



**Applied Mathematics**

... linking Mathematics with  
Science, Engineering, and Medicine

# **MODELS OF NONLINEAR VISCOELASTICITY**

**Katerina Papoulia**

**with**

**Amin Eshraghi, Pritam Ganguly, Mircea Grigoriu,  
Ching-Hang Sam, Steve Vavasis**



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# Overview

**Hyperelastic material models**

**Hypoelastic material models**

**Rate form hyperelastic model**

**Examples**

**Finite nonlinear viscoelasticity**

**Finite linear viscoelasticity**

**Other related topics**

# Hyperelastic material models

$$\mathbf{S} = 2\partial W / \partial \mathbf{C}$$

$$\mathbf{C} = \mathbf{F}^T \mathbf{F}$$

$$\mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X} \quad \mathbf{X} = \text{original, } \mathbf{x} = \text{current position of material point}$$

$$\mathbf{S} = \text{2nd Piola-Kirchhoff stress,}$$

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} = \text{Kirchhoff stress}$$

$$\boldsymbol{\sigma} = \mathbf{F}\mathbf{S}\mathbf{F}^T / J, \quad J = \det \mathbf{F}$$

$$W = W(I_i) \quad \text{or} \quad W = W(\lambda_i)$$

$$I_i = \text{Invariants of } \mathbf{C}$$

$$I_1 = \text{tr} \mathbf{C}, \quad I_2 = \left[ (\text{tr} \mathbf{C})^2 - \text{tr} \mathbf{C}^2 \right] / 2, \quad I_3 = \det \mathbf{C}, \quad \text{higher}$$

$$I_1, I_2, I_3 \quad \text{needed for isotropy}$$

$$\lambda_i^2 = \text{eigenvalues of } \mathbf{C}$$

# Hypoelastic material models

$$\overset{\circ}{\boldsymbol{\tau}} = \mathbf{H}(\boldsymbol{\tau}) : \mathbf{d}$$

where  $\overset{\circ}{\phantom{x}}$  denotes a frame indifferent (objective) rate.

$\boldsymbol{\tau}$  = Kirchoff stress,  $\boldsymbol{\tau} = J\boldsymbol{\sigma}$

$\mathbf{d}$  = Rate of deformation  $= (\mathbf{l} + \mathbf{l}^T) / 2$        $\mathbf{l} = \partial \mathbf{v} / \partial \mathbf{x}$

Velocity gradient

$\boldsymbol{\tau}, \mathbf{d}$  are functions of the current geometry  $\mathbf{x}$  and time (Eulerian framework),  
whereas for hyperelastic models,

$\mathbf{S}, \mathbf{C}$  are functions of the original geometry  $\mathbf{X}$  and time (Lagrangian framework)

## Hyperelastic models:

- consistent with reversible thermodynamics
- naturally objective

## Hypoelastic models:

- simple, ideally suited for incremental solutions
- direct relation to actual physical stress makes modeling more physically motivated

Yet, several problems identified with hypoelastic models:

- spurious shear oscillations
- residual elastic stress in closed elastic loading path
- unconditionally (exactly) integrable only if the logarithmic (D) rate of stress is used

# Rate form hyperelastic model

**Theorem:** An Eulerian rate model of the form

$$\overset{J}{\boldsymbol{\tau}} = \mathbf{H}(\mathbf{b}) : \overset{J}{\mathbf{b}},$$

in which  $\overset{J}{\boldsymbol{\tau}}$  denotes the Jaumann objective rate,  $\overset{J}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} + \boldsymbol{\tau}\mathbf{W} - \mathbf{W}\boldsymbol{\tau}$ ,  $\mathbf{W} = (\mathbf{l} - \mathbf{l}^T) / 2$  and  $\mathbf{b} = \mathbf{F}\mathbf{F}^T$

- a) is a hypoelastic material model, i.e., linear in the rate of deformation tensor  $\mathbf{d}$ .
- b) is exactly integrable to yield  $\boldsymbol{\tau} = \psi(\mathbf{b})$  iff  $\mathbf{H}(\mathbf{b}) = \nabla \psi(\mathbf{b})$ .
- c) is integrable to yield a hyperelastic material if, in addition, it satisfies the following integrability condition:

$$b_{ks}^{-1} H_{slpq} + b_{ls}^{-1} H_{skpq} = b_{ps}^{-1} H_{sqkl} + b_{qs}^{-1} H_{spkl}$$

## Deviatoric-volumetric decomposition

$$\mathbf{F} = J^{1/3} \bar{\mathbf{F}} \quad J = dV / dV_0$$

$\bar{\mathbf{F}}$  describes isochoric deformation (  $\det \bar{\mathbf{F}} = 1$  )

$$W = \bar{W}(\bar{I}_i) + U(J) \quad \bar{I}_i = \text{invariants of } \bar{\mathbf{C}} = \bar{\mathbf{F}}^T \bar{\mathbf{F}}$$

The hyperelastic model becomes

$$\boldsymbol{\tau} = J \frac{\partial U}{\partial J} \mathbf{1} + 2 \text{dev} \left( \bar{\mathbf{F}} \frac{\partial \bar{W}}{\partial \bar{\mathbf{C}}} \bar{\mathbf{F}}^T \right), \quad \text{dev}(\cdot) = (\cdot) - \frac{1}{3} \text{tr}(\cdot) \mathbf{1}$$

and in rate form is

$$\text{dev} \overset{J}{\boldsymbol{\tau}} = \mathbf{H}(\bar{I}_i, \bar{\mathbf{b}}) : \overset{J}{\dot{\mathbf{b}}},$$

## Construction

For any hyperelastic model  $\mathbf{S} = 2\partial W / \partial \mathbf{C}$ , compute  $\boldsymbol{\tau} = \mathbf{F}\mathbf{S}\mathbf{F}^T$  and write it in the representation form

$$\boldsymbol{\tau}(\mathbf{b}) = \varphi_1 \mathbf{1} + \varphi_2 \mathbf{b} + \varphi_3 \mathbf{b}^2, \quad \varphi_k = \varphi_k(I_i).$$

In isotropy, this is always possible according to the representation theory of Truesdell & Noll, others. Then  $\boldsymbol{\tau}$  is of the form  $\boldsymbol{\tau} = \psi(\mathbf{b})$  and  $\mathbf{H}(\mathbf{b}) = \nabla \psi(\mathbf{b})$ .

