Consumption and Annuitization

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1. Introduction

Yaari (1965):

$$\max_{c(t)\in\Phi} \int_0^{\overline{T}} \Omega(t)\alpha(t)g(c(t))dt \tag{1}$$

subject to

$$c(t) \geq 0, \tag{2}$$

$$S(t) \geq 0, \tag{3}$$

$$S'(t) = j(t)S(t) + m(t) - c(t), \tag{4}$$

$$S(0) = S_0. \tag{5}$$

$\Omega(t)$ = survival probability at time t, $\alpha(t)$ = discount function, $c(t)$ = consumption, $g(c)$ = utility function, $S(t)$ = wealth (accumulated savings), $m(t)$ = income, $j(t)$ = interest rate, $\overline{T}$ = maximum lifetime, S_0 = initial wealth at time 0, and Φ = space of piecewise continuous functions that take values on $[0, \infty)$.

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- A minute increase in S_0 can change the model from one with an optimal solution to one without.
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- What is the cause of the existence problem?

THEOREM 1 (Leung (1994)). *If either* $\lim_{c \to 0^+} g'(c) < \infty$ *or* $m(\overline{T}) > 0$, *then there exists a* $t^* \in [0, \overline{T})$ *such that* $S^*(t) = 0$ *and* $c^*(t) = m(t)$ *for all* $t \in [t^*, \overline{T}]$.

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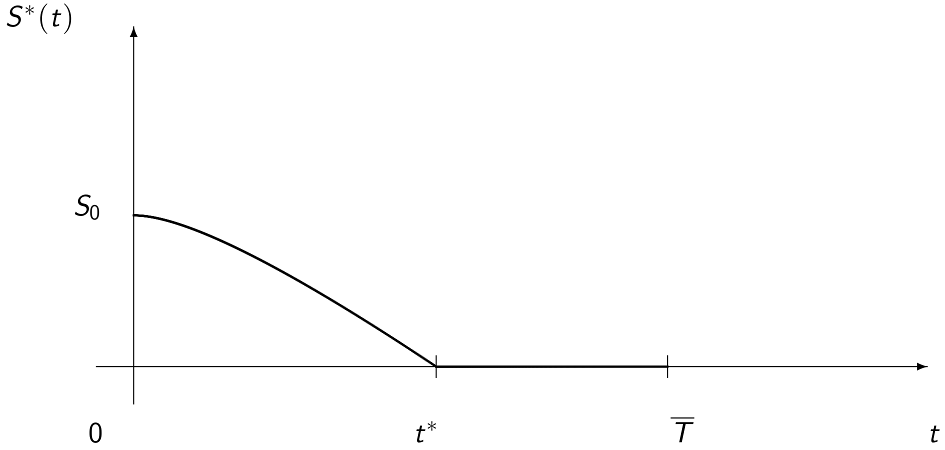
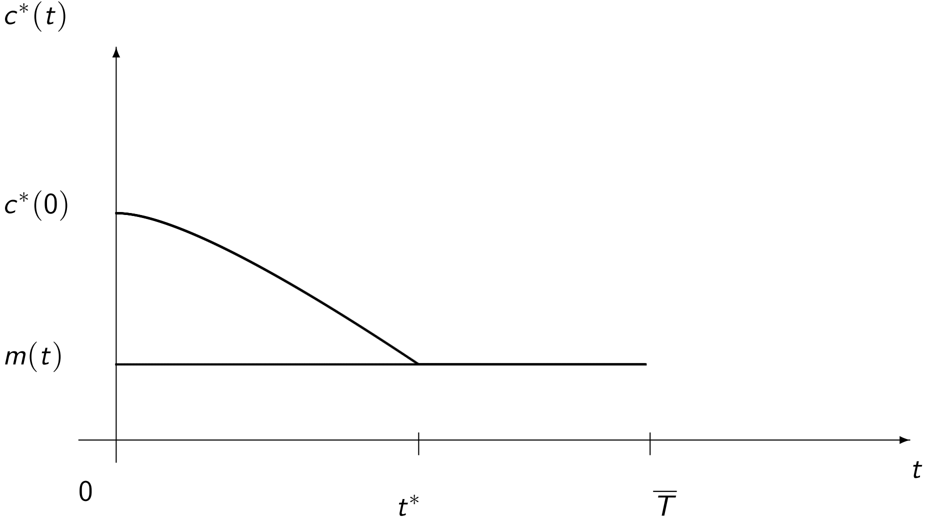
- The entire wealth is exhausted at t^* and the depletion is terminal (as opposed to temporary).

