Optimal Life Insurance Purchase, Consumption and Investment in a Financial Market with Multi-Dimensional Diffusive Terms

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A Financial Planning Problem

- A young wage earner is just starting a career and a family
- He wants to allocate anticipated income among
  - Consumption
  - Investments
  - Life Insurance Purchase
- His “concerns” are
  - Level of consumption during working years
  - Size of estate at retirement (if he lives that long)
  - Size of estate (bequest) if he dies before retirement

What is the optimal allocation of income?

Literature Review

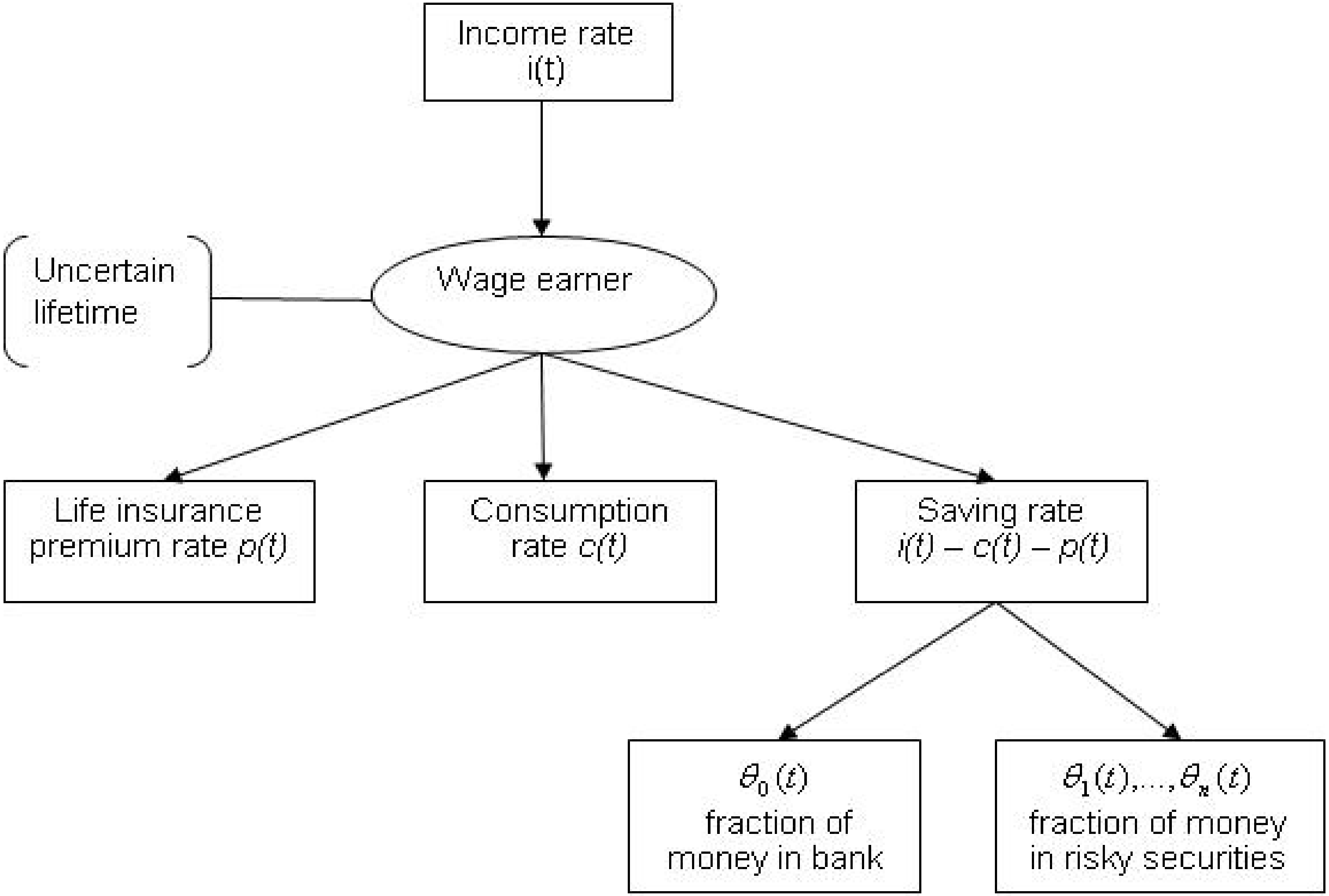
- Yaari (1965) considered a problem of optimal financial planning (including insurance purchase) decisions for an individual with an uncertain lifetime. A key idea was showing how to convert the optimization problem having a random planning horizon to one with a fixed time horizon.

- Merton (1969, 1971) studied the problem of optimal consumption and portfolio investment for a fixed planning horizon, but with life insurance playing no role. He formulated a stochastic model in a continuous-time setting and used dynamic programming to derive the optimal consumption and portfolio strategies for particular cases.

- Richard (1975) extended Merton's work to include a random lifetime and optimal decisions about life insurance purchase in addition to decisions about consumption and investment
  - He used dynamic programming (like Merton) with a fixed, finite planning horizon that served as an upper bound for the lifetime of the wage earner
  - Using sophisticated methods, he obtained explicit solutions for certain cases

- Ye (2005) studied a similar model, except that, in contrast to Richard's set-up, the planning horizon T is fixed and should be interpreted as the time when the wage earner retires
  - He obtained explicit solutions for certain cases
  - But he assumed only a single risky security, in addition to the riskless bank account, was available for investment

Our Basic Model



Income rate i(t)

Uncertain lifetime

Wage earner

Life insurance premium rate *p(t)*

Consumption rate *c(t)*

Saving rate $i(t) - c(t) - p(t)$

$\theta_0(t)$ fraction of money in bank

$\theta_1(t),\ldots,\theta_n(t)$ fraction of money in risky securities

Lifetime of the Wage Earner

- The wage earner is alive and starts working at time $t = 0$
- His lifetime is a non-negative random variable τ which is independent of the risky securities
- This random variable has the distribution function F and the density function f, which are known to the wage earner
- Corresponding to τ is the hazard rate function λ, that is

$$\lambda(\mathrm{t}) = f(t)/[1 - F(t)]$$

The (Continuous-Time) Insurance Market

- At any point in time t the wage earner will be covered by a life insurance policy if he is paying the premium payment rate $p(t)$ to the insurance company
- Then if he dies at time t, his beneficiaries will collect

$$p(t)/\eta(t),$$

where η is a strictly positive, continuous function called the insurance premium payout ratio

- $\eta(t)$ is known in advance by the wage earner; $p(t)$ is chosen by him (or her)
- No insurance will be purchased after the start of retirement

The Decision Variables

- $c(t)$ = consumption rate at time t
- $p(t)$ = premium payment rate at time t
- $\theta_n(t)$ = proportion of portfolio value invested in security n, $n = 0, 1, \ldots, N$ (so $\theta_0(t) + \theta_1(t) + \ldots + \theta_N(t) = 1$)
  - Security $n = 0$ is a riskless bank account having interest rate $r(t)$, a specified, deterministic function
  - Securities $n = 1, \ldots, N$ are modeled as multivariate, geometric Brownian motions, the dynamics of which are specified and known to the wage earner