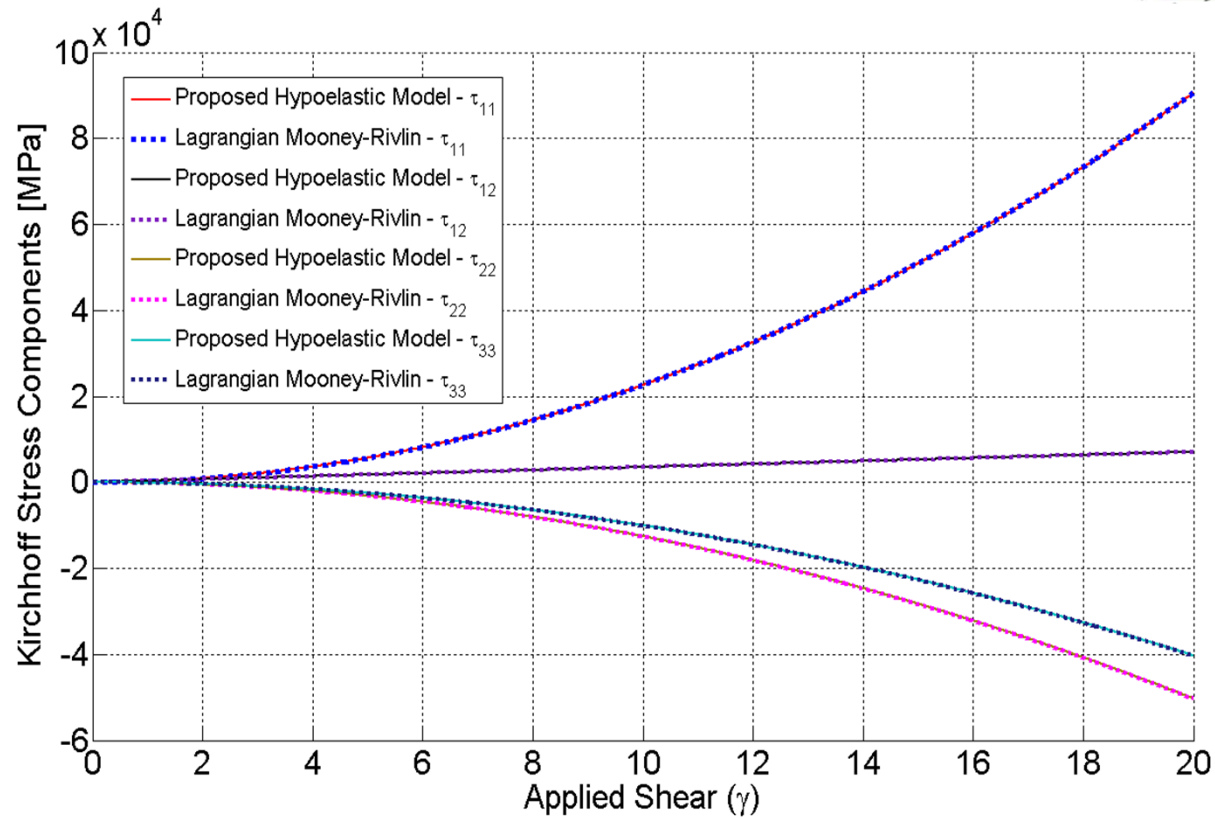
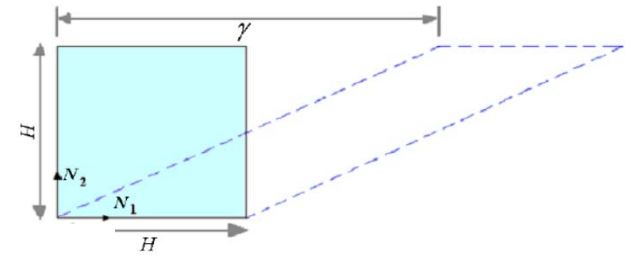
**Example:** 2 term Mooney-Rivlin hyperelastic material

$$\begin{aligned} W &= C_{10}(\bar{I}_1 - 3) + C_{01}(\bar{I}_2 - 3) + (J - 1)^2 / D \\ H_{ijkl}(\bar{I}_1, \bar{I}_2, \bar{\mathbf{b}}) &= 2[(C_{10} + C_{01}\bar{I}_1)\delta_{ij}\delta_{kl} - C_{01}\bar{b}_{ij}\delta_{kl}] \\ &\quad - \frac{2}{3}[C_{10}\bar{I}_1 + 2C_{01}(\bar{I}_1\delta_{kl} - \bar{b}_{kl})]\delta_{ij} \end{aligned}$$

# Examples

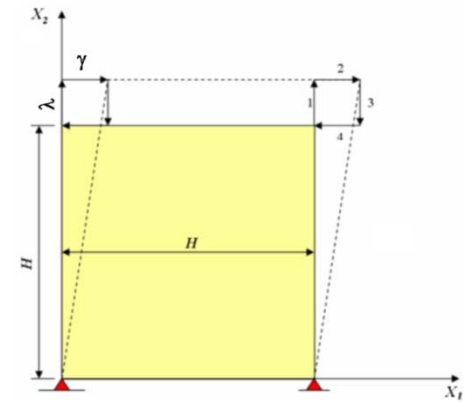
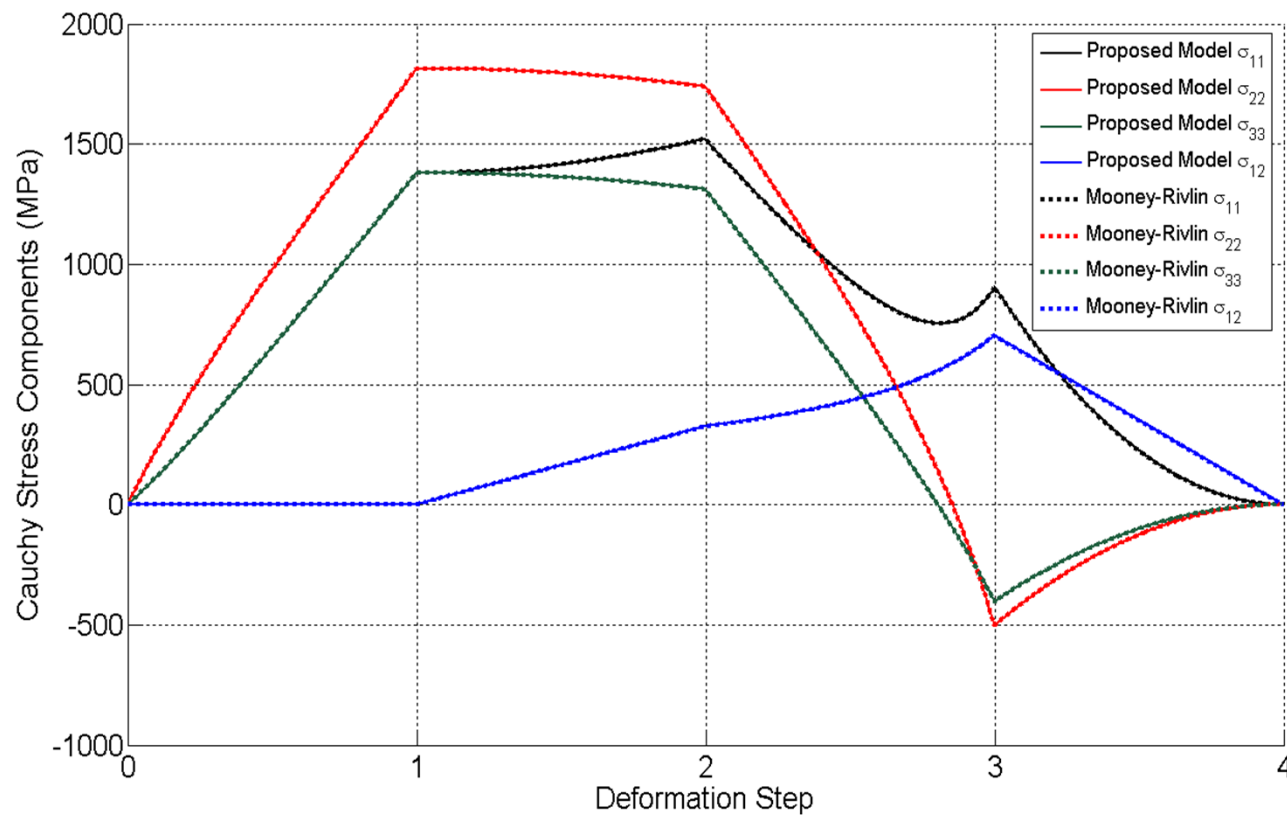
$$C_{10} = 163\text{MPa} \quad C_{01} = 12.5\text{MPa} \quad D = 0.0026\text{MPa}^{-1}$$

