Homeomorphic Measures on a Cantor Set

S. Bezuglyi

October 13, 2010
Workshop on concentration phenomenon,
transformation groups and Ramsey theory
Fields Institute

Based on joint paper with O. Karpel

Plan of the talk

- Motivation
- Bernoulli measures
- Good and refinable measures on a Cantor set
- Measures on stationary Bratteli diagrams
- Main results

Homeomorphic measures

Definition

Two probability Borel measures μ and ν defined on a topological space Y are called homeomorphic or topologically equivalent (notation $\mu \sim \nu$) if there exists a self-homeomorphism f of Y such that $\mu = \nu \circ f$, i.e. $\mu(A) = \nu(f(A))$ for every Borel subset A of Y.

Homeomorphic measures

Definition

Two probability Borel measures μ and ν defined on a topological space Y are called homeomorphic or topologically equivalent (notation $\mu \sim \nu$) if there exists a self-homeomorphism f of Y such that $\mu = \nu \circ f$, i.e. $\mu(A) = \nu(f(A))$ for every Borel subset A of Y.

Notation:

$H(Y) =$ the group of all homeomorphisms of Y,
$M(Y) =$ Borel probability non-atomic measures on Y.

Homeomorphic measures

Definition

Two probability Borel measures μ and ν defined on a topological space Y are called homeomorphic or topologically equivalent (notation $\mu \sim \nu$) if there exists a self-homeomorphism f of Y such that $\mu = \nu \circ f$, i.e. $\mu(A) = \nu(f(A))$ for every Borel subset A of Y.

Notation:
$H(Y) =$ the group of all homeomorphisms of Y,
$M(Y) =$ Borel probability non-atomic measures on Y.

Problems:
(1) Classify measures from $M(Y)$ (or some natural subsets of $M(Y)$) with respect to $\sim$.
(2) Given $\mu \in M(Y)$, describe the class of measures equivalent to μ.

Oxtoby-Ulam Theorem

Theorem

Oxtoby - Ulam (1941): A non-atomic Borel probability measure μ on the finite-dimensional cube $[0,1]^n$ is homeomorphic to the Lebesgue measure if and only if every nonempty open set has a positive measure (in other words, μ is full) and the boundary of the cube has measure 0.

Any two probability measures on $[0,1]^n$ satisfying the Oxtoby-Ulam theorem are *homeomorphic*.

Oxtoby-Ulam Theorem

Theorem

Oxtoby - Ulam (1941): A non-atomic Borel probability measure μ on the finite-dimensional cube $[0, 1]^n$ is homeomorphic to the Lebesgue measure if and only if every nonempty open set has a positive measure (in other words, μ is full) and the boundary of the cube has measure 0.

Any two probability measures on $[0, 1]^n$ satisfying the Oxtoby-Ulam theorem are *homeomorphic*.

Oxtoby - Prasad (1978) extended the previous result to the Hilbert cube $[0, 1]^{\mathbb{N}}$. More results about measures on manifolds can be found in the book by *Alpern - Prasad (2004)*.

Oxtoby-Ulam Theorem

Theorem

Oxtoby - Ulam (1941): A non-atomic Borel probability measure μ on the finite-dimensional cube $[0,1]^n$ is homeomorphic to the Lebesgue measure if and only if every nonempty open set has a positive measure (in other words, μ is full) and the boundary of the cube has measure 0.

Any two probability measures on $[0,1]^n$ satisfying the Oxtoby-Ulam theorem are *homeomorphic*.

Oxtoby - Prasad (1978) extended the previous result to the Hilbert cube $[0,1]^{\mathbb{N}}$. More results about measures on manifolds can be found in the book by *Alpern - Prasad (2004)*.

E.Akin (1999, 2005) initiated the systematic study of homeomorphic measures on a *Cantor set*, i.e. on a 0-dimensional compact metric space without isolated points.

Motivation

1. “Thing-in-itself” (c)

Motivation

1. “Thing-in-itself” (c)

2. Orbit equivalence.
Let $f \in H(X)$ and $g \in H(Y)$ be two homeomorphisms of Cantor sets X and Y. They are called *orbit equivalent* if there exists a homeomorphism $h : X \to Y$ such that $h(Orb_f(x)) = Orb_g(hx)$ for all $x \in X$.

Motivation

1. “Thing-in-itself” (c)

2. Orbit equivalence.
Let $f \in H(X)$ and $g \in H(Y)$ be two homeomorphisms of Cantor sets X and Y. They are called *orbit equivalent* if there exists a homeomorphism $h : X \to Y$ such that $h(Orb_f(x)) = Orb_g(hx)$ for all $x \in X$.

Giordano - Putnam - Skau (1995): For a *minimal* homeomorphism f of X, the set $M_f(X)$ of probability ergodic invariant measures is a *complete* invariant of orbit equivalence. So, if there is a homeomorphism $F : X \to Y$ that sends $M_g(Y)$ onto $M_f(X)$, then the minimal homeomorphisms f and g are orbit equivalent.

