Classification of C*-algebras and set theory

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(joint work with Andrew Toms and Asger Törnquist)

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A (concrete) *C*-algebra* is a norm-closed subalgebra of $\mathcal{B}(H)$.

Theorem (Gelfand–Naimark–Segal, 1942)

A Banach algebra with involution A is isomorphic to a concrete C-algebra if and only if*

$$\|aa^*\| = \|a\|^2$$

for all $a \in A$.

Examples

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(2) If X is a compact metric space, $C(X)$.

$$C(X) \cong C(T) \qquad \Leftrightarrow \qquad X \cong Y.$$

(3) If (X, α) is a minimal dynamical system, $C(X) \rtimes_\alpha \mathbb{Z}$.

$$(X, \alpha) \cong (Y, \beta) \qquad \Rightarrow \qquad C(X) \rtimes_\alpha \mathbb{Z} \cong C(Y) \rtimes_\beta \mathbb{Z}.$$

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AF algebras are direct limits of finite-dimensional C*-algebras.

Theorem (Elliott, 1975)

Separable AF algebras are classified by the ordered group $(K_0(A), K_0(A)^+, 1)$.

Classification results, II

Theorem (Kirchberg–Phillips, 1995)

All purely infinite, nuclear, separable, simple, unital C-algebras with UCT are classified by their K-theoretic invariant.*

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Theorem (Elliott–Evans, 1993)

Irrational rotation algebras are are classified by their K-theoretic invariant.

Elliott program

All nuclear, separable, simple, unital C*-algebras are classified by the Elliott invariant,

$$((K_0(A), K_0(A)^+, 1), K_1(A), T(A), \rho_A).$$

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$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Ell}(A) \arrow[d] \\ B \arrow[r] & \mathrm{Ell}(B) \end{tikzcd}$$

Rørdam, Toms, 2004 - counterexamples.

New directions

1. Classification of nuclear, simple, unital, separable, $\mathcal{Z}$-stable C*-algebras.
2. Cuntz semigroup as an invariant.

Descriptive set theory: Abstract classification

Assume the collection X of objects we are trying to classify forms a 'nice' space, typically a Polish space or a standard Borel space and the equivalence relation E is a Borel or analytic subset of X^2. (*Analytic* set is a continuous image of a Borel set.)

The basic concept of abstract classification

Definition

If (X, E) and (Y, F) are equivalence relations, E is *Borel-reducible* to F, in symbols

$$E \leq_B F,$$

if there is a Borel-measurable map $f \colon X \to Y$ such that

$$x \, E \, y \Leftrightarrow f(x) \, E \, f(y).$$

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Example

Spectral theorems.

The big picture: Borel equivalence relations

If $E \leq_B =^{\mathbb{R}}$ we say E is *smooth*.

OER: Orbit equivalence relation of a continuous Polish group action on a Polish space.

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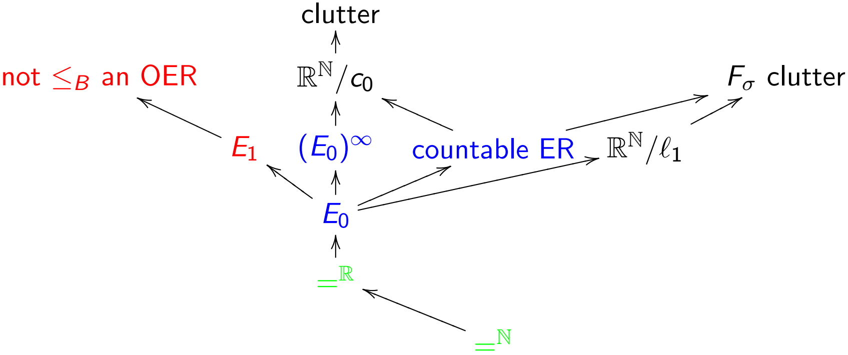

Modelling classification problems I

Example (The Polish space of countable groups)

A countable group G is coded by
$(\mathbb{N}, e_G, x_G, {}^{-1}{}_G)$, for $e \in \mathbb{N}$, $\times_G : \mathbb{N}^2 \longrightarrow \mathbb{N}$, ${}^{-1}{}_G : \mathbb{N} \longrightarrow \mathbb{N}$.

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$\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}^3) \times \mathcal{P}(\mathbb{N}^2)$.

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The isomorphism $\cong^G$ is an S_∞-orbit equivalence relation.

Modelling classification problems II

In general, a given concrete classification problem for category $\mathcal{C}$ is modelled by a standard Borel space (X, Σ) and $F\colon X \twoheadrightarrow \mathcal{C}$ such that the relation E on X,

$$x \, E \, y \Leftrightarrow F(x) \cong F(y)$$

is analytic (i.e., a continuous image of a Borel set).

Classification by countable structures

An equivalence relation (X, E) is classified by countable structures if there is a countable language L and a Borel map from X into countable L-models such that

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Lemma (Sasyk–Törnquist 2009, after Hjorth)

If $G \subsetneq F$ are separable Banach spaces, G is dense in F, and $\mathrm{id}\colon G \to F$ is bounded, then the coset equivalence F/G cannot be classified by countable structures.

