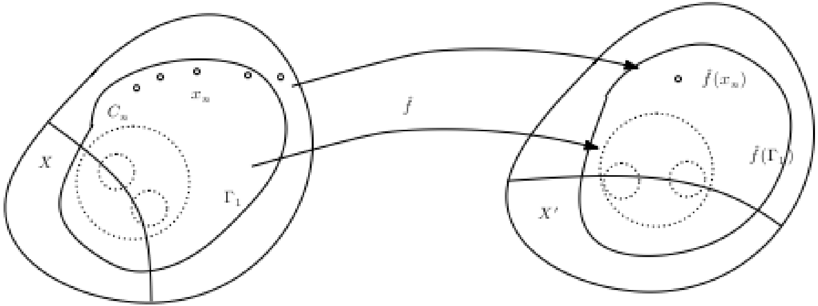
1. $dom(f_n) = C_n$
2. There is $k(n)$ so that for all $k \geq k(n)$ the type of x_k over C_n is the same as $\hat{f}(x_k)$ over $\hat{f}(C_n)$.



Proof.

(Cont'd) To show that it works, let Γ_1 be a copy of Γ, we show that $\hat{f}(\Gamma_1)$ is also a copy.

We need to realize every type in $\hat{f}(\Gamma_1)$.



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Question

- *What about the Urysohn space, or other homogeneous structures?*
- *If f is a bijection preserving copies, does the inverse also preserve copies?*
- *What about automorphisms of the poset $(\mathcal{I}so(G), \subseteq)$? Are they induced by automorphisms of the hypergraph?*