Motivation

1. “Thing-in-itself” (c)

2. Orbit equivalence.
Let $f \in H(X)$ and $g \in H(Y)$ be two homeomorphisms of Cantor sets X and Y. They are called *orbit equivalent* if there exists a homeomorphism $h : X \to Y$ such that $h(Orb_f(x)) = Orb_g(hx)$ for all $x \in X$.

Giordano - Putnam - Skau (1995): For a *minimal* homeomorphism f of X, the set $M_f(X)$ of probability ergodic invariant measures is a *complete* invariant of orbit equivalence. So, if there is a homeomorphism $F : X \to Y$ that sends $M_g(Y)$ onto $M_f(X)$, then the minimal homeomorphisms f and g are orbit equivalent.

For aperiodic homeomorphisms, the problem of orbit equivalence is open.

Homeomorphic Bernoulli trial measures

Let $X = 2^{\mathbb{N}}$, $\mu_r = \otimes_i \mu_i$ where $\mu_i(\{1\}) = r$ and $\mu_i(\{0\}) = 1 - r$.

Homeomorphic Bernoulli trial measures

Let $X = 2^{\mathbb{N}}$, $\mu_r = \otimes_i \mu_i$ where $\mu_i(\{1\}) = r$ and $\mu_i(\{0\}) = 1 - r$.

Navarro-Bermudez (1979), Huang (1984): (1) Let r be a rational, transcendental, or an algebraic integer number of degree 2. Then $\mu_r \sim \mu_s \Leftrightarrow r = s$ or $r = 1 - s$. (2) There are continuum equivalence classes of Bernoulli measures on the Cantor space $2^{\mathbb{N}}$. Each class is at most countable.

In particular, $\mu_{1/2}$ and $\mu_{1/4}$ are not homeomorphic.

Homeomorphic Bernoulli trial measures

Let $X = 2^{\mathbb{N}}$, $\mu_r = \otimes_i \mu_i$ where $\mu_i(\{1\}) = r$ and $\mu_i(\{0\}) = 1 - r$.

Navarro-Bermudez (1979), Huang (1984): (1) Let r be a rational, transcendental, or an algebraic integer number of degree 2. Then $\mu_r \sim \mu_s \Leftrightarrow r = s$ or $r = 1 - s$. (2) There are continuum equivalence classes of Bernoulli measures on the Cantor space $2^{\mathbb{N}}$. Each class is at most countable.

In particular, $\mu_{1/2}$ and $\mu_{1/4}$ are not homeomorphic.

Oxtoby, Navarro-Bermudez (1988): Let t be the solution of $x^3 + x^2 - 1 = 0$ taken from $(0, 1)$. Then $\mu_t \sim \mu_{t^2} \sim \mu_{1-t} \sim \mu_{1-t^2}$.

Homeomorphic Bernoulli trial measures

Let $X = 2^{\mathbb{N}}$, $\mu_r = \otimes_i \mu_i$ where $\mu_i(\{1\}) = r$ and $\mu_i(\{0\}) = 1 - r$.

Navarro-Bermudez (1979), Huang (1984): (1) Let r be a rational, transcendental, or an algebraic integer number of degree 2. Then $\mu_r \sim \mu_s \Leftrightarrow r = s$ or $r = 1 - s$. (2) There are continuum equivalence classes of Bernoulli measures on the Cantor space $2^{\mathbb{N}}$. Each class is at most countable.

In particular, $\mu_{1/2}$ and $\mu_{1/4}$ are not homeomorphic.

Oxtoby, Navarro-Bermudez (1988): Let t be the solution of $x^3 + x^2 - 1 = 0$ taken from $(0, 1)$. Then $\mu_t \sim \mu_{t^2} \sim \mu_{1-t} \sim \mu_{1-t^2}$.

It is unknown whether there are other measures homeomorphic to μ_t.

Homeomorphic Bernoulli trial measures

Problem: Is it true that for every $0 < r < 1$

$$|\{0 < s < 1 : \mu_s \sim \mu_r\}| < \infty?$$

Homeomorphic Bernoulli trial measures

Problem: Is it true that for every $0 < r < 1$

$$|\{0 < s < 1 : \mu_s \sim \mu_r\}| < \infty?$$

Yingst (2008): For any algebraic integer $0 < r < 1$ there are only finitely many s such that μ_s is homeomorphic to μ_r.

Homeomorphic Bernoulli trial measures

Problem: Is it true that for every $0 < r < 1$

$$|\{0 < s < 1 : \mu_s \sim \mu_r\}| < \infty?$$

Yingst (2008): For any algebraic integer $0 < r < 1$ there are only finitely many s such that μ_s is homeomorphic to μ_r.

Dougherty - Mauldin - Yingst (2007): For every $n \in \mathbb{N}$ there exists an algebraic integer number r such that

$$|\{s : \mu_s \sim \mu_r\}| \geq n.$$

Homeomorphic Bernoulli trial measures

Problem: Is it true that for every $0 < r < 1$

$$|\{0 < s < 1 : \mu_s \sim \mu_r\}| < \infty?$$

Yingst (2008): For any algebraic integer $0 < r < 1$ there are only finitely many s such that μ_s is homeomorphic to μ_r.