## 1. Rectilinear shear motion

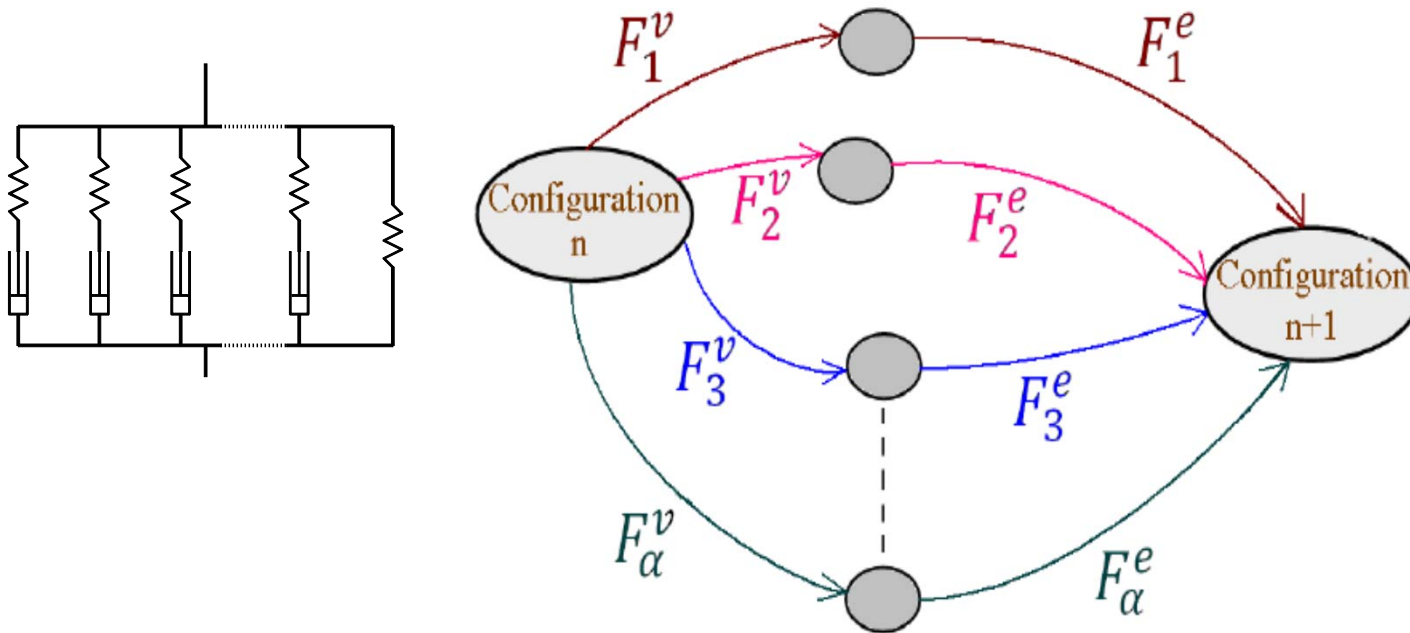


# Examples

## 2. Four-step closed path loading



# A paradigm for viscoelasticity: Generalized Maxwell model



A multiplicative decomposition of the deformation is assumed for each Maxwell element in parallel

$$\mathbf{F}_\alpha = \mathbf{F}_\alpha^e \mathbf{F}_\alpha^v$$

# Thermodynamic basis

Free energy (purely mechanical)

$$\Psi(\mathbf{b}, \mathbf{b}^e) = \Psi^{eq}(\mathbf{b}) + \sum_{\alpha=1}^N \Psi_{\alpha}^{neq}(\mathbf{b}_{\alpha}^e)$$

where  $\mathbf{b}_{\alpha}^e = \mathbf{F}_{\alpha} \left( \mathbf{F}_{\alpha}^{vT} \mathbf{F}_{\alpha}^v \right)^{-1} \mathbf{F}_{\alpha}^T$  follows from the kinematic decomposition

$$\boldsymbol{\tau} = \boldsymbol{\tau}^{\infty} + \sum_{\alpha=1}^N \boldsymbol{\tau}_{\alpha}^v$$

$$\boldsymbol{\tau}^{\infty} = \frac{\partial \Psi^{eq}(\mathbf{b})}{\partial \mathbf{b}}, \quad \boldsymbol{\tau}_{\alpha}^v = \frac{\partial \Psi_{\alpha}^{neq}(\mathbf{b}_{\alpha}^e)}{\partial \mathbf{b}_{\alpha}^e}$$

In accordance with previous theorem, we write in Eulerian form

$$\overset{J}{\boldsymbol{\tau}}^{\infty} = \mathbf{H}(\mathbf{b}) : \overset{J}{\mathbf{b}}, \quad \overset{J}{\boldsymbol{\tau}}_{\alpha}^v = \mathbf{H}(\mathbf{b}_{\alpha}^e) : \overset{J}{\mathbf{b}}_{\alpha}^e$$

## Strain-based constitutive model

(Given  $\mathbf{F}$ ,  $\mathbf{v} \Rightarrow \mathbf{b} = \mathbf{F}\mathbf{F}^T$ ,  $\mathbf{d} = \nabla^s \mathbf{v}$ , update the stress)

$$\overset{J}{\boldsymbol{\tau}}^\infty = \mathbf{H}(\mathbf{b}) : \overset{J}{\mathbf{b}}$$