Figure 1
The path of $S^*(t)$



- Consumption after t^* will be exactly equal to income.

Figure 2
The path of $c^*(t)$



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- Let $\pi(t)$ and $\pi_t(t)$ denote the probability density function of T and the hazard rate of death, respectively. By definition, $\Omega(t) = \Pr(T > t) = \int_t^{\overline{T}} \pi(x)dx$ and $\pi_t(t) = \pi(t)/\Omega(t)$.

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- The key factor that causes wealth depletion is uncertain lifetime because it eventually drives the effective discount rate for the future (i.e., $-\pi_t(t) + \frac{\alpha'(t)}{\alpha(t)}$) to infinity (as $\Omega(\overline{T}) = 0$ means that $\pi_t(\overline{T}) = \infty$).

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- As emphasized in Rae (1834) and Fisher (1930), uncertain lifetime increases impatience and tends to reduce saving.
- What is new here is the analytical and unambiguous result that the uncertainty of lifetime will eventually lead to terminal wealth depletion before the maximum lifetime.

Simulation

- $g(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $\gamma > 0$ ($g(c) = \log(c)$ if $\gamma = 1$), $t \in [65, \overline{T}]$, $j = 0.03$
- $m(t) = M$ for $t \geq 65$, where M is a positive constant. It is a good approximation (e.g., Hurd (1989)).
- t^* is determined by

$$\frac{S(65)}{M} = \int_{65}^{t^*} e^{-j(t-65)} \left\{ \left[\frac{\Omega(t) e^{(j-\alpha)t}}{\Omega(t^*) e^{(j-\alpha)t^*}} \right]^{\frac{1}{\gamma}} - 1 \right\} dt.$$

- Mortality follows the Gompertz Law

$$\Omega(t) = e^{-0.00093\phi(e^{0.087t}-1)}, \ \phi = 1, 2.$$

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- Table 1 reports the simulation results for three different values of $\frac{S(65)}{M}$.

Table 1

Terminal Wealth Depletion Time t^*

		t^*					
		$\phi = 1$ $\frac{S(65)}{M}$			$\phi = 2$ $\frac{S(65)}{M}$		
γ	α	1	5	10	1	5	10
4	0.10	74	82	87	73	80	84
	0.05	77	86	90	75	82	86
	0.03	79	87	92	75	83	86
	0.01	81	89	93	76	84	87
1	0.10	70	74	76	69	73	75
	0.05	71	77	80	70	74	77
	0.03	73	79	82	71	75	78
	0.01	75	81	84	72	77	79
0.5	0.10	68	71	73	68	70	72
	0.05	70	73	76	69	72	74
	0.03	71	75	78	69	73	75
	0.01	73	78	80	70	74	76
0.1	0.10	66	67	68	66	67	68
	0.05	67	69	70	67	68	69
	0.03	68	70	71	67	68	69
	0.01	69	72	73	68	69	70

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  4. Other factors such as poor health, low risk aversion, low interest rate, or high income could also be responsible.
  5. The simulation results support that the model can account for low wealth holdings and early terminal wealth depletion observed in a number of empirical studies.

- For the general model (1) - (5), t^* is obtained by solving the equation $\varphi(t^*) = 0$, where

$$\varphi(t) = S_0 - \int_0^t e^{-\int_0^z j(x)dx} \left[(g')^{-1} \left(\frac{\Omega(t)\alpha(t)g'(m(t))e^{\int_z^t j(x)dx} + \int_z^t e^{\int_z^w j(x)dx}\mu(w)dw}{\Omega(z)\alpha(z)} \right) - m(z) \right] dz, \tag{6}$$

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  1. Existence: Whether $\varphi(t)$ has a root?
  2. Uniqueness: Whether it is possible for $\varphi(t)$ to have multiple roots?
  3. Optimality: If $\varphi(t)$ has a unique root, is it necessarily optimal? If $\varphi(t)$ has multiple roots, which root (if any) is optimal?