Dougherty - Mauldin - Yingst (2007): For every $n \in \mathbb{N}$ there exists an algebraic integer number r such that

$$|\{s : \mu_s \sim \mu_r\}| \geq n.$$

Yingst (2008) gave conditions which determine whether two Bernoulli trial measures are homeomorphic.

Very few results are known about homeomorphism of Bernoulli measures with more than two states.

Clopen values set

Let X be a Cantor set and $\mu \in M(X)$, then $S(\mu) := \{\mu(U) : U \text{ is clopen in } X\}$ is called the clopen values set.

Clopen values set

Let X be a Cantor set and $\mu \in M(X)$, then $S(\mu) := \{\mu(U) : U \text{ is clopen in } X\}$ is called the clopen values set.

$S(\mu)$ is a countable dense subset of $[0, 1]$ containing 0 and 1. Clearly, $S(\mu) = S(\mu \circ f)$ for any $f \in H(X)$.

Clopen values set

Let X be a Cantor set and $\mu \in M(X)$, then
$S(\mu) := \{\mu(U) : U \text{ is clopen in } X\}$ is called the clopen values set.

$S(\mu)$ is a countable dense subset of $[0, 1]$ containing 0 and 1.
Clearly, $S(\mu) = S(\mu \circ f)$ for any $f \in H(X)$.

Akin (1999): There exists an uncountable subset $\mathcal{M} \subset M(X)$ such that for $\mu_1, \mu_2 \in \mathcal{M}$ and $f \in H(X)$

$$\mu_1 = \mu_2 \circ f \implies (\mu_1 = \mu_2 \text{ and } f = id_X)$$

Clopen values set

Let X be a Cantor set and $\mu \in M(X)$, then $S(\mu) := \{\mu(U) : U \text{ is clopen in } X\}$ is called the clopen values set.

$S(\mu)$ is a countable dense subset of $[0, 1]$ containing 0 and 1. Clearly, $S(\mu) = S(\mu \circ f)$ for any $f \in H(X)$.

Akin (1999): There exists an uncountable subset $\mathcal{M} \subset M(X)$ such that for $\mu_1, \mu_2 \in \mathcal{M}$ and $f \in H(X)$

$$\mu_1 = \mu_2 \circ f \Longrightarrow (\mu_1 = \mu_2 \text{ and } f = id_X)$$

The set $S(\mu)$ provides an *invariant* for homeomorphic measures (i.e. $S(\mu) = S(\mu \circ f)$, $\forall f \in H(X)$) although it is not a *complete invariant*, in general. For example, $S(\mu(1/3, 1/3, 1/3)) = S(\mu(1/3, 2/3))$ but the Bernoulli measures $\mu(1/3, 1/3, 1/3)$ and $\mu(1/3, 2/3)$ are not homeomorphic.

Good measures

Definition

Akin (2005) A full non-atomic probability measure μ on a Cantor set X is called good if whenever U, V are clopen sets with $\mu(U) < \mu(V)$, there exists a clopen subset W of V such that $\mu(W) = \mu(U)$.

Good measures

Definition

Akin (2005) A full non-atomic probability measure μ on a Cantor set X is called good if whenever U, V are clopen sets with $\mu(U) < \mu(V)$, there exists a clopen subset W of V such that $\mu(W) = \mu(U)$.

Akin (2005): For the class of good measures, $S(\mu)$ **is** a complete invariant for homeomorphism.
Akin (2005), Glasner - Weiss (1995): Good measures are exactly invariant measures of uniquely ergodic minimal homeomorphisms of Cantor sets.

Good measures

Definition

Akin (2005) A full non-atomic probability measure μ on a Cantor set X is called good if whenever U, V are clopen sets with $\mu(U) < \mu(V)$, there exists a clopen subset W of V such that $\mu(W) = \mu(U)$.

Akin (2005): For the class of good measures, $S(\mu)$ **is** a complete invariant for homeomorphism.
Akin (2005), Glasner - Weiss (1995): Good measures are exactly invariant measures of uniquely ergodic minimal homeomorphisms of Cantor sets.

A subset S of the unit interval $[0, 1]$ is called group-like if $S = G \bigcap [0, 1]$ where G is an additive subgroup of $\mathbb{R}$.

For a good measure μ, the set $S(\mu)$ is *group-like*.

Refinable measures

Definition

Dougherty - Mauldin - Yingst (2007): (1) A measure $\mu \in M(X)$ on a Cantor set X is called refinable if for any clopen set U such that $\mu(U) = \sum_{i=1}^{n} \mu(U_i)$ with clopen sets U_i, there exist a clopen partition $\{U'_1, \ldots, U'_n\}$ of U with $\mu(U'_i) = \mu(U_i)$.
(2) μ is weakly refinable if (i) X is refinable and (ii) every clopen set can be partitioned into (finitely many) refinable clopen sets.