Other Variables of Interest

- $X(t)$ = current wealth = initial wealth + cumulative income - cumulative consumption - cumulative insurance payments + time-t value of investments

- $Z(t) = X(t) + p(t)/\eta(t)$ = wage earner's legacy or bequest upon death if $t = \tau < T$

- $b(t)$ = time-t value of human capital, that is, a discounted value of the wage earner's future income, with the discount rate being the interest rate r combined with the insurance premium payout ratio η

The Wage Earner's Objective

- Maximize the expected utility of:

  1. Cumulative consumption while working
  2. The time-T value of the investments, if $\tau > T$
  3. The time- τ value of the bequest, if $\tau < T$

- $V(t,x)$ denotes the maximum expected utility if the time-t value of the investments is x

Stochastic Dynamic Programming

- With mild assumptions the maximum expected utility $V(t,x)$ will be the unique solution of a Hamilton-Jacobi-Bellman equation (details later, if time permits)
- From such a solution, the optimal strategy (c^*, p^*, θ^*) can readily be determined
- Unfortunately, it is almost impossible to obtain explicit solutions, except for some rare special cases
- Hence one usually needs to use numerical methods to compute $V(t,x)$

Discounted CRRA Utilities

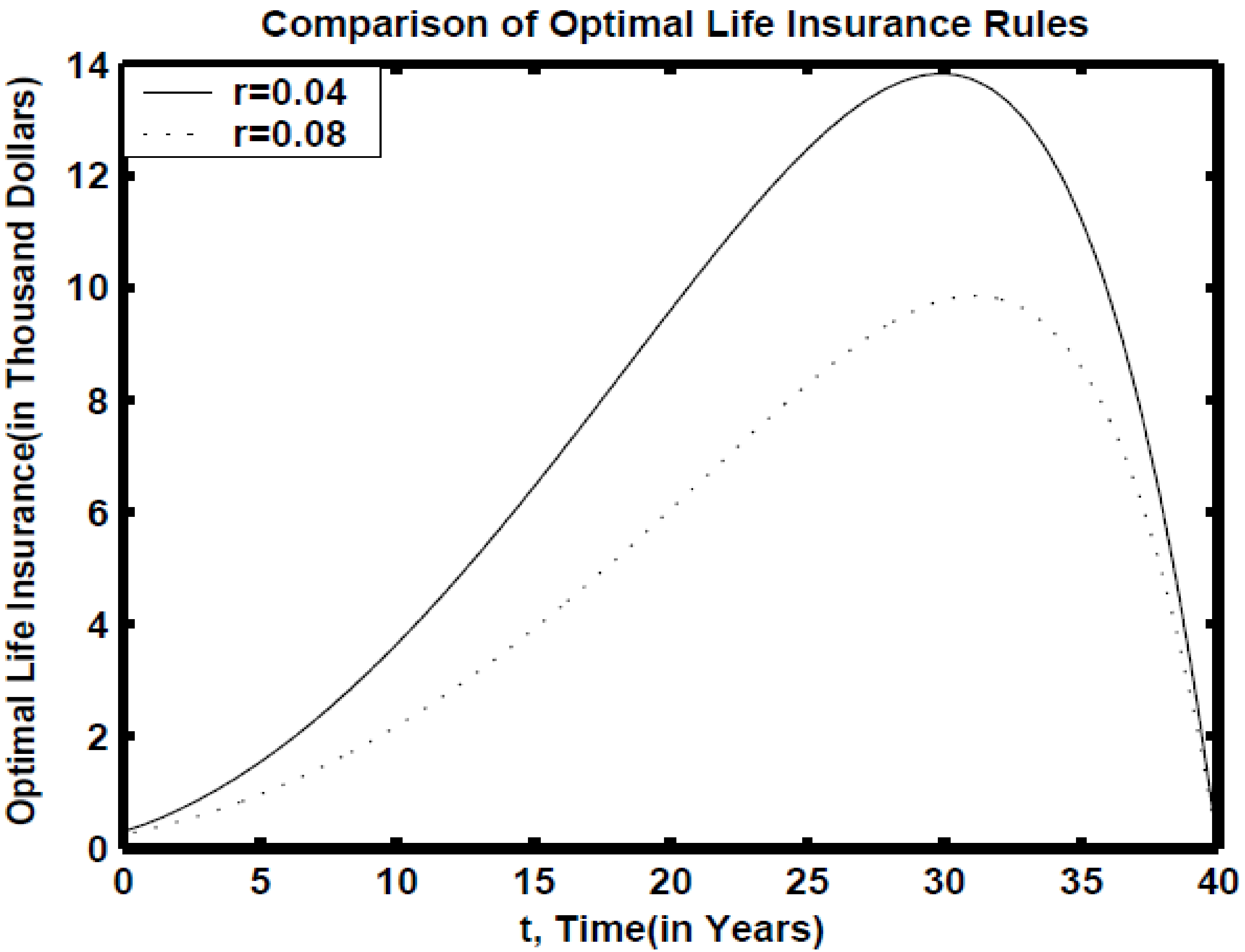
- From now on we restrict ourselves to the special case where the wage earner has the same discounted Constant Relative Risk Aversion (CRRA) utility function for:
  - the consumption of his family
  - the size of his legacy
  - his wealth at retirement