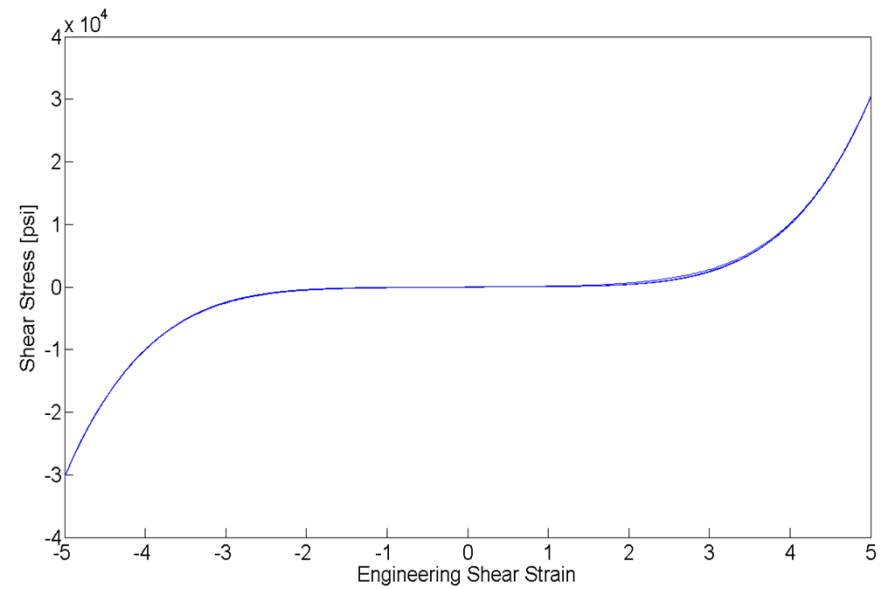
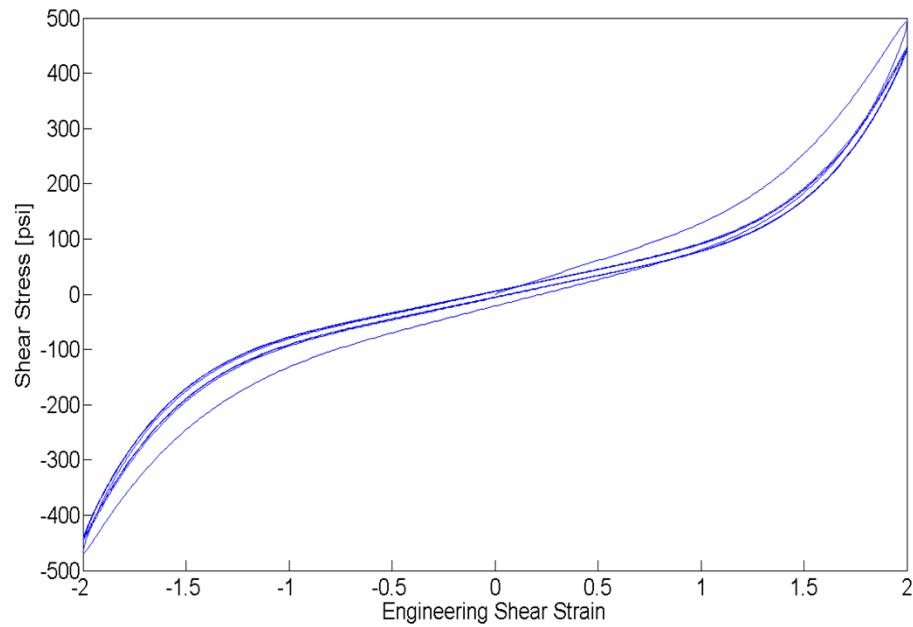
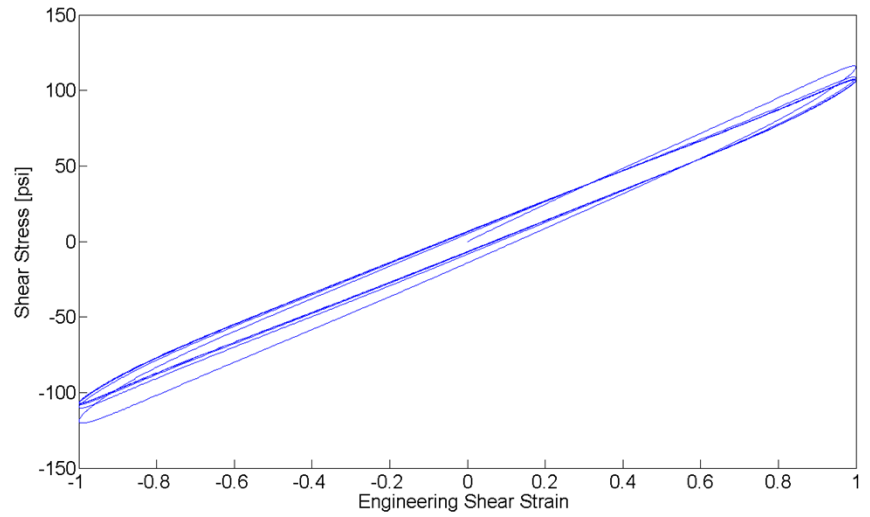
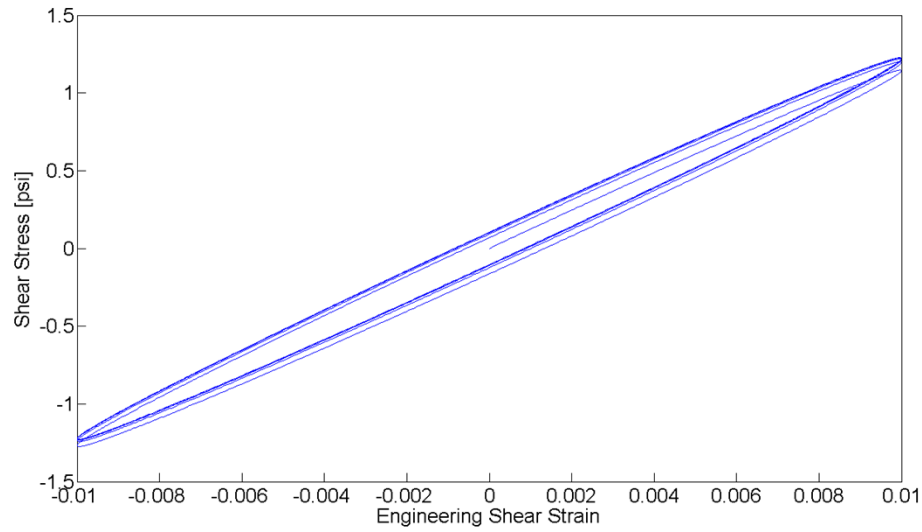
$$\overset{J}{\boldsymbol{\tau}}_\alpha^\nu = \mathbf{H}(\mathbf{b}_\alpha^e) : \overset{J}{\mathbf{b}}_\alpha^e$$

$$\mathbf{b}_\alpha^e = \mathbf{F}_\alpha \left( \mathbf{F}_\alpha^{\nu T} \mathbf{F}_\alpha^\nu \right)^{-1} \mathbf{F}_\alpha^T \Rightarrow \overset{J}{\mathbf{b}}_\alpha^e = \mathbf{b}_\alpha^e (\mathbf{d} - \mathbf{d}_\alpha^\nu) + (\mathbf{d} - \mathbf{d}_\alpha^\nu) \mathbf{b}_\alpha^e$$

