From Discrete to Continuous Arguments in Model Theory

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Here we focus on model-theoretic frameworks.

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- First-order logic is not adequate as a model-theoretic language for certain structures, e.g., Banach spaces. (The first-order theory of the class of Banach spaces is essentially equivalent to a second order logic [Shelah-Stern, 1978].)
- Nevertheless, ideas adapted from first-order model theory have provided powerful applications to functional analysis. [Krivine, 1976], [Krivine-Maurey, 1981].
- Currently in model theory, there is considerable activity in trying to replicate in non first-order contexts the successful development of Shelah's stability and classification theory.

Classical model theory

- First-order syntax

  Logical symbols Connectives ($\wedge$, $\vee$, $\rightarrow$ $\neg$), quantifiers ($\exists$, $\forall$).

  Nonlogical symbols symbols for functions, relations, and constants (these depend on the classes of structures being considered).

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- Semantics

  If φ is sentence, or a set of sentences in a first-order language L, and M is an L-structure, we write

  $$M \models \varphi$$

  φ is true when interpreted in M. We say that M *satisfies* φ or that M is a *model* of φ.

Let L be a first-order language and let M, N be L-structures. We say that M and N are *elementary equivalent*, written

$$M \equiv N,$$

if M and N satisfy the same L-sentences.
If M is a substructure of N, we say that M is an *elementary substructure* of N, written

$$M \prec N,$$

if

$$(M, a \mid a \in M) \equiv (M, a \mid a \in M).$$

Compactness and Löwenheim-Sklolem

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The (Downward) Löwenheim-Sklolem Theorem

Let L be a first-order language and let M be an L-structure. Then there exists a countable structure M_0 such that $M_0 \prec M$ and $|M_0| \leq |M| + |L|$.

Lindström's Theorem

Theorem (P. Lindström, 1969)

Let $\mathcal{L}$ be a logic such that

1. $\mathcal{L}$ extends first-order logic
2. $\mathcal{L}$ satisfies the Compactness and Downward Löwenheim-Skolem properties.

Then $\mathcal{L}$ is equivalent to first-order logic. That is, every sentence of $\mathcal{L}$ is equivalent to a first-order sentence.

Realizing types

If Σ is a set of L-formulas and $x_1, \ldots, x_n$ are variables, we write Σ as

$$\Sigma(x_1, \ldots, x_n)$$

to indicate that the free variables of every formula in Σ are among $x_1, \ldots, x_n$.

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Let $\Sigma(x_1, \ldots, x_n)$ be a set of sentences. If there exists a structure M and elements $a_1, \ldots, a_n \in M$ such that

$$M \models \Sigma[a_1, \ldots, a_n]$$

we say that Σ is *consistent*, or that Σ is a *type* and that $(a_1, \ldots, a_n)$ *realizes* Σ in M.

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Principal Types

Let $\Gamma(x_1, \ldots, x_n), \Sigma(x_1, \ldots, x_n)$ be sets of formulas. We write

$$\Gamma(x_1, \ldots, x_n) \models \Sigma(x_1, \ldots, x_n)$$

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Let T be a theory and let $\Sigma(x_1, \ldots, x_n)$ be a type consistent with T. We say that Σ is *principal* if there exists a formula $\varphi(x_1, \ldots, x_n)$ consistent with T such that

$$T, \varphi(x_1, \ldots, x_n) \models \Sigma(x_1, \ldots, x_n).$$

In this case, we say that φ is a *generator* of Σ.

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Corollary

Let L be countable. Let T be a complete L-theory (i.e., if for every L-formula φ, either φ or $\neg\varphi$ is consistent with T) and let Σ be a type consistent with T. Then Σ is realized in all the models of T if and only if Σ is principal.

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Remark: The countability assumption here cannot be removed.

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- It is adequate for wider classes of structures (e.g., metric spaces, C^*-algebras)
- The "nice" characteristics of first-order model theory (e.g, compactness, omitting types) are preserved?

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- Ben Yaacov-Usvyatsov (2007): Continuous logic.

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For the purposes of this talk, I will use the name "continuous logic" to refer to any of these frameworks.

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2. Is there a more powerful approach, i.e., is there a logic with more expressive power than those listed above which
   - expands first-order model theory to include these structures, and yet
   - preserves desirable characteristics of first-order model theory?

All of these frameworks satisfy:

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All of them are “positive” in the sense that the logic does not have a classical negation. (In fact, if one adds negation, the expressive power becomes equivalent to that of first-order logic.)
However, they have a “weak negation” which, through approximations, serves as a replacement of the classical negation for many purposes.

Questions

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Answer

No. Recently, Caicedo has exhibited examples of proper extensions of continuous logic that satisfy the Compactness Theorem and proper extensions of continuous logic that satisfy the classical Omitting Types Theorem.

Abstrac Model theory

If L and L' are multi-sorted languages, a *renaming* is a bijection $r\colon L \to L'$ that maps sort symbols onto sort symbols, relation symbols onto relation symbols, and function symbols onto function symbols, and respects sorts and arities. If $r\colon L \to L'$ is a renaming and $\mathcal{M}$ is an L-structure, $\mathcal{M}^r$ denotes the structure that results from converting $\mathcal{M}$ into an L'-structure through r. We call the map $\mathcal{M} \mapsto \mathcal{M}^r$, too, a renaming.