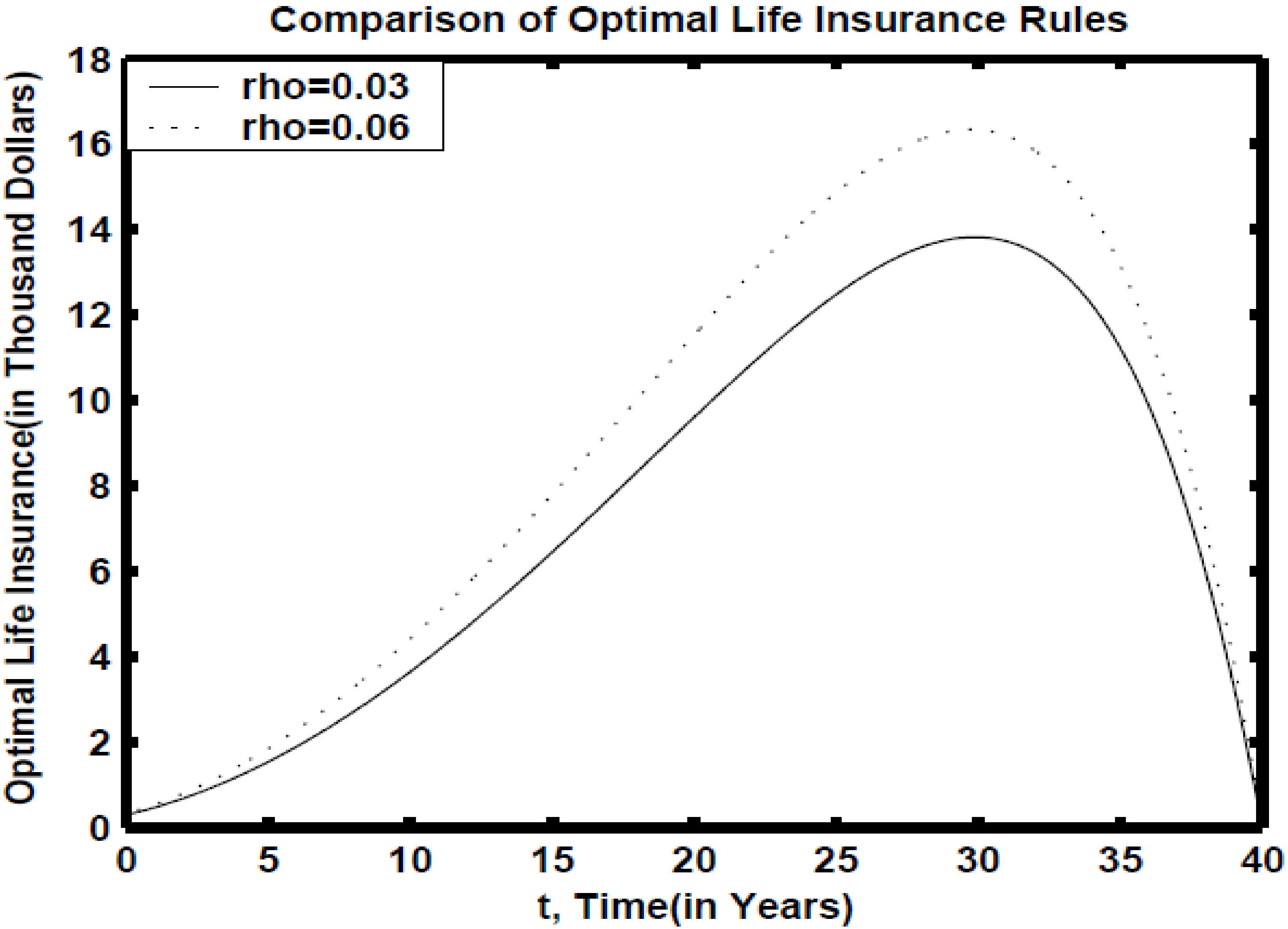
- We assume that $\gamma < 1$, $\gamma \neq 0$ and $\rho > 0$ and let
  - $U(c; t) = e^{-\rho t} c^{\gamma}/\gamma$ (utility of consumption rate *c*)
  - $B(Z; t) = e^{-\rho t} Z^{\gamma}/\gamma$ (utility of the bequest *Z*)
  - $W(X) = e^{-\rho T} X^{\gamma}/\gamma$ (utility of wealth *X* at retirement)

The Optimal Strategies

$$\begin{aligned}
c^*(t,x) &= \frac{1}{e(t)}(x+b(t)) \\
p^*(t,x) &= \eta(t)\left((D(t)-1)x+D(t)b(t)\right) \\
\theta^*(t,x) &= \frac{1}{x(1-\gamma)}(x+b(t))\xi\alpha(t)\ ,
\end{aligned}$$

where *b(t)* (the human capital), *D(t)*, and *e(t)* are various deterministic, scalar-valued functions, α(t) is a vector-valued, deterministic function, and ξ is a matrix


Comparison of Optimal Life Insurance Rules
r=0.04
r=0.08
Optimal Life Insurance(in Thousand Dollars)
0
2
4
6
8
10
12
14
t, Time(in Years)
0
5
10
15
20
25
30
35
40