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c_0/ℓ_2.

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Theorem (Sasyk–Törnquist, 2009)

Type II_1 factors are not classifiable by countable structures. The same result applies to II_∞ factors and III_λ factors for $0 \leq \lambda \leq 1$, to injective III_0 factors and to ITPFI factors.

Effros Borel space

For a Polish space X let X^* be the space of closed subsets of X. The σ-algebra Σ on X^* is generated by sets

$$\{A \in X^* : A \subseteq U\}$$

where U ranges over open subsets of X.

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Proposition

(X^*, Σ) is a standard Borel space. If X is a separable C-algebra then*

$$S(X) = \{B \in X^* : B \text{ is a subalgebra of } X\}$$

is a Borel subspace of X^*.

Examples

Theorem (Kirchberg, 1994)

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Theorem (Pisier–Junge, 1995)

$S(A)$ is not the space of all separable C-algebras for any separable C*-algebra A.*

Borel space of separable C*-algebras

Definition (Kechris, 1996)

Let Γ *be* $\mathcal{B}(\ell_2)^{\mathbb{N}}$, *with respect to the weak operator topology. Then*

$$\Gamma \ni \gamma \mapsto C^*(\gamma)$$

maps Γ *onto the space of all separable C*-algebras represented on* H, *and*

$$\gamma_0 \, E \, \gamma_1 \Leftrightarrow C^*(\gamma_0) \cong C^*(\gamma_1)$$

is analytic.

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is analytic.

There is also a space Δ of abstract separable C*-algebras.

Two representations are *equivalent*.

Lemma (Kechris, 1996)

There are Borel maps $\Phi_j \colon \Gamma \to \Gamma$ ($j = 1, 2, 3$) such that

1. $\Phi_1(\gamma)$ *enumerates a norm-dense subset of* $C^*(\gamma)$,
2. $\Phi_2(\gamma)$ *enumerates a norm-dense subset of* $C^*(\gamma)_+$,
3. $\Phi_3(\gamma)$ *enumerates a norm-dense subset of the projections of* $C^*(\gamma)$.

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Proposition (Effros, 1996)

The set $\{\gamma \in \Gamma : C^*(\gamma) \text{ is nuclear}\}$ *is Borel.*

Classification problem of C*-algebras

Lemma (Farah–Toms–Törnquist, 2009)

There is a Borel map $\Psi\colon \Gamma^2 \to \Gamma$ *such that*

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Actually we can do this for AI algebras.

Classifiable C*-algebras are not classifiable. . . by countable structures

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Classifiable C*-algebras are not classifiable... by countable structures

Theorem (Elliott, 1993)

AI algebras are classified by the Elliott invariant.

Theorem (Farah–Toms–Törnquist, 2009)

If L is a countable language, then the isomorphism of countable L-models is $\leq_B$ *to the isomorphism of AI algebras.*

The top

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The Cuntz semigroup, $W(A)$

Theorem (Perera–Toms, 2007)

Upon restriction to $\mathcal{Z}$-stable (simple, separable, nuclear, unital) C-algebras,* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ *if and only if* $(W(A), K_1(A)) \cong (W(B), K_1(B))$.

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On positive elements of a C*-algebra A define the relation $\precsim$ by

$$a \precsim b \Leftrightarrow (\forall \varepsilon > 0)(\exists x \in A)\|a - xbx^*\| < \varepsilon$$

and let $a \sim b$ iff $a \precsim b$ and $b \precsim a$.

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and let $a \sim b$ iff $a \precsim b$ and $b \precsim a$.
Cuntz semigroup, $W(A)$, is the quotient structure of $(A \oplus \mathcal{K}, \precsim, +, \ll)$ with respect to $\sim$, where

$$a + b := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

and $\ll$ is a subrelation of $\precsim$ called *compact containment*.

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Proposition (Farah–Toms–Tørnquist, 2010)

There is a Borel map $\Phi\colon \Gamma \to \mathbf{Cu}$ *such that the equivalence relation* E *on* **Cu**

$$\Phi(\gamma) \, \mathrm{E} \, \Phi(\gamma') \textit{ if and only if } \mathrm{Cu}(C^*(\gamma)) \cong \mathrm{Cu}(C^*(\gamma'))$$

is analytic.

Sketch of the proof: $\gamma \mapsto \mathrm{Cu}(C^*(\gamma)$ is Borel

Replace $\gamma \in \Gamma$ with γ' such that

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Replace $\gamma \in \Gamma$ with γ' such that

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By Kechris, we have effective enumerations:
(x_n) of a dense subset of $C^*(\gamma) \otimes \mathcal{K}$ and
(a_n) of a dense sequence of positive elements of $C^*(\gamma) \otimes \mathcal{K}$.