Refinable measures

Definition

Dougherty - Mauldin - Yingst (2007): (1) A measure $\mu \in M(X)$ on a Cantor set X is called refinable if for any clopen set U such that $\mu(U) = \sum_{i=1}^{n} \mu(U_i)$ with clopen sets U_i, there exist a clopen partition $\{U'_1, ..., U'_n\}$ of U with $\mu(U'_i) = \mu(U_i)$.
(2) μ is weakly refinable if (i) X is refinable and (ii) every clopen set can be partitioned into (finitely many) refinable clopen sets.

Goodness $\Longrightarrow$ Refinability $\Longrightarrow$ Weak refinability.

Refinable measures

Definition

Dougherty - Mauldin - Yingst (2007): (1) A measure $\mu \in M(X)$ on a Cantor set X is called refinable if for any clopen set U such that $\mu(U) = \sum_{i=1}^{n} \mu(U_i)$ with clopen sets U_i, there exist a clopen partition $\{U'_1, ..., U'_n\}$ of U with $\mu(U'_i) = \mu(U_i)$.
(2) μ is weakly refinable if (i) X is refinable and (ii) every clopen set can be partitioned into (finitely many) refinable clopen sets.

Goodness $\Longrightarrow$ Refinability $\Longrightarrow$ Weak refinability.

Open question: Is it true that weak refinability implies refinability?

Refinable measures

Definition

Dougherty - Mauldin - Yingst (2007): (1) A measure $\mu \in M(X)$ on a Cantor set X is called refinable if for any clopen set U such that $\mu(U) = \sum_{i=1}^{n} \mu(U_i)$ with clopen sets U_i, there exist a clopen partition $\{U'_1, ..., U'_n\}$ of U with $\mu(U'_i) = \mu(U_i)$.
(2) μ is weakly refinable if (i) X is refinable and (ii) every clopen set can be partitioned into (finitely many) refinable clopen sets.

Goodness $\Longrightarrow$ Refinability $\Longrightarrow$ Weak refinability.

Open question: Is it true that weak refinability implies refinability?

Theorem

Akin - Dougherthy - Mauldin - Yingst (2008): Let μ and ν be weakly refinable measures on X. Then $\mu \sim \nu \Longleftrightarrow S(\mu) = S(\nu)$.

Definition of a Bratteli diagram

Definition

A Bratteli diagram is an infinite graph $B = (V, E)$ with the vertex set $V = \bigcup_{i\geq 0} V_i$ and edge set $E = \bigcup_{i\geq 1} E_i$:
1) $V_0 = \{v_0\}$ is a single point;
2) V_i and E_i are finite sets;
3) edges E_i connect V_i to V_{i+1}: there exist a range map r and a source map s from E to V such that $r(E_i) = V_i, s(E_i) = V_{i-1}$, and $s^{-1}(v) \neq \emptyset$; $r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$.

Definition of a Bratteli diagram

Definition

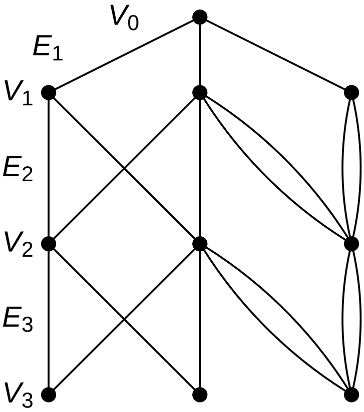
A Bratteli diagram is an infinite graph $B = (V, E)$ with the vertex set $V = \bigcup_{i \geq 0} V_i$ and edge set $E = \bigcup_{i \geq 1} E_i$:
1) $V_0 = \{v_0\}$ is a single point;
2) V_i and E_i are finite sets;
3) edges E_i connect V_i to V_{i+1}: there exist a range map r and a source map s from E to V such that $r(E_i) = V_i, s(E_i) = V_{i-1}$, and $s^{-1}(v) \neq \emptyset$; $r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$.

F_n = incidence matrix of size $|V_{n+1}| \times |V_n|$.

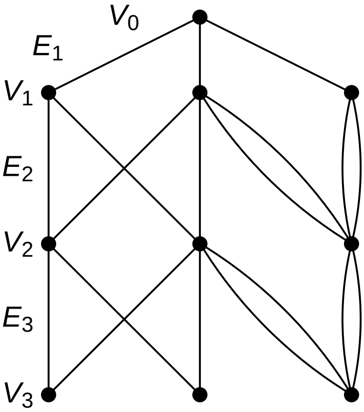
B is stationary if $F_n = F_1$ for $n \geq 2$.

Forrest (1997), Durand - Host - Skau (1999), B. - Kwiatkowski - Medynets (2009): The class of of stationary Bratteli diagrams describes exactly aperiodic substitution dynamical systems.