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- For Yaari's (1964) example, it can be shown that $\varphi(t)$ does not have a root if $S_0 \geq \log 2$. If $S_0 < \log 2$, then $\varphi(t)$ has a root.
- A major factor that renders $\varphi(t)$ rootless is the property of the utility function, namely $g'(c) \geq 1$, which prevents $\varphi(t)$ from diminishing to $-\infty$.

Uniqueness

- For the uniqueness issue, the objective functional in (1) is strictly concave in $c(\cdot)$, and (2) - (5) imply that the set of feasible $c(\cdot)$ is convex. Therefore, if the solution to (1) - (5) exists, it must be unique.

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- This implies that, even if $\varphi(t)$ has multiple roots, only one of them can be optimal. The optimal root must be unique.

Existence Theorems

THEOREM 2 (Leung (2009)). *Assume that* (i) $g'(0^+) < \infty$ *or* $m(\overline{T}) > 0$, *and* (ii) $S_0 > 0$. *Let* $g'(\infty) = \lim_{c\to\infty} g'(c)$. *If* $g'(\infty) = 0$, *then the optimization problem* (1) - (5) *has a solution.*

- Theorem 2 shows that the optimization problem has a solution if the marginal utility of infinite consumption is zero.
- Many popular utility functions satisfy this condition, e.g., logarithmic ($\ln c$), CRRA ($c^{1-\delta}/(1-\delta)$), and exponential ($-e^{-c}$).
- Examples of a utility function that does not satisfy $g'(\infty) = 0$ are $c - e^{-c}$ (the one studied in Yaari (1964)), $g(c) = c + \ln c$, $c + \frac{c^\delta}{\delta}$ ($\delta < 1$), and $\sqrt{c^2 - 1}$ ($c \geq 1$). For these utility functions, $g'(\infty) = 1$.
- The following theorem offers some sufficient conditions to deal with the case where $g'(\infty) \neq 0$.

THEOREM 3 (Leung (2009)). *Assume that* (i) $g'(0^+) < \infty$ *or* $m(\overline{T}) > 0$, *and* (ii) $S_0 > 0$. *Suppose* $g'(\infty) = k > 0$. *Without loss of generality, assume that* $k = 1$. *Then there exists a* $\tau \in (0, 1)$ *such that*

$$\Omega(\tau)\alpha(\tau)g'\left(m(\tau)\right) e^{\int_0^\tau j(x)\,dx} = 1. \tag{7}$$

If there is more than one τ *that satisfies* (7), *assume that*

$$\tau = \max\left\{\theta \in (0,1) \,\middle|\, \Omega(\theta)\alpha(\theta)g'\left(m(\theta)\right) e^{\int_0^\theta j(x)\,dx} = 1\right\} \tag{8}$$

exists. Then the optimization problem (1) - (5) *has a solution if*

$$\int_0^\tau e^{-\int_0^t j(x)\,dx} (g')^{-1}\left(\frac{1}{\Omega(t)\alpha(t)e^{\int_0^t j(x)\,dx}}\right) dt > S_0 + \int_0^\tau e^{-\int_0^t j(x)\,dx} m(t)\,dt. \tag{9}$$

In particular, if

$$\int_0^\tau e^{-\int_0^t j(x)\,dx} (g')^{-1}\left(\frac{1}{\Omega(t)\alpha(t)e^{\int_0^t j(x)\,dx}}\right) dt = \infty, \tag{10}$$

then (9) *is satisfied and* (1) - (5) *has a solution.*

- Condition (10) is relatively simple and verifiable.
- In the context of Yaari's (1964) example, it is different from his condition, namely,

$$\alpha(0)g'(\infty) < \alpha(t^{\#})g'\left(\frac{S_0}{t^{\#}}\right) \tag{11}$$

holds for some $t^{\#} \in [0, 1]$.