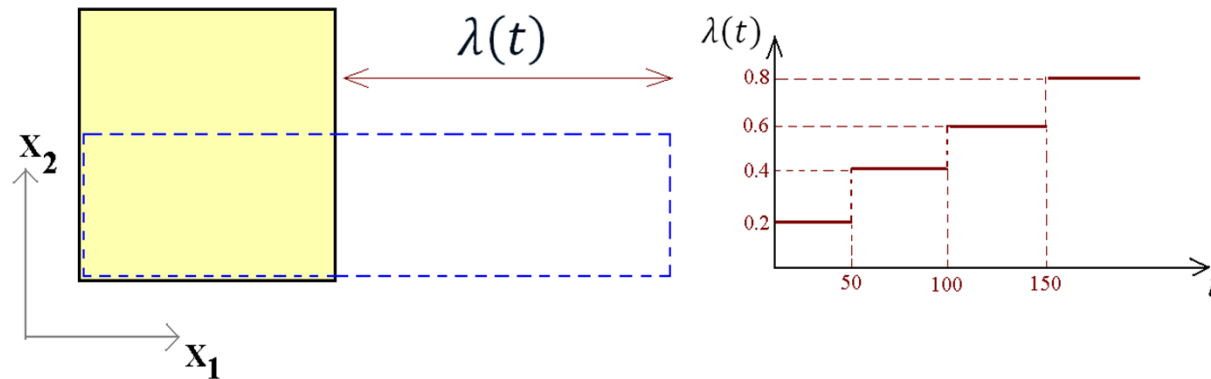
General non-Newtonian fluid flow:

$$\mathbf{d}_\alpha^\nu = \mathbf{M}(\boldsymbol{\tau}_\alpha^\nu) : \boldsymbol{\tau}_\alpha^\nu, \quad \text{e.g.,} \quad \mathbf{d}_\alpha^\nu = \frac{1}{2\eta_\alpha} \boldsymbol{\tau}_\alpha^\nu$$