Example



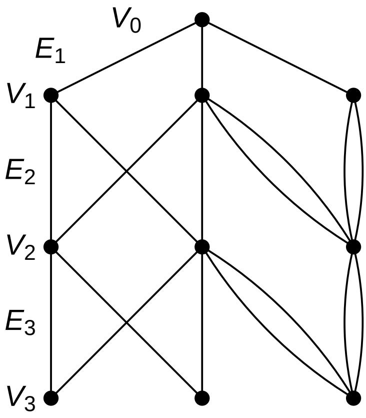
Example



The diagram is *stationary*

$$F = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

Example

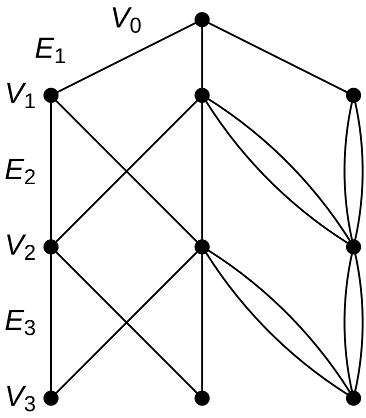


The diagram is *stationary*

$$F = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

Topology on the path space
X_B: two paths are close if they agree on a large initial segment.
X_B is a Cantor set if it has no isolated points.

Example



The diagram is *stationary*

$$F = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

Topology on the path space X_B: two paths are close if they agree on a large initial segment.
X_B is a Cantor set if it has no isolated points.
There is one minimal component on the diagram.

Measures on Bratteli diagrams

Definition

Two infinite paths $x = (x_i)$ and $y = (y_i)$ from the path space X_B of a Bratteli diagram $B = (V, E)$ are called tail (cofinal) equivalent if there exists i_0 such that $x_i = y_i$ for all $i \geq i_0$.

Denote by $\mathcal{R}$ the tail equivalence relation on X_B.

Measures on Bratteli diagrams

Definition

Two infinite paths $x = (x_i)$ and $y = (y_i)$ from the path space X_B of a Bratteli diagram $B = (V, E)$ are called tail (cofinal) equivalent if there exists i_0 such that $x_i = y_i$ for all $i \geq i_0$. Denote by $\mathcal{R}$ the tail equivalence relation on X_B.

A Bratteli diagram is called simple if the tail equivalence relation $\mathcal{R}$ is minimal.

Measures on Bratteli diagrams

Definition

Two infinite paths $x = (x_i)$ and $y = (y_i)$ from the path space X_B of a Bratteli diagram $B = (V, E)$ are called tail (cofinal) equivalent if there exists i_0 such that $x_i = y_i$ for all $i \geq i_0$. Denote by $\mathcal{R}$ the tail equivalence relation on X_B.

A Bratteli diagram is called simple if the tail equivalence relation $\mathcal{R}$ is minimal.

$E(v_0, v)$ is the set of all path that connect v_0 and $v \in V$. Set $h_v^{(n)} = |E(v_0, v)|$, $v \in V_n$ and

$$X_w^{(n)}(\overline{e}) := \{x = (x_i) \in X_B : x_i = e_i, i = 1, ..., n\}$$

where $\overline{e} = (e_1, \ldots, e_n) \in E(v_0, w)$, $n \geq 1$.

Measures on Bratteli diagrams

Definition

Two infinite paths $x = (x_i)$ and $y = (y_i)$ from the path space X_B of a Bratteli diagram $B = (V, E)$ are called tail (cofinal) equivalent if there exists i_0 such that $x_i = y_i$ for all $i \geq i_0$. Denote by $\mathcal{R}$ the tail equivalence relation on X_B.

A Bratteli diagram is called simple if the tail equivalence relation $\mathcal{R}$ is minimal.

$E(v_0, v)$ is the set of all path that connect v_0 and $v \in V$. Set $h_v^{(n)} = |E(v_0, v)|$, $v \in V_n$ and

$$X_w^{(n)}(\overline{e}) := \{x = (x_i) \in X_B : x_i = e_i, i = 1, ..., n\}$$

where $\overline{e} = (e_1, \ldots, e_n) \in E(v_0, w)$, $n \geq 1$.

A measure μ is $\mathcal{R}$-invariant on X_B if and only if $\mu(X_v^{(n)}(\overline{e})) = \mu(X_v^{(n)}(\overline{e}'))$ for any $\overline{e}, \overline{e}' \in E(v_0, v)$.

Measures on stationary Bratteli diagrams

Theorem

B. - Kwiatkowski - Medynets - Solomyak (2010): Let B be a stationary Bratteli diagram and $A = F^T$ is the matrix transposed to the incidence matrix of B. Then there is a one-to-one correspondence between vectors of the cone

$$core(A) = \bigcap_{k \geq 1} A^k(\mathbb{R}^n_+)$$

and $\mathcal{R}$-invariant measures on X_B. The ergodic measures correspond to the extreme vectors of $core(A)$. Some of the ergodic measures may be infinite.

Frobenius normal form

Let B be a stationary Bratteli diagram and A the matrix transpose to the incidence matrix of B. Then A can be transformed to the *Frobenius normal form*:

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 & Y_{1,s+1} & \cdots & Y_{1,m} \\ 0 & A_2 & \cdots & 0 & Y_{2,s+1} & \cdots & Y_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & A_s & Y_{s,s+1} & \cdots & Y_{s,m} \\ 0 & 0 & \cdots & 0 & A_{s+1} & \cdots & Y_{s+1,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & A_m \end{pmatrix}$$

where all A_i are primitive matrices, $A_1, ..., A_s$ determine minimal components of $\mathcal{R}$, non-zero matrices $Y_{i,j}$ show how non-minimal components "interact" with minimal ones.