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3. A binary relation $\models$, called *satisfaction*, between structures and sentences of $\mathcal{L}$ such that:

(a) If $\mathcal{M}$ is an *L*-structure of $\mathcal{L}$ and $\mathcal{M} \models \varphi$, then $\varphi \in \mathcal{L}[L]$.

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(c) *Reduct Property.* If $L \subseteq L'$, $\mathcal{M}$ is a L'-structure of $\mathcal{L}$ and $\varphi \in \mathcal{L}[L]$, then $\mathcal{M} \models \varphi$ if and only if $\mathcal{M} \restriction L \models \varphi$;

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(d) *Renaming Property.* Suppose that $r\colon L \to L'$ is a renaming. Then for each sentence $\varphi \in \mathcal{L}[L]$ there exists a sentence $\varphi^r \in \mathcal{L}[L]$ such that $\mathcal{M} \models \varphi$ if and only if $\mathcal{M}^r \models \varphi^r$.

If $\mathcal{L}$ is a logic,

- The class of sentences of $\mathcal{L}$ is denoted $\mathrm{Sent}(\mathcal{L})$
- The class of structures of $\mathcal{L}$ is denoted $\mathrm{Str}(\mathcal{L})$.

A *theory* is a subclass of $\mathrm{Sent}(\mathcal{L})$. If T is a theory,

$$\mathrm{Mod}(T) = \{ M \in \mathrm{Str}(\mathcal{L}) \mid M \models T \}.$$

The classes $\mathrm{Mod}(T)$ form the closed sets for a topology on $\mathrm{Str}(\mathcal{L})$. We will refer to this topology as the *logical topology* of $\mathcal{L}$.

A logic $\mathcal{L}$ is said to have *negations* if for every sentence $\varphi \in \mathrm{Sent}(\mathcal{L})$ there exists a sentence $\varphi \in \mathrm{Sent}(\mathcal{L})$ such that

$$\mathcal{M} \models \psi \qquad \text{if and only if} \qquad \mathcal{M} \not\models \varphi.$$

Note that a logic has negations if and only if its logical topology has a base consisting of clopen sets.

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Recall that continuous logic does not have negations. Furthermore, adding classical negations to it results in classical (discrete) first-order logic, which, for continuous structures, has too high an expressive power. [Shelah-Stern, op. cit.]

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However, continuous logic does have a feature that, for practical applications, takes the role of negation:

Regular Logics

Definition

We will say that a logic is *regular* if its logical topology is regular.

Definition

A logic $\mathcal{L}$ is *compact* if it satisfies the Compactness Theorem. A logic is *locally compact* if for every structure $\mathcal{M}$ there is a sentence φ such that $\mathcal{M} \models \varphi$ and the Compactness Theorem holds for types containing φ.

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Corollary

There are proper extensions of continuous logic that satisfy compactness and the classical Omitting Types Theorem.

κ-principal types

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Let κ be an infinite cardinal and let T be a theory. If $\Sigma(x_1, \ldots, x_n)$ is a type consistent with T, we say that Σ is *κ-principal* if there exists a set of formulas $\Gamma(x_1, \ldots, x_n)$, consistent with T and satisfying $|\Gamma| < \kappa$, such that

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If $|T|, |\Sigma| \leq \kappa$ and Σ is not κ-principal, then there is a model of T that omits Σ.

Theorem (Brucks, Caicedo, Iovino)

Let $\mathcal{L}$ be a locally compact regular logic. Then, if κ is regular, $|T|, |\Sigma| \leq \kappa$, and Σ is not κ-principal, then there is a model of T that omits Σ.

The proof of this theorem is topological. It uses an uncountable version of the Baire category theorem.

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The version of this result for logics with negation (and existential quantifier) is not new; it is known as the Chang-Kreisel-Krivine omitting types theorem for uncountable languages.

Let $\mathcal{L}, \mathcal{L}'$ be locally compact regular logics and let $\mathcal{U}, \mathcal{U}'$ be uniform structures compatible with their respective logical topologies. We will say that $\mathcal{L}'$ *extends* $\mathcal{L}$ if for every $\varphi \in$ Sent($\mathcal{L}'$) and every $U \in \mathcal{U}'$ there exist $\psi \in \mathcal{L}$ and $V \in \mathcal{U}$ such that

$$\text{Mod}(\varphi) \subseteq \text{Mod}(\psi)$$

and

$$V\text{-thickening of } \text{Mod}(\psi) \subseteq U\text{-thickening of } \text{Mod}(\varphi).$$

Intuitively, this means that every sentence in $\mathcal{L}'$ is a uniform limit of sentences in $\mathcal{L}$.

We will say that two logics are *equivalent* if they extend each other.

The Main Result

Theorem (Brucks, Caicedo, Iovino)

Let $\mathcal{L}$ be a regular logic such that $\mathcal{L}$

- extends continuous logic,
- is locally compact,
- satisfies the κ-Omitting Types Theorem for some regular uncountable cardinal κ.

Then $\mathcal{L}$ is equivalent to continuous logic.