Point realizations of near-actions of groups of isometries

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Definitions

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Let $Aut(X, \lambda)$ denote the group of measure preserving automorphisms of (X, λ). Equipped with the weakest topology such that evey map (for every A)

$$g \to \lambda(A \triangle gA)$$

is continuous is a Polish group.

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A measure preserving Borel action of G on (X, λ) induces a near-action.

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Remark

A measure preserving Borel action of G *on* (X, λ) *induces a near-action.*

Question (Main Question)

Given a Polish group G*. Does every near-action of* G *admit a spatial model (i.e. is it induced from some Borel measure preserving action of* G*)?*

Positive examples

Theorem (Mackey, 1962)

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Definition

By S_∞ we denote the group of all permutations of natural numbers. This is a Polish group with the pointwise topology.

Positive examples 2

Theorem

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Theorem (Glasner-Weiss, 2005)

Every near-action of every closed subgroup of S_∞ admits a spatial model.

Negative examples and whirly actions

Definition (Glasner-Tsirelson-Weiss)

A near-action of G on (X, λ) is whirly if for any A of positive measure and any neighbourhood V of the identity in G, $\lambda(VA) = 1$.

Negative examples and whirly actions

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Proposition (Glasner-Tsirelson-Weiss)

No whirly action can admit a spatial model.

Groups of isometries

Examples of Polish groups of isometries of separable locally compact metric spaces:

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Theorem (Gao-Kechris)

Polish groups of isometries of separable locally compact metric spaces are precisely closed subgroups of

$$\prod_{n\in\mathbb{N}} S_\infty \ltimes H_n^{\mathbb{N}},$$

where H_n are locally compact Polish.

Main Theorem

Theorem (K.-Solecki)

Every near-action of a Polish group of isometries of a separable locally compact metric space admits a spatial model.

Characterization of the group of isometries

Theorem (K.-Solecki)

Let G be Polish. Then, the following are equivalent.

1. *G is a group of isometries of a separable locally compact metric space.*
2. *Each neighbourhood V of 1 in G contains a closed subgroup H such that $N(H)$ (the normalizer of H) is open and $N(H)/H$ is a locally compact group.*

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“$N(H)/H$ is a locally compact group“ can be replaced by ”$N(H)/H$ is a Lie group“.

Projective Lie groups

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Theorem (Montgomery-Zippin)

Suppose that G is a locally compact separable group. Then there is an open subgroup $G' < G$ that is Lie projective.

Closure properties of a group of isometries

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Groups of isometries are closed under taking:

1. countable products,
2. closed subgroups,
3. images of continuous homomorphisms onto Polish groups.

About the proof

Let G be Polish. Suppose that each neighbourhood V of 1 in G contains a closed normal subgroup H such that G/H is locally compact. Then, G is a group of isometries of a separable locally compact metric space.

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1. Take $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$ with diameters going to 0.
2. Take $H_i < G$ with $H_i \subseteq V_i$.
3. Let $f\colon G \to \mathrm{Iso}(\bigoplus_i (G/H_i))$ be given by

$$f(g)(xH_i) = (gx)H_i.$$