Clopen values set for ergodic measures

Let λ_i be the spectral radius of A_i. Then λ_i is a distinguished eigenvalue if $\lambda_i > \lambda_j$ for any j with $Y_{i,j} \neq 0$. Then there exists a non-negative eigenvector $x = (x_1, ..., x_K)^T$ with $Ax = \lambda_i x$ such that $x_v > 0$ if the vertex v is accessible from A_i.

Clopen values set for ergodic measures

Let λ_i be the spectral radius of A_i. Then λ_i is a distinguished eigenvalue if $\lambda_i > \lambda_j$ for any j with $Y_{i,j} \neq 0$. Then there exists a non-negative eigenvector $x = (x_1, ..., x_K)^T$ with $Ax = \lambda_i x$ such that $x_v > 0$ if the vertex v is accessible from A_i.
Let λ be a distinguished eigenvalue and x the corresponding probability non-negative eigenvector. The ergodic probability $\mathcal{R}$-invariant measure μ defined by λ and x satisfies the relation:

$$\mu(X_i^{(n)}(\overline{e})) = \frac{x_i}{\lambda^{n-1}}$$

where $i \in V_n$ and $\overline{e}$ is a finite path that ends at i. Thus,

$$S(\mu) = \left\{ \sum_{i=1}^{K} k_i^{(n)} \frac{x_i}{\lambda^{n-1}} : 0 \leq k_i^{(n)} \leq h_i^{(n)};\ n = 1, 2, \ldots \right\}.$$

Clopen values set for ergodic measures

Let λ_i be the spectral radius of A_i. Then λ_i is a distinguished eigenvalue if $\lambda_i > \lambda_j$ for any j with $Y_{i,j} \neq 0$. Then there exists a non-negative eigenvector $x = (x_1, ..., x_K)^T$ with $Ax = \lambda_i x$ such that $x_v > 0$ if the vertex v is accessible from A_i.

Let λ be a distinguished eigenvalue and x the corresponding probability non-negative eigenvector. The ergodic probability $\mathcal{R}$-invariant measure μ defined by λ and x satisfies the relation:

$$\mu(X_i^{(n)}(\overline{e})) = \frac{x_i}{\lambda^{n-1}}$$

where $i \in V_n$ and $\overline{e}$ is a finite path that ends at i. Thus,

$$S(\mu) = \left\{ \sum_{i=1}^{K} k_i^{(n)} \frac{x_i}{\lambda^{n-1}} : 0 \leq k_i^{(n)} \leq h_i^{(n)};\ n = 1, 2, \ldots \right\}.$$

Non-distinguished eigenvalues determine infinite ergodic invariant measures.

Main results

Let $\mathcal{S}$ be the set of all ergodic probability $\mathcal{R}$-invariant measures on stationary Bratteli diagrams.

Main results

Let $\mathcal{S}$ be the set of all ergodic probability $\mathcal{R}$-invariant measures on stationary Bratteli diagrams.

Questions: 1. Is $S(\mu)$ *group-like* for every $\mu \in \mathcal{S}$?

Main results

Let $\mathcal{S}$ be the set of all ergodic probability $\mathcal{R}$-invariant measures on stationary Bratteli diagrams.

Questions: 1. Is $S(\mu)$ *group-like* for every $\mu \in \mathcal{S}$?
2. Which measures from $\mathcal{S}$ are *good*?

Main results

Let $\mathcal{S}$ be the set of all ergodic probability $\mathcal{R}$-invariant measures on stationary Bratteli diagrams.

Questions: 1. Is $S(\mu)$ *group-like* for every $\mu \in \mathcal{S}$?
2. Which measures from $\mathcal{S}$ are *good*?
3. What is the *cardinality* of the set $\{\nu \in \mathcal{S} : \nu \sim \mu\}$?

Main results

Let $\mathcal{S}$ be the set of all ergodic probability $\mathcal{R}$-invariant measures on stationary Bratteli diagrams.

Questions: 1. Is $S(\mu)$ *group-like* for every $\mu \in \mathcal{S}$?
2. Which measures from $\mathcal{S}$ are *good*?
3. What is the *cardinality* of the set $\{\nu \in \mathcal{S} : \nu \sim \mu\}$?

THEOREM 1 (B. - Karpel)

Let μ be an ergodic invariant measure on a stationary diagram B defined by a distinguished eigenvalue λ of the matrix $A = F^T$. Let $x = (x_1, \ldots, x_n)^T$ be the corresponding vector and H the additive subgroup of $\mathbb{R}$ generated by $\{x_1, \ldots, x_n\}$. Then the clopen values set $S(\mu)$ is group-like and

$$S(\mu) = \left(\bigcup_{N=0}^{\infty} \frac{1}{\lambda^N} H \right) \cap [0, 1].$$

Idea of the proof of Theorem 1

The proof is divided into two parts depending on the properties of λ. The first part deals with rational (hence integer) λ, and the second one contains the proof of the case of irrational (hence algebraic integer) λ.