Comparison of Optimal Life Insurance Rules
rho=0.03
rho=0.06
Optimal Life Insurance(in Thousand Dollars)
0
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16
18
t, Time(in Years)
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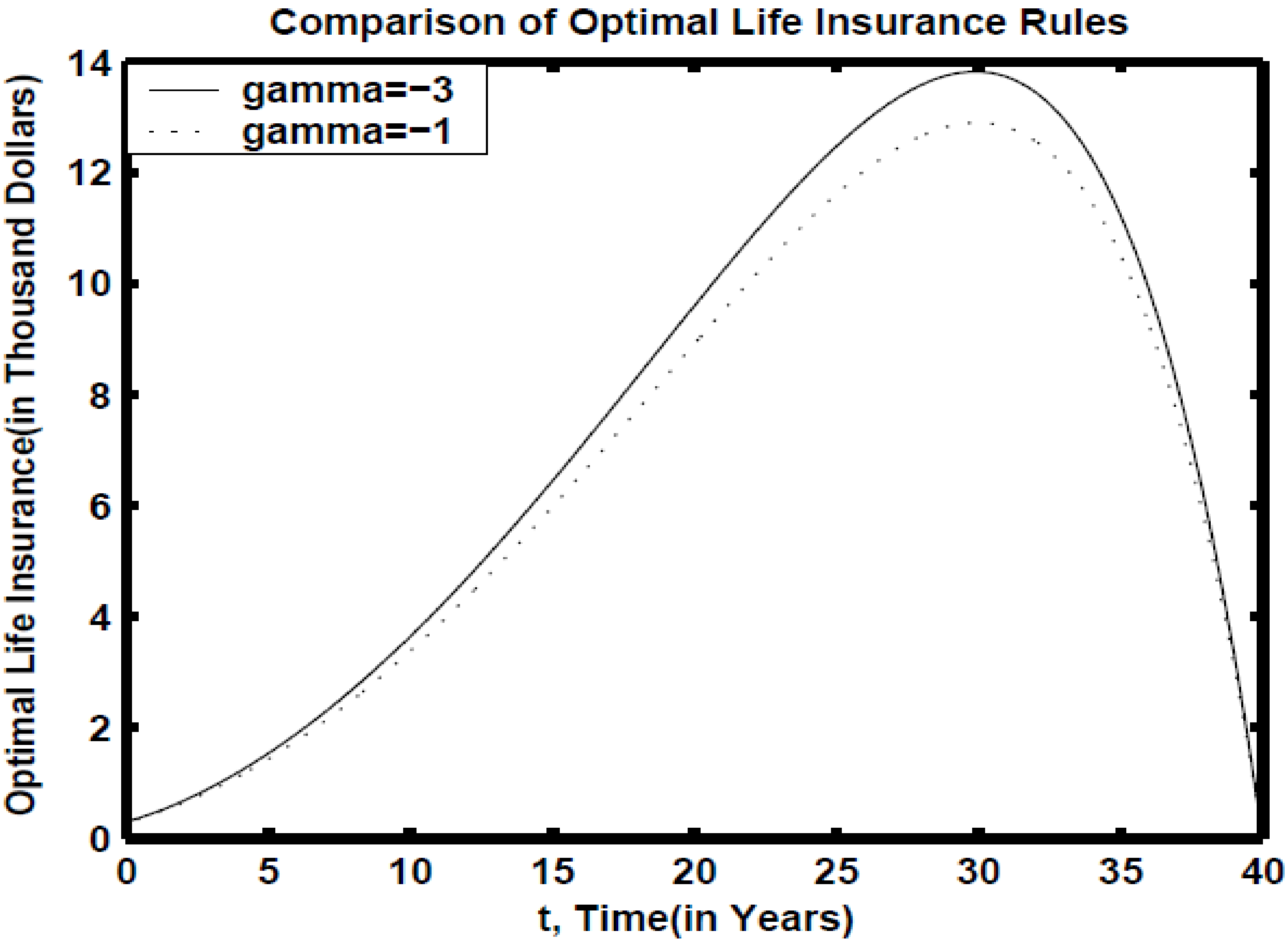
Comparison of Optimal Life Insurance Rules
gamma=−3
gamma=−1
Optimal Life Insurance(in Thousand Dollars)
t, Time(in Years)
0
2
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8
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12