# Stress- strain curves



# Multi step relaxation in isochoric stretching



5 parameter Mooney-Rivlin strain energy function

$$C_{10} = 0.2362 \text{ psi}$$

$$C_{01} = 0.0493 \text{ psi}$$

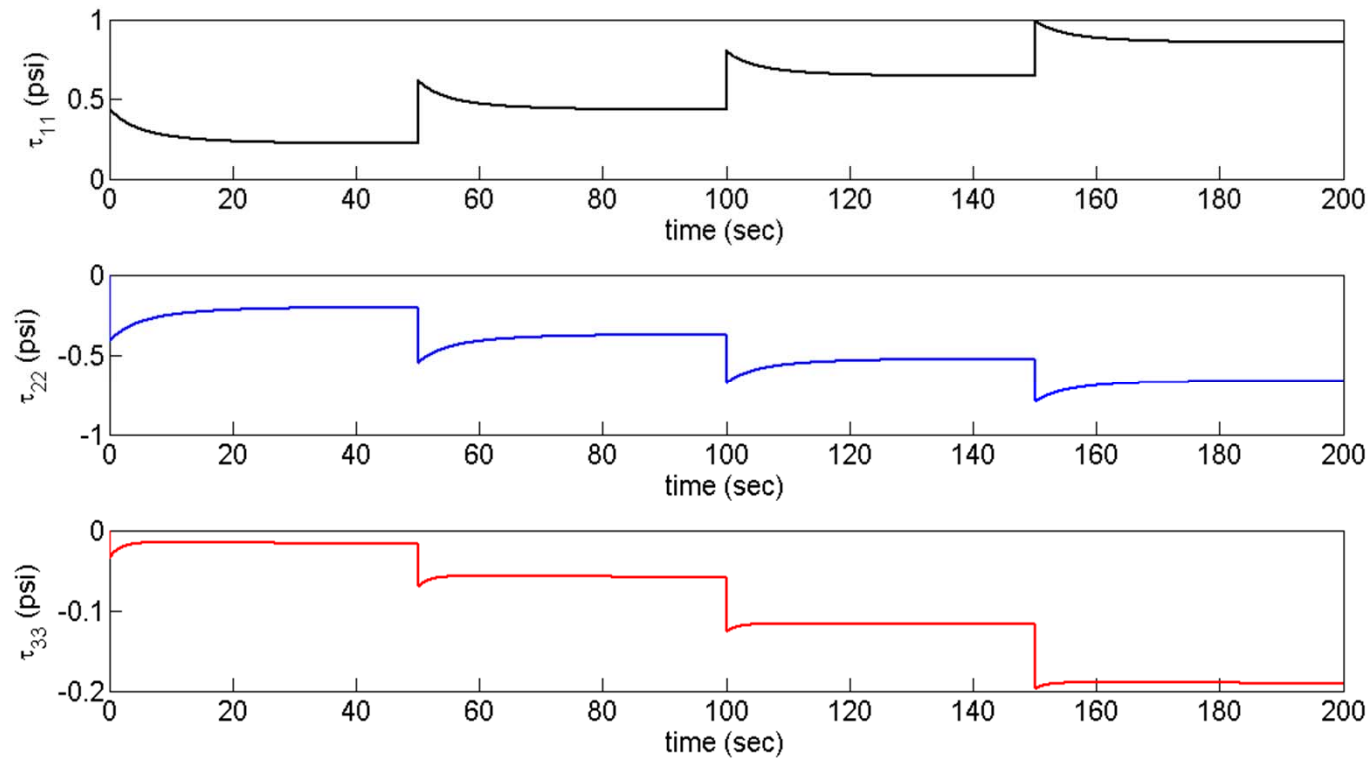
$$C_{20} = 0.0014 \text{ psi}$$

$$C_{11} = -0.0123 \text{ psi}$$

$$C_{02} = 0.0022 \text{ psi}$$

$$\eta_1 = 6 \text{ psi s}, \quad \eta_2 = 2 \text{ psi s}$$

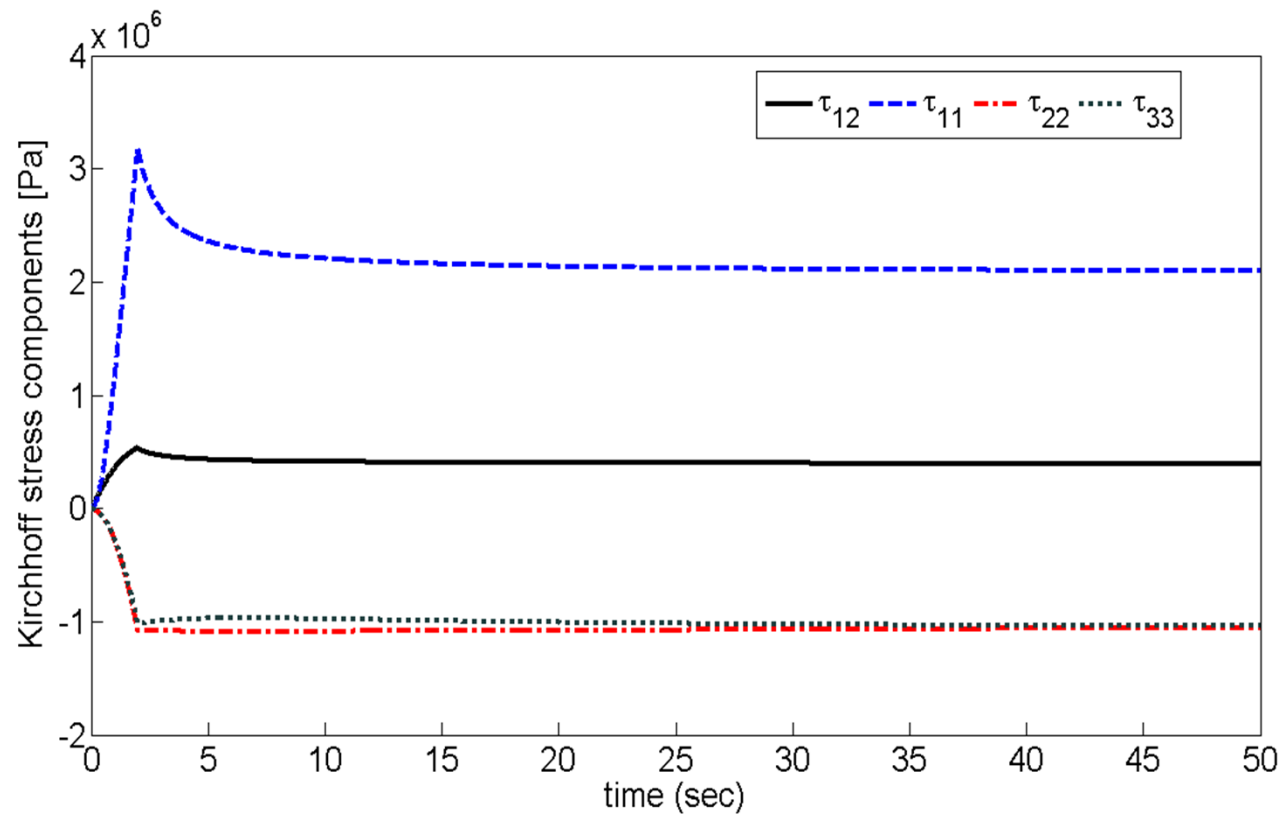
# Multi step relaxation in isochoric stretching



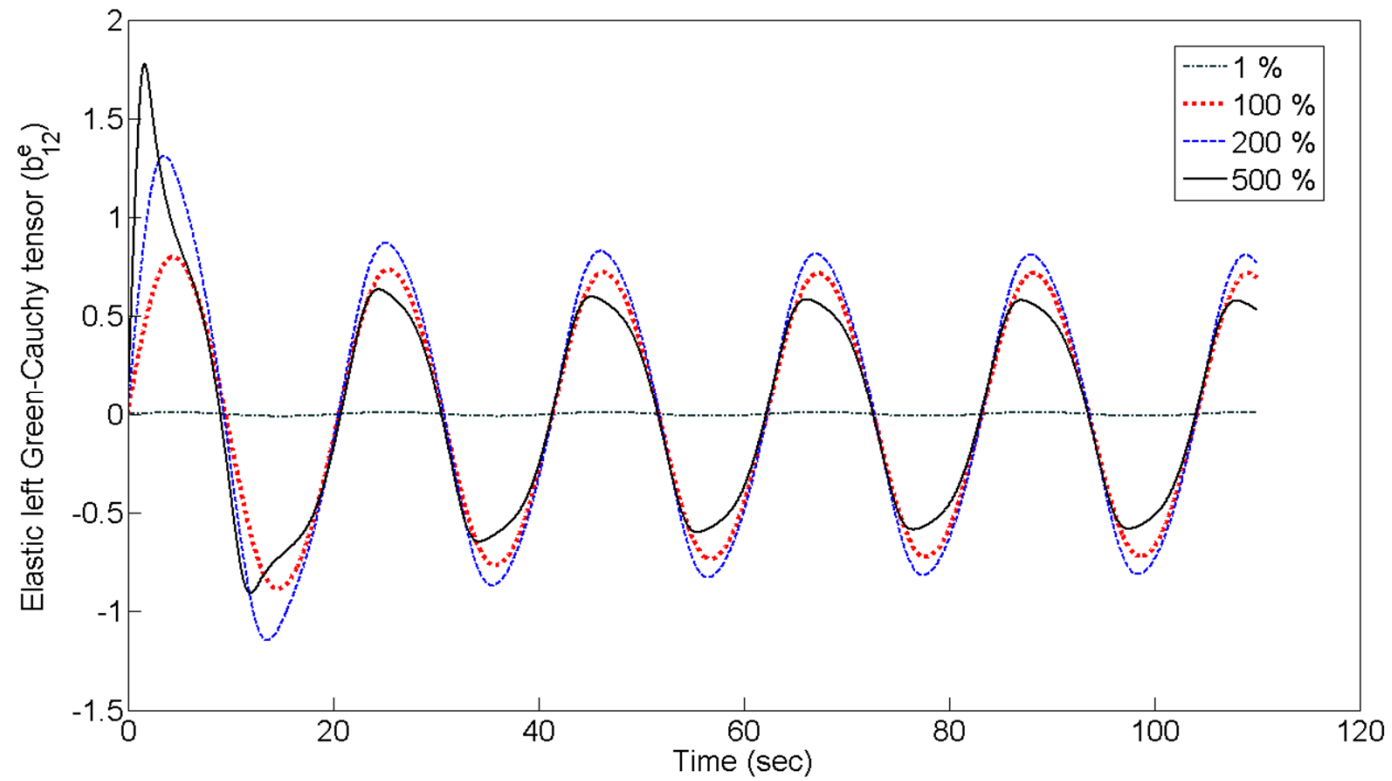
Kirchhoff stress relaxation under multistep relaxation loading

# Stress relaxation under simple shear motion

Applied shear linearly increases to  $\gamma_{\max} = 8$  over 2 s, then remains constant for 48 s.



# Effect of nonlinear viscoelasticity



# A finite linear viscoelastic model

based on Simo (1987), Holzapfel (1996 & subsequent)

$$\boldsymbol{\tau} = J \frac{\partial \Psi^{eq}}{\partial J} \mathbf{1} + \text{dev} \left( \bar{\mathbf{F}} \left( 2 \frac{\partial \Psi^{eq}}{\partial \bar{\mathbf{C}}} - \mathbf{S}^v \right) \bar{\mathbf{F}}^T \right)$$

Suppose  $\mathbf{S}^v$  evolves according to the following fractional order DE corresponding to the fractional Zener model:

$$D^\beta \mathbf{S}^v + \frac{1}{\tau^\beta} \mathbf{S}^v = \frac{1-\zeta}{\tau^\beta} \left( \bar{\mathbf{F}}^{-1} \text{dev} \left( 2 \bar{\mathbf{F}} \frac{\partial \Psi^{eq}}{\partial \bar{\mathbf{C}}} \bar{\mathbf{F}}^T \right) \bar{\mathbf{F}}^{-1T} \right)$$