1. $\lambda \in \mathbb{Q}$ and $x = (\frac{p_1}{q}, \dots, \frac{p_n}{q})$, where $p_1, \dots, p_n, q \in \mathbb{N}$ and $gcd(p_1, \dots, p_n) = 1$. We prove that

$$S(\mu) = \left\{ \frac{m}{q\lambda^N} \mid m, N \in \mathbb{N},\ 0 \leq m \leq q\lambda^N \right\}.$$

We use the fact that every clopen set can be represented as a finite disjoint union of cylinder sets with arbitrary large length. We also use the fact that the Bratteli diagram is not simple and the formula for asymptotic behavior of $h_i^{(N)} \sim \lambda^N$ as $N \to +\infty$.

Idea of the proof of Theorem 1

The proof is divided into two parts depending on the properties of λ. The first part deals with rational (hence integer) λ, and the second one contains the proof of the case of irrational (hence algebraic integer) λ.

1. $\lambda \in \mathbb{Q}$ and $x = (\frac{p_1}{q}, \dots, \frac{p_n}{q})$, where $p_1, \dots, p_n, q \in \mathbb{N}$ and $gcd(p_1, \dots, p_n) = 1$. We prove that

$$S(\mu) = \left\{ \frac{m}{q\lambda^N} \mid m, N \in \mathbb{N},\ 0 \leq m \leq q\lambda^N \right\}.$$

We use the fact that every clopen set can be represented as a finite disjoint union of cylinder sets with arbitrary large length. We also use the fact that the Bratteli diagram is not simple and the formula for asymptotic behavior of $h_i^{(N)} \sim \lambda^N$ as $N \to +\infty$.

2. $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ and $x = (x_1, ..., x_n)$. Then $S(\mu) \subset \mathbb{Q}(\lambda) = \mathbb{Q}[\lambda]$. Let k be the degree of the minimal polynomial for λ. Then $\mathbb{Q}[\lambda] = \{\sum_{i=0}^{k-1} a_i\lambda^i\}$, $a_i \in \mathbb{Q}$.

Idea of the proof of Theorem 1

There is a one-to-one correspondence:

$$a_0 + a_1\lambda + \ldots + a_{k-1}\lambda^{k-1} \leftrightarrow (a_0, a_1, ..., a_{k-1})^T.$$

Every element of $S(\mu) \subset \mathbb{Q}(\lambda)$ can be considered as a vector in $\mathbb{Q}^k$. Denote by $\{\mathbf{e}_1, ..., \mathbf{e}_k\}$ the standard basis in $\mathbb{R}^k$ (or $\mathbb{Q}^k$). Let $\mathbf{n} = (1, \lambda, ..., \lambda^{k-1})^T$. Denote by $\pi = \{\mathbf{y} : \langle \mathbf{y}, \mathbf{n} \rangle = 0\}$ the hyperplane in $\mathbb{R}^k$. We prove that all points of $S(\mu)$ "uniformly" fill the gap between π and $\pi + \mathbf{e}_1$.

$$S(\mu) = \left\{ D^{N-1} \left(\sum_{i=1}^{n} k_i^{(N)} \mathbf{x}_i \right) \mid 0 \le k_i^{(N)} \le h_i^{(N)};\ N = 1, 2, ... \right\},$$

where $D \in Mat(k \times k, \mathbb{Q})$ which corresponds to the multiplication by $\frac{1}{\lambda}$ in $\mathbb{Q}(\lambda)$. The entries of D are obtained from the coefficients of the minimal polynomial for λ.

Main results

THEOREM 2 (B. - Karpel)

Let μ be an ergodic $\mathcal{R}$-invariant probability measure on a stationary Bratteli diagram B defined by a distinguished eigenvalue λ of the matrix $A = F^T$. Denote by $x = (x_1, ..., x_n)^T$ the corresponding probability eigenvector. Let the vertices $m+1, \dots, n$ belong to the distinguished class α corresponding to μ. Then μ is good if and only if there exists $R \in \mathbb{N}$ such that $\lambda^R x_1, ..., \lambda^R x_m$ belong to the additive group generated by $\{x_{m+1}, ..., x_n\}$.