14
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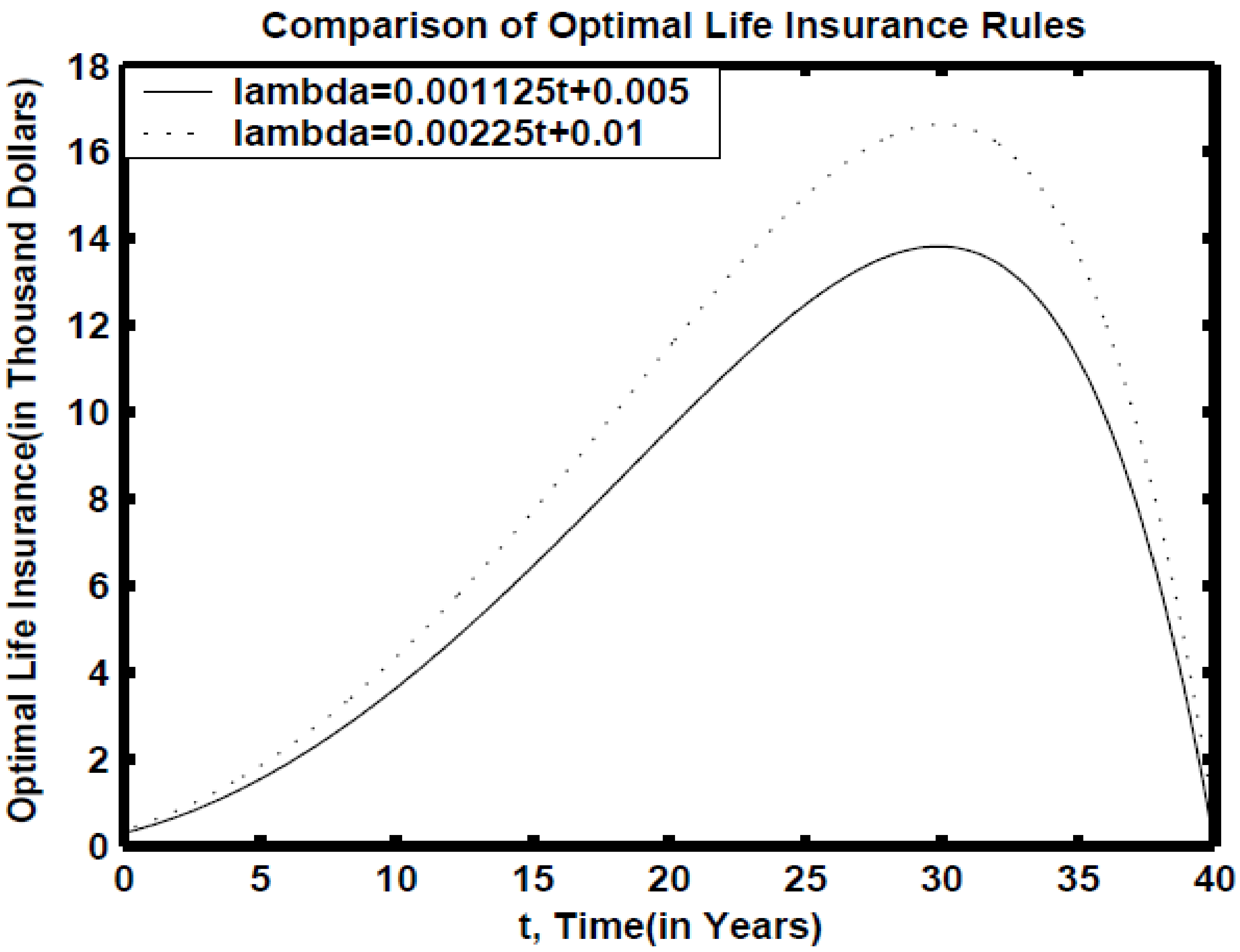
Comparison of Optimal Life Insurance Rules
lambda=0.001125t+0.005
lambda=0.00225t+0.01
Optimal Life Insurance(in Thousand Dollars)
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2
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t, Time(in Years)
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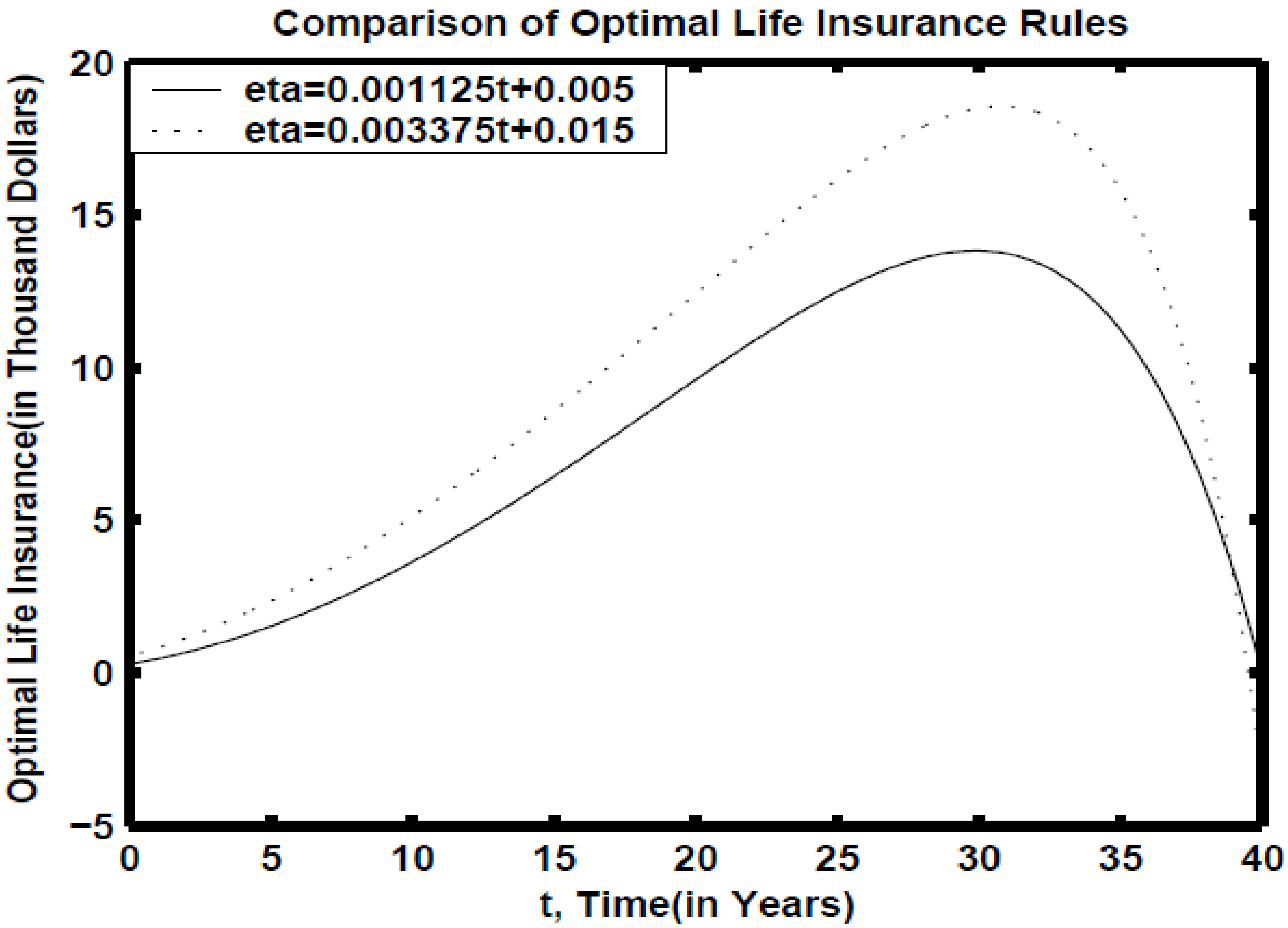
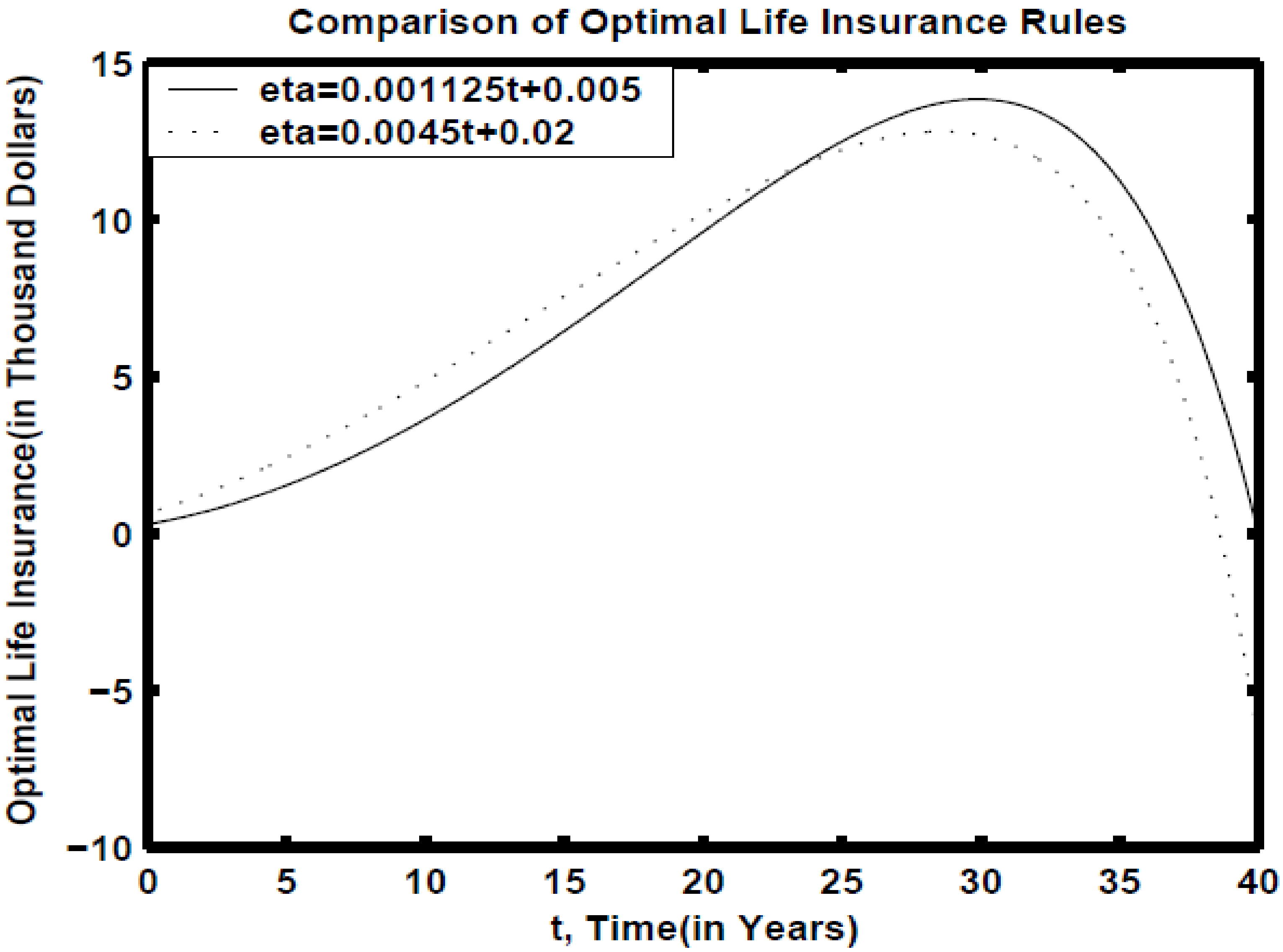