The above equations (stress-strain and evolution) combine to

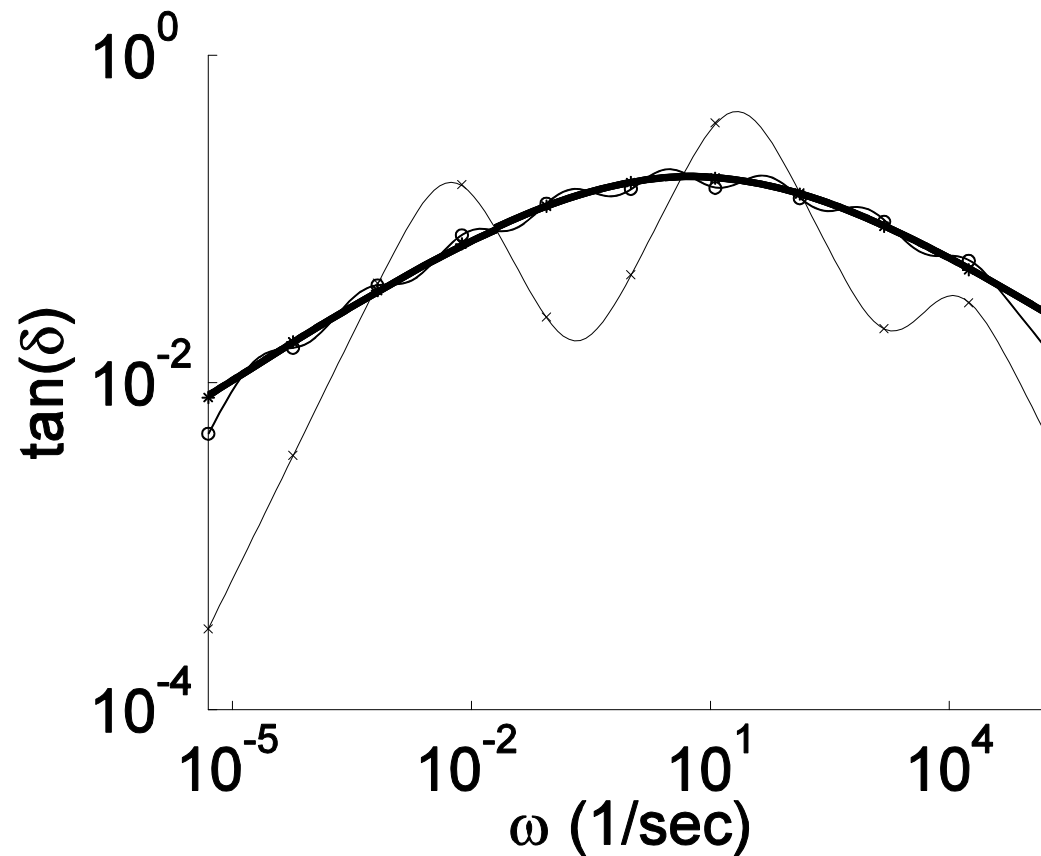
$$\text{dev } \boldsymbol{\tau} = \text{dev} \int_0^t G(t-t') \frac{d}{dt'} \left( \bar{\mathbf{F}}^{-1} \text{dev} \left( 2 \bar{\mathbf{F}} \frac{\partial \Psi^{eq}}{\partial \bar{\mathbf{C}}} \bar{\mathbf{F}}^T \right) \bar{\mathbf{F}}^{-1T} \right) dt'$$

where  $G(t) = \zeta + (1-\zeta) E_\beta \left( -\left(\frac{t}{\tau}\right)^\beta \right)$  is the relaxation function of the fractional Zener model

# Fractional derivative models of linear viscoelasticity

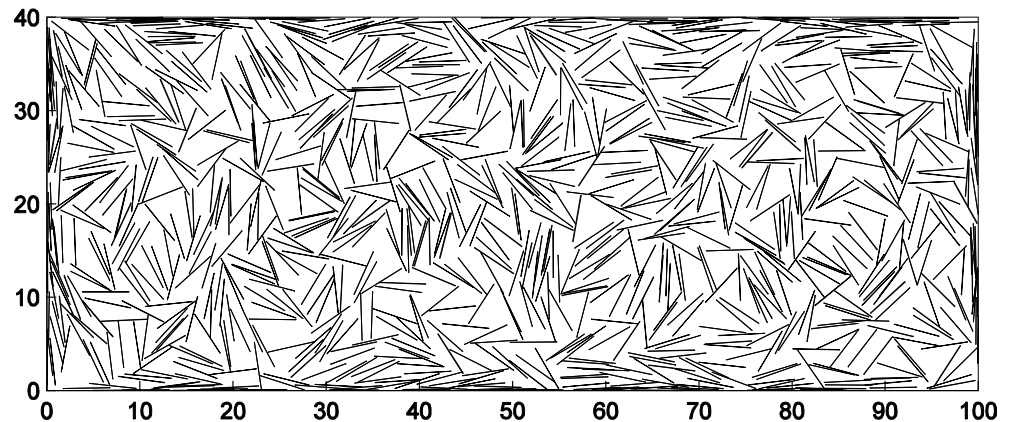
- Fit data over a large range of frequencies with a minimal set of parameters
- Are hard to integrate

**Rheological approximation** (Papoulia *et al.*, Rheol. Acta , 2010)



## Other related topics

Are neurons fibers?  
Answer: Is brain rubber?

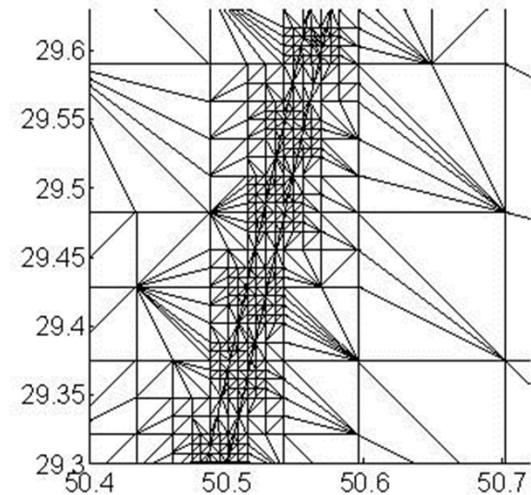


### Traditional fiber modeling:

Select a cubical sample of the matrix and insert fibers using a random number generator, i.e., center and orientation chosen uniformly at random