Sketch of the proof: $\gamma \mapsto \mathrm{Cu}(C^*(\gamma)$ is Borel, II

Define $\precsim_\gamma$ on $\mathbb{N}$ by

$$m \precsim_\gamma n \qquad \Leftrightarrow \qquad (\forall i)(\exists j)\|x_j a_n x_j^* - a_m\| < 1/i$$

The map $\Gamma \ni \gamma \mapsto \precsim_\gamma \in \mathcal{P}(\mathbb{N})^2$ is Borel.

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Map $\Gamma \ni \gamma \mapsto +_\gamma \in \mathcal{P}(\mathbb{N})^3$ is similarly Borel...

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...and so is $\Gamma \ni \gamma \mapsto \ll_\gamma \in \mathcal{P}(\mathbb{N})^2$ (here $\ll$ is the compact containment relation).

The quotient structure on $\mathbb{N}$, wrt $\approx_\gamma$, $+_\gamma$ and $\precsim_\gamma$, belongs to **Cu**.

Sketch of the proof: $\gamma \mapsto \mathrm{Cu}(C^*(\gamma))$ is Borel, III: Recovering Cu

Given $D = (\mathbb{N}, \precsim, +, C) \in \mathbf{Cu}$, define $\tilde{D}$ and $D^{\nearrow}$ as follows.

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Given $D = (\mathbb{N}, \precsim, +, C) \in \mathbf{Cu}$, define $\tilde{D}$ and $D^{\nearrow}$ as follows. Let $D^{\nearrow}$ be the set of $\ll$-increasing sequences in D, and let

$$(x_n) \leq (y_n) \quad \Leftrightarrow \quad (\forall m)(\exists n) x_m \ll y_n \text{ and } y_m \ll x_n$$

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Let $(x_n) \sim (y_n)$ iff $(x_n) \leq (y_n)$ and $(y_n) \leq (x_n)$.
Then $\tilde{D} = D^{\nearrow} / \sim$ is the Cuntz semigroup of $C^*(\gamma)$.

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The isomorphism relation is analytic

Lemma

The relation on $\mathbf{Cu}$ *defined by* $D_1 \mathrel{\mathsf{E}} D_2$ *iff* $\tilde{D}_1 \cong \tilde{D}_2$ *is analytic.*

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Lemma

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Proof.

$\tilde{D}_1 \cong \tilde{D}_2$ iff:

$$\begin{aligned} (\exists \Phi_1)\colon D_1 &\to D_2^{\nearrow}, \ \text{homomorphism} \\ (\exists \Phi_2)\colon D_2 &\to D_1^{\nearrow}, \ \text{homomorphism} \\ \Phi_1 \circ \Phi_2 &= \mathrm{id}_{D_2} / \sim \\ \Phi_2 \circ \Phi_1 &= \mathrm{id}_{D_1} / \sim \end{aligned}$$

$\square$

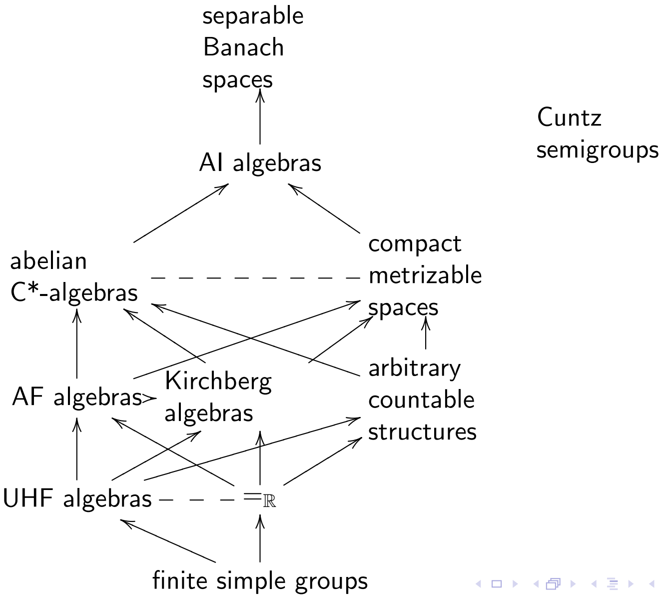
Corollary

The set $\{\gamma \in \Gamma : C^*(\gamma)$ *is simple* $\}$ *is Borel.*

Proof.

A is simple if and only if $\mathrm{Cu}(A \otimes \mathcal{O}_2)$ has only one positive element. $\square$

Relative complexity of some isomorphism relations



Some problems

Question

Is the isomorphism of all separable, unital C-algebras* $\leq_B$ *orbit equivalence relation?*

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Is the isomorphism of all separable, unital C-algebras* $\leq_B$ *orbit equivalence relation?*
What about the not necessarily simple nuclear C-algebras? Exact C*-algebras? Arbitrary C*-algebras?*

Some problems

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Is the isomorphism of all separable, unital C-algebras* $\leq_B$ *orbit equivalence relation?*
What about the not necessarily simple nuclear C-algebras? Exact C*-algebras? Arbitrary C*-algebras?*

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Develop set-theoretic framework for Elliott's functorial classification.

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Problem

Is the isomorphism of countably determined Cuntz semigroups $\leq_B$ *orbit equivalence relation?*