Comparison of Optimal Life Insurance Rules
eta=0.001125t+0.005
eta=0.003375t+0.015
Optimal Life Insurance(in Thousand Dollars)
20
15
10
5
0
−5
0
5
10
15
20
25
30
35
40
t, Time(in Years)


Comparison of Optimal Life Insurance Rules
eta=0.001125t+0.005
eta=0.0045t+0.02
Optimal Life Insurance(in Thousand Dollars)
15
10
5
0
−5
−10
0
5
10
15
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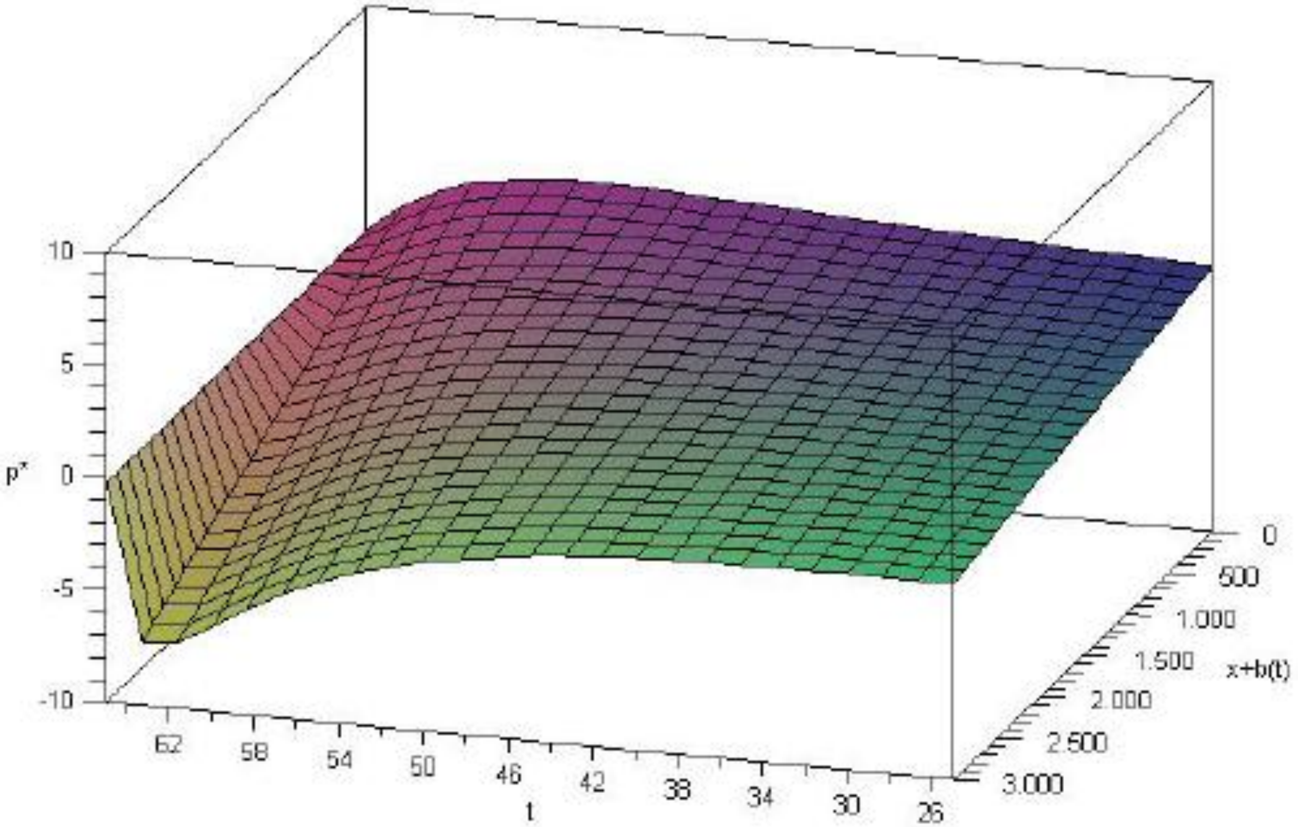
40
t, Time(in Years)

In the next slide:

- $N = M = 2$ risky securities
- income rate $i(t) = 50{,}000 \exp(0.03t)$
- interest rate $r = 4\%$
- discount rate $\rho = 3\%$
- risk aversion parameter $\gamma = -3$
- hazard rate $\lambda(t) = 0.001 + \exp(-9.5 + 0.1t)$
- insurance premium payout ratio $\eta(t) = 1.05\,\lambda(t)$
- appreciation rates $\mu_1 = 0.07, \ \mu_2 = 0.11$
- volatilities $\sigma = 0.19, \ \sigma = 0.21$
- correlation between risky assets 80%

Optimal Life Insurance Purchase (2 Risky Securities)



Some Conclusions About $p^*(t,x)$ for the CRRA Case (with some modest assumptions)

- The optimal insurance purchase strategy $p^*(t,x)$ is:
  - A decreasing function of the total wealth x
  - An increasing function of the human capital $b(t)$
  - For all small enough values of the wealth x it is a unimodal function of the age t
  - A function that can be negative for some x and t
  - A decreasing function of the interest rate r in some neighborhood of $(t,x) = (0,0)$
  - A decreasing function of the risk aversion parameter γ except in some neighborhood of $t = T$
  - An increasing function of the utility discount rate ρ
  - An increasing function of the hazard rate λ in some neighborhood of $x = 0$

Some Conclusions About The Optimal Risky Proportions (With an Additional, Modest Assumption)

- For each risky security n the optimal proportion $\theta_n(t)$ is a decreasing function of the total wealth x
- For each risky security n the optimal proportion $\theta_n(t)$ is an increasing function of the human capital $b(t)$
- For small enough wealth it can be optimal to borrow money from the riskless bank and buy the risky securities on margin

A “Mutual Fund” Result

Recall the optimal strategy:

$$\theta^*(t,x) = \frac{1}{x(1-\gamma)}(x+b(t))\xi\alpha(t)$$

Actually, this vector has only N components, corresponding to the N risky securities, so the optimal proportion for the bank is

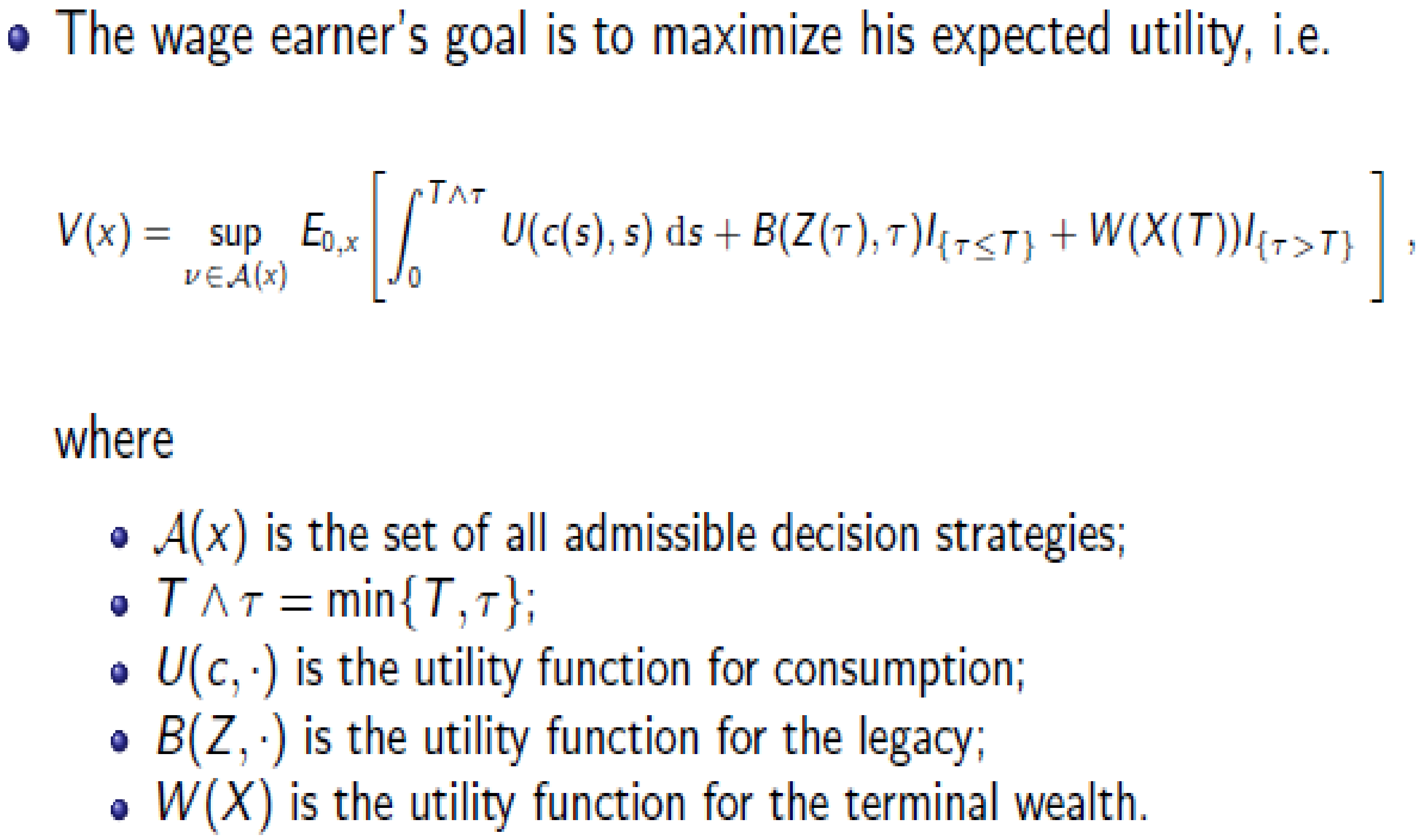
$$\theta_0 = 1 - \theta_1 - \ldots - \theta_N$$

Here ξ is the covariance matrix and α is the column vector of excess returns describing the dynamics of the risky securities. Since the preceding factor is a scalar, we see that the relative proportions among the risky securities are independent of the insurance and mortality data (intuition: due to no correlation).