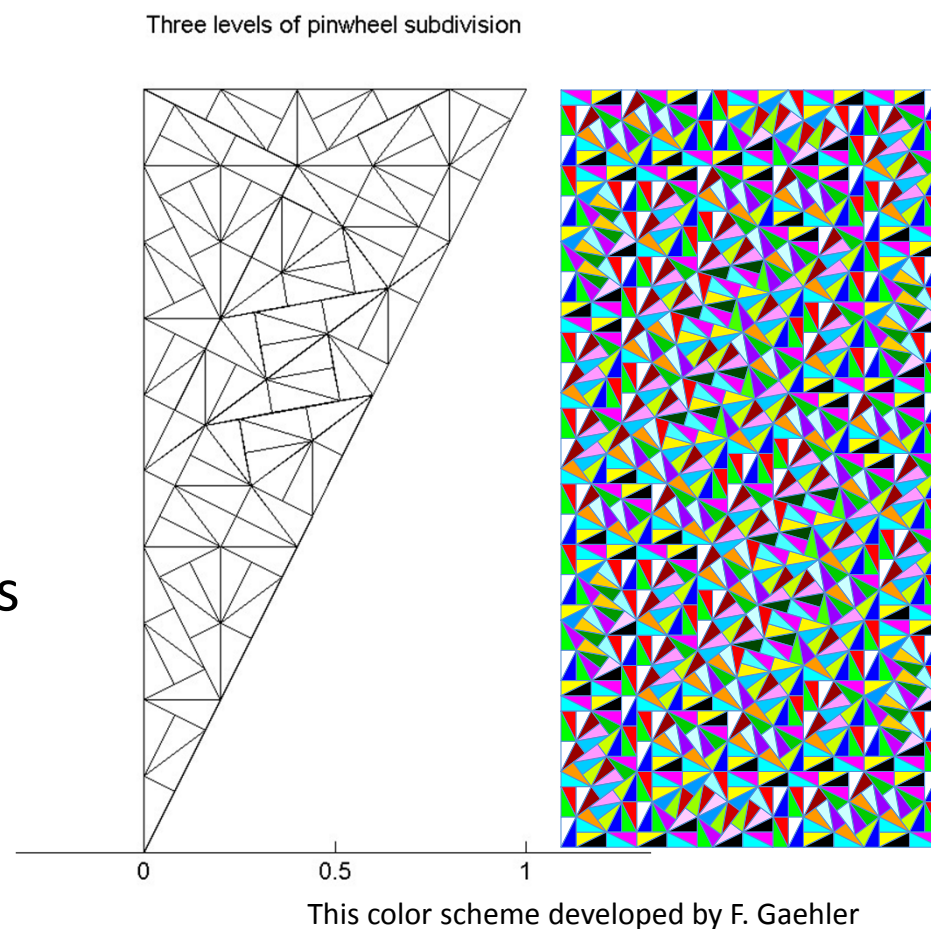
Finite element mesh is generated after the fibers are placed.

The finite element mesh is not easy to generate and is huge



# Pinwheel tiling

- Pinwheel tilings (Radin, 1994 & Conway) can geometrically represent all possible curves in the limit. (“Isoperimetry”)
- A 1:2 right triangle can be subdivided into five mutually congruent smaller 1:2 right triangles.
- Tiling is obtained by using this rule recursively
- Isotropic pinwheel meshes (Ganguly *et al.*, *SIAM J. Sci Comp*, 2006)



# Isoperimetry

- Any line segment  $L$  contained in the initial triangle has an approximation  $L_n$  in the level- $n$  pinwheel tiling (using only tile edges) such that  $L_n \rightarrow L$  and  $\text{length}(L_n) \rightarrow \text{length}(L)$  as  $n \rightarrow \infty$ .
- More formal statement: for any line segment  $(p,q)$  and for any  $\varepsilon > 0$ , there exists a refinement  $T$  of the pinwheel tiling such that the shortest  $(p,q)$  path using only mesh edges of  $T$  has length  $\|p-q\| + \varepsilon$ .
- This result follows because the pinwheel tiling has an infinite number of angles represented in the neighborhood of any point in the limit  $n \rightarrow \infty$  (unique property of these geometries.)

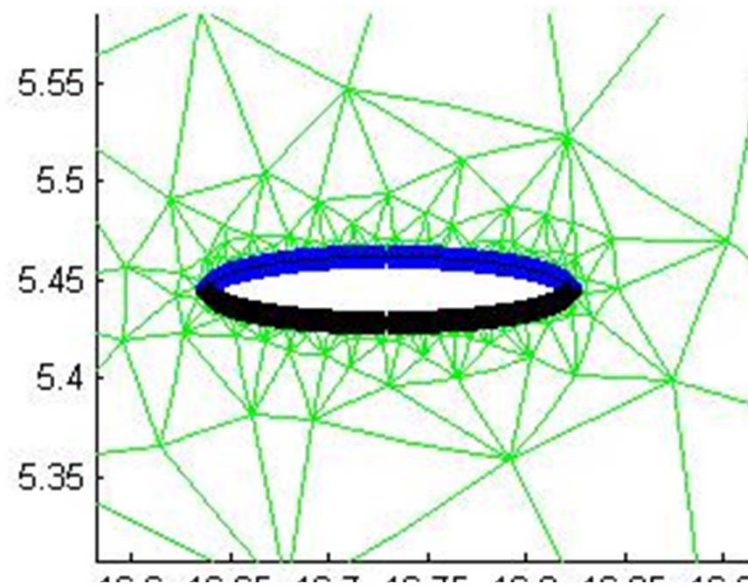
We explore the possibility of fibers selected as randomly chosen edges of the matrix tetrahedra.

Number of fibers	Resulting number of tetrahedral elements in mesh	
	QMG (Vavasis, 1997)	Isoperimetric
200		
2000	2,990,404	65,536
20000	30,953,434	524,288

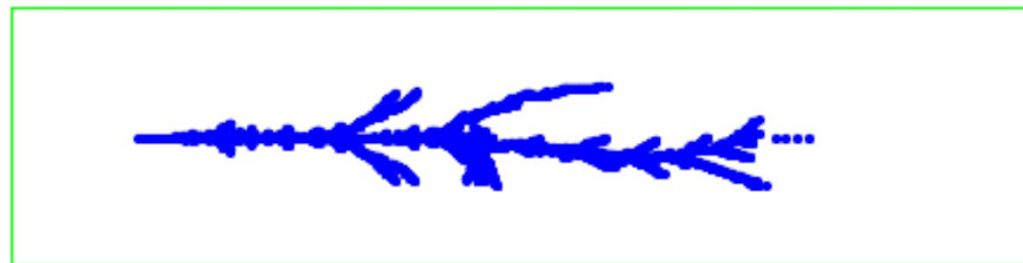
- The fibers can be selected as randomly chosen edges of isoperimetric tetrahedra or prisms.
- Preceding theorem guarantees isotropy of fiber orientations in the limit
- We first generate the mesh and then insert the fibers

## Other related topics

State of stress at “tip” and propagation of “crack”



Distort-2, numElements  $\approx$  201000, Pinwheel mesh



# Conclusions

- A rate form hyperelastic model is a good candidate for the simulation of brain tissue at finite strains
- The model is exactly integrable and free of spurious oscillations.
- Difficulty in incorporating material anisotropy.
- A thermodynamic framework will prove useful in modeling the multi-physics processes in the brain
- Finite linear models are extensively used but will fail to model mechanical response at large amplitudes and high frequencies.
- Fractional derivative models predict response to mechanical loads well over a large range of frequencies with a minimal number of parameters
- Rheological approximations to FD models maintain the number of parameters while able to be calibrated to the desired range of frequencies
- Isoperimetric grids do well in modeling random orientation of 1D microstructures as well as the evolution of fracture paths.