Corollaries

COROLLARY 1

If the clopen values set of μ *is rational and* $(\frac{p_1}{q}, \ldots, \frac{p_n}{q})$ *is the corresponding eigenvector, then* μ *is good if and only if* $\mathrm{gcd}(p_{m+1}, ..., p_n) | \lambda^R$ *for some* $R \in \mathbb{R}$*. If* $\mathrm{gcd}(p_{m+1}, ..., p_n) = 1$*, then* μ *is good.*

Corollaries

COROLLARY 1

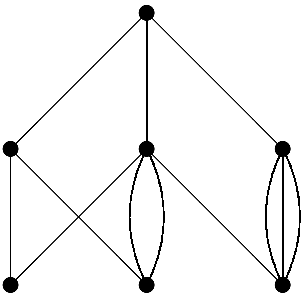
If the clopen values set of μ is rational and $(\frac{p_1}{q}, \ldots, \frac{p_n}{q})$ is the corresponding eigenvector, then μ is good if and only if $\mathrm{gcd}(p_{m+1}, ..., p_n) \mid \lambda^R$ *for some $R \in \mathbb{R}$. If* $\mathrm{gcd}(p_{m+1}, ..., p_n) = 1$, *then μ is good.*

COROLLARY 2

Let $\mu \in \mathcal{S}$. The following are equivalent:

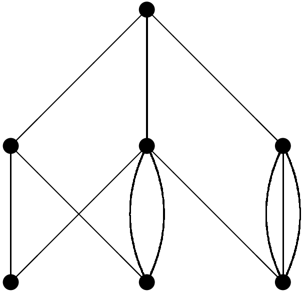
- *μ is good;*
- *μ is refinable;*
- *μ is weakly refinable.*

Example 1



. . . .

Example 1



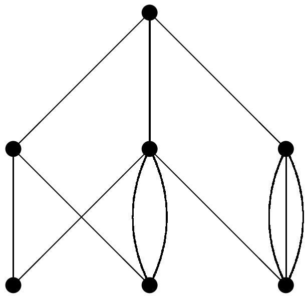
. . . .

For the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix},$$

the eigenvectors
$x = (\frac{3-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}, 0)^T$ and
$y = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})^T$ correspond to the eigenvalues $\lambda_1 = \frac{3+\sqrt{5}}{2}$ and $\lambda_2 = 3$, respectively.

Example 1



. . . .

For the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix},$$

the eigenvectors
$x = (\frac{3-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}, 0)^T$ and
$y = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})^T$ correspond to the eigenvalues $\lambda_1 = \frac{3+\sqrt{5}}{2}$ and
$\lambda_2 = 3$, respectively.
It gives two ergodic *good* measures μ_1 and μ_2. But for any $t \in (0,1)$ the measure
$\nu_t = t\mu_1 + (1-t)\mu_2$ is not *good*.

Example 2

Fix an integer $N \geq 3$ and let

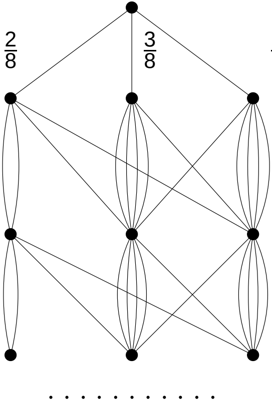
$$F_N = \begin{pmatrix} 2 & 0 & 0 \\ 1 & N & 1 \\ 1 & 1 & N \end{pmatrix}.$$

The Perron-Frobenius eigenvalue $\lambda = N + 1$ and the corresponding probability eigenvector

$$x = \left(\frac{1}{N}, \frac{N-1}{2N}, \frac{N-1}{2N} \right)^T.$$

The full ergodic measure μ_N is a good measure if and only if $N = 2^k + 1$.

Example 2 (cont'd)



For $N = 4$, we have $\lambda = 5$ and $x = (\frac{2}{8}, \frac{3}{8}, \frac{3}{8})$. For any $m \in \mathbb{N}$, $3 \nmid 5^m$.

The cylinder set U of the length 1 that ends in the first vertex has the measure $\frac{2}{8}$. The cylinder set V of the length 1 that ends in the second vertex has the measure $\frac{3}{8}$.

There is no clopen subset $W \subset V$ such that $\mu(W) = \mu(U) = \frac{2}{8}$. Hence, the measure μ_4 is not good.

Main results

Theorem 3 (B. - Karpel)

Let μ be a good ergodic $\mathcal{R}$-invariant probability measure on a stationary (non-simple) Bratteli diagram B. Then there exist stationary Bratteli diagrams $\{B_i\}_{i=0}^{\infty}$ and good ergodic $\mathcal{R}_i$-invariant probability measures μ_i on B_i such that each measure μ_i is homeomorphic to μ and the dynamical systems $(B_i, \mathcal{R}_i)$, $(B_j, \mathcal{R}_j)$ are topologically orbit equivalent if and only if $i = j$. Moreover, the diagram B_i has exactly i minimal components for the tail equivalence relation $\mathcal{R}_i, i \in \mathbb{N}$.

Idea of the proof of Theorem 3

1. Let $S(\mu) \subset \mathbb{Q}$. Then $S(\mu) = \{\frac{m}{q\lambda^N} \mid m, N \in \mathbb{N},\ 0 \leq m \leq q\lambda^N\}$.

We construct a simple Bratteli diagram B_0 and an ergodic probability invariant measure μ_0 such that $S(\mu_0) = S$. Then, on the base of B_0, we construct Bratteli diagrams B_i with i minimal components and full measures μ_i homeomorphic to μ.

2. Let $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. We construct a stationary Bratteli diagram B' such that:
(i) there is an ergodic invariant probability good measure ν on B' such that $S(\nu) = S(\mu)$;
(ii) B' has one more minimal component in comparison with B