Comparison With an Otherwise Identical Wage Earner Who Does Not Have the Opportunity To Purchase Life Insurance

- In the paper we goofed up because we ignored the utility of the bequest (equal to value of investments) in the event of premature death
- The problem for this second wage earner should be identical to that of the first with the added constraint that $p(t,x) = 0$
- So ignore this part of the present paper
- We will deal with this in a revised version

Now For Some Mathematics



- The wage earner's goal is to maximize his expected utility, i.e.

$$V(x) = \sup_{\nu \in \mathcal{A}(x)} E_{0,x}\left[\int_0^{T \wedge \tau} U(c(s), s)\, \mathrm{d}s + B(Z(\tau), \tau) I_{\{\tau \leq T\}} + W(X(T)) I_{\{\tau > T\}} \right],$$

  where

  - $\mathcal{A}(x)$ is the set of all admissible decision strategies;
  - $T \wedge \tau = \min\{T, \tau\}$;
  - $U(c, \cdot)$ is the utility function for consumption;
  - $B(Z, \cdot)$ is the utility function for the legacy;
  - $W(X)$ is the utility function for the terminal wealth.

The Dynamic Programming Principle

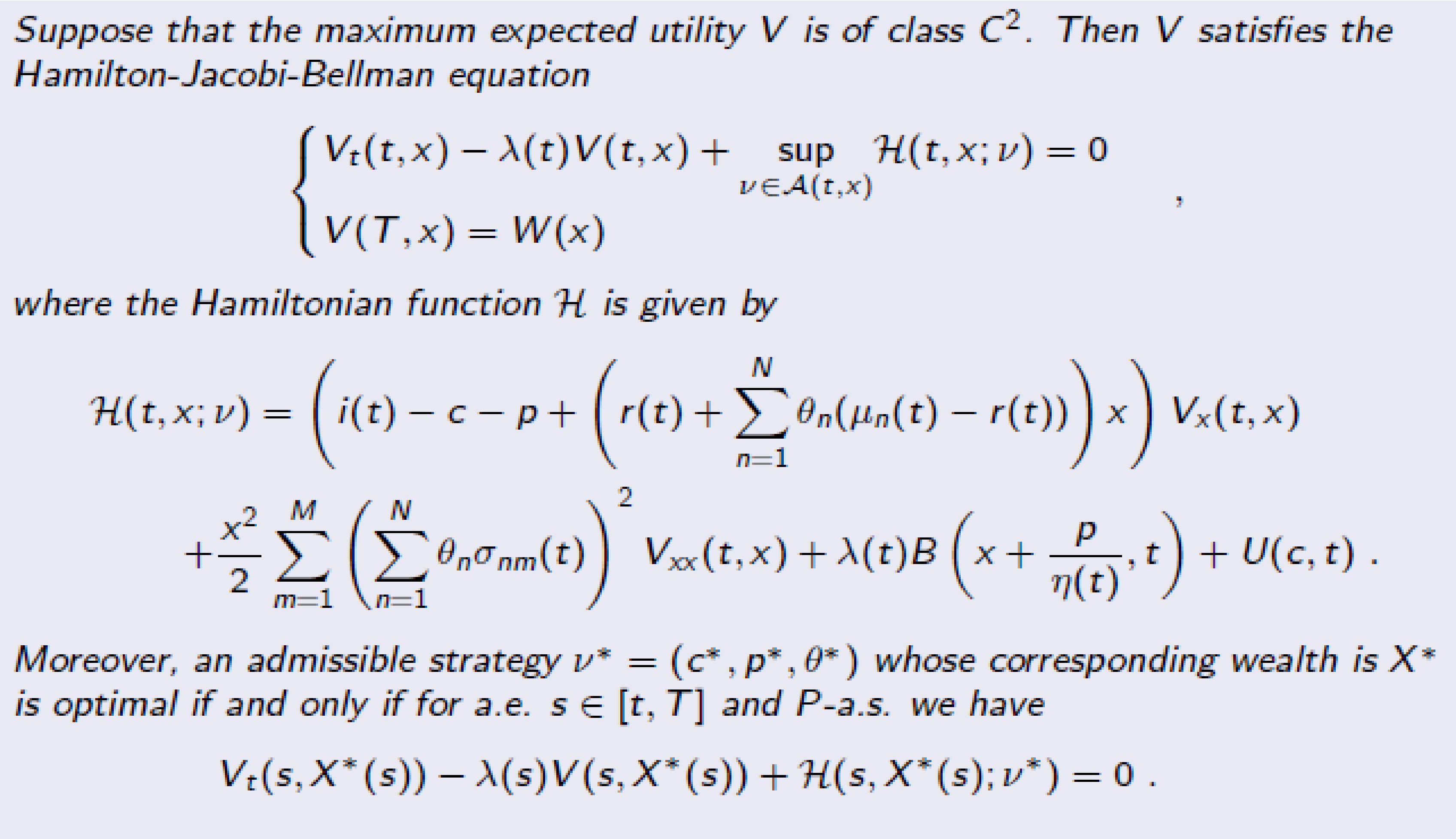
- Let $\mathcal{A}(t,x)$ be the set of admissible decision strategies $\nu = (c,p,\theta)$ for the dynamics of the wealth process with boundary condition $X(t) = x$.
- For any $\nu \in \mathcal{A}(t,x)$, define the functional

$$J(t,x;\nu) = E_{t,x}\left[\int_t^{T\wedge\tau} U(c(s),s)\,\mathrm{d}s + B(Z(\tau),\tau)I_{\{\tau\le T\}} + W(X(T))I_{\{\tau>T\}}\,\middle|\,\tau > t, \mathcal{F}_t\right].$$

- The optimal control problem can be restated in dynamic programming form as

$$V(t,x) = \sup_{\nu\in\mathcal{A}(t,x)} J(t,x;\nu).$$

The Hamilton-Jacobi-Bellman Equation



Suppose that the maximum expected utility V is of class C^2. Then V satisfies the Hamilton-Jacobi-Bellman equation

$$\begin{cases} V_t(t,x) - \lambda(t)V(t,x) + \sup\limits_{\nu \in \mathcal{A}(t,x)} \mathcal{H}(t,x;\nu) = 0 \\ V(T,x) = W(x) \end{cases} ,$$

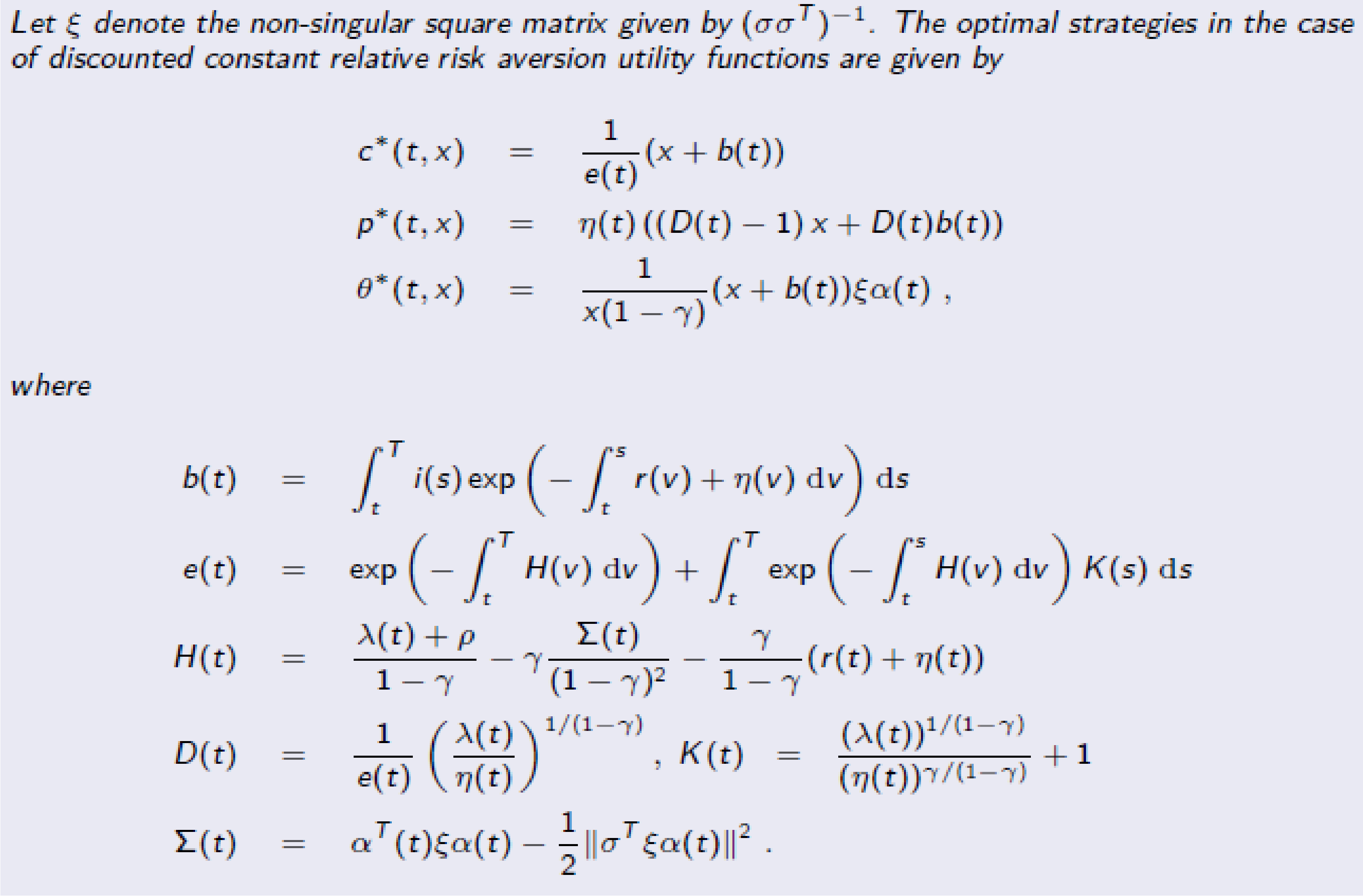
where the Hamiltonian function $\mathcal{H}$ is given by

$$\mathcal{H}(t,x;\nu) = \left(i(t) - c - p + \left(r(t) + \sum_{n=1}^{N} \theta_n(\mu_n(t) - r(t)) \right) x \right) V_x(t,x)$$
$$+ \frac{x^2}{2} \sum_{m=1}^{M} \left(\sum_{n=1}^{N} \theta_n \sigma_{nm}(t) \right)^2 V_{xx}(t,x) + \lambda(t) B \left(x + \frac{p}{\eta(t)}, t \right) + U(c,t) \ .$$

Moreover, an admissible strategy $\nu^* = (c^*, p^*, \theta^*)$ whose corresponding wealth is X^* is optimal if and only if for a.e. $s \in [t,T]$ and P-a.s. we have

$$V_t(s, X^*(s)) - \lambda(s)V(s, X^*(s)) + \mathcal{H}(s, X^*(s); \nu^*) = 0 \ .$$

Optimal Strategies For the CRRA Case



Let ξ denote the non-singular square matrix given by $(\sigma\sigma^T)^{-1}$. The optimal strategies in the case of discounted constant relative risk aversion utility functions are given by

$$\begin{aligned}
c^*(t,x) &= \frac{1}{e(t)}(x+b(t)) \\
p^*(t,x) &= \eta(t)\left((D(t)-1)\,x + D(t)b(t)\right) \\
\theta^*(t,x) &= \frac{1}{x(1-\gamma)}(x+b(t))\xi\alpha(t)\ ,
\end{aligned}$$

where

$$\begin{aligned}
b(t) &= \int_t^T i(s)\exp\left(-\int_t^s r(v)+\eta(v)\,\mathrm{d}v\right)\mathrm{d}s \\
e(t) &= \exp\left(-\int_t^T H(v)\,\mathrm{d}v\right) + \int_t^T \exp\left(-\int_t^s H(v)\,\mathrm{d}v\right)K(s)\,\mathrm{d}s \\
H(t) &= \frac{\lambda(t)+\rho}{1-\gamma} - \gamma\frac{\Sigma(t)}{(1-\gamma)^2} - \frac{\gamma}{1-\gamma}(r(t)+\eta(t)) \\
D(t) &= \frac{1}{e(t)}\left(\frac{\lambda(t)}{\eta(t)}\right)^{1/(1-\gamma)},\quad K(t) = \frac{(\lambda(t))^{1/(1-\gamma)}}{(\eta(t))^{\gamma/(1-\gamma)}}+1 \\
\Sigma(t) &= \alpha^T(t)\xi\alpha(t) - \frac{1}{2}\|\sigma^T\xi\alpha(t)\|^2\ .
\end{aligned}$$

Final Remarks

- Our model provides a variety of conclusions about optimal decisions for financial planning
  - Some conclusions simply confirm what we always thought
  - Other conclusions provide new insight and intuition

- There are considerable opportunities to develop various extensions, such as:
  - Other utility functions
  - Uncertain future income and/or hazard rate
  - A constraint ruling out selling one’s own insurance policy
  - Alternative insurance products

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