- In contrast, condition (10) requires that

$$\int_0^{\tau} (g')^{-1}\left(\frac{1}{\alpha(t)}\right) dt = \infty. \tag{12}$$

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- Clearly, (12) bears no resemblance to (11).
- When applied to Example 1, Yaari (1964, p.587) acknowledges that his condition (11) is too strong.
- Condition (12) is considerably weaker than condition (11).

Multiple roots

Theorem 4 (Leung (2007)). *Assume that $m(t)$ is continuously differentiable and $\eta(t) = 0$ for $t \in (t^*, \overline{T})$.*
(i) If t^* is optimal, then $\varphi(t)$ must satisfy the condition

$$\varphi'(t) \leq 0 \text{ for } t > t^*. \tag{13}$$

(ii) If t_1 and t_2 are roots of $\varphi(t)$ with $t_1 < t_2$ and there exists a $\tau \in (t_1, t_2)$ such that $\varphi(\tau) \neq 0$, then t_1 is not optimal.
(iii) Suppose there exists a $t^\#$ such that $\varphi(t^\#) = 0$ and $\varphi'(t) \leq 0$ for $t > t^\#$, then

$$t^* = \inf\{t^\# \in [0, \overline{T}) \mid \varphi(t^\#) = 0 \text{ and } \varphi'(t) \leq 0 \text{ for } t > t^\#\} \tag{14}$$

will be optimal.

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- The location of the optimal root depends on the configuration of the roots.
- If the roots are isolated, Theorem 4(ii) demonstrates that the largest root will be the only optimum. This result holds regardless of whether the roots are finitely many or countably infinitely many.
- If the roots are in a continuum, Theorem 4(iii) reveals that the smallest root will be the only optimum.
- The selection of the optimal root is not a straightforward matter. In contrast to the isolated case in which condition (13) eliminates all but the largest root, the continuum case requires a different approach.

- As long as the largest of the continuum of roots satisfies condition (13), then all the roots in the continuum will also satisfy (13). In this case, condition (13) cannot help further eliminate the roots in the continuum. In order to locate the optimal root, it is necessary to draw on the continuum itself, which will occur only if a special condition is satisfied. It turns out that this special condition imposes a restriction on the optimal consumption path such that it eliminates all but the smallest of the continuum of roots.

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- The condition (13) requires that $\varphi(t)$ be non-increasing for all t greater than the candidate optimum. If the condition is not satisfied, then the candidate root will not be optimal. In this case, there is no optimal root. Accordingly, the optimal wealth depletion time t^* does not exist and the optimization problem (1) - (5) does not have an optimal solution. Thus, condition (13) provides a pivotal test for the optimality of the roots.

3. Annuitization

I. Baseline model

Assume that $j(t) = 0$. In the absence of annuities, the individual's decision problem is given by:

$$\max_{c(t)} \int_0^1 (1-t)\log\left(c(t)\right) dt \tag{15}$$

subject to

$$c(t) \geq 0, \tag{16}$$

$$S(t) \geq 0, \tag{17}$$

$$S'(t) = -c(t), \tag{18}$$

and

$$S(0) = S_0 > 0. \tag{19}$$

Let the Hamiltonian H and the Lagrangian L be

$$H(c(t), S(t), t) = (1-t)\log\left(c(t)\right) - \lambda(t)c(t) \tag{20}$$

and

$$L(c(t), S(t), t) = (1-t)\log c(t) - \lambda(t)c(t) + \eta(t)c(t) + \mu(t)S(t), \tag{21}$$

Let $c^*(t)$ and $S^*(t)$ denote the optimal solution. The necessary optimality conditions are

$$\frac{1-t}{c^*(t)} = \lambda(t) \tag{22}$$

and

$$-\lambda'(t) = 0. \tag{23}$$

It is straightforward to verify that the optimal solution is given by

$$c^*(t) = 2S_0(1-t) \text{ and } S^*(t) = S_0(1-t)^2. \tag{24}$$

Substituting (24) into (15),

$$\begin{aligned} V_0^* &= \int_0^1 (1-t)\log\left(c^*(t)\right) dt \\ &= \frac{\log(2S_0)}{2} - \frac{1}{4}. \end{aligned} \tag{25}$$

- For this model, $S^*(t) > 0$ for all $t \in [0,1)$ and there is no terminal wealth depletion before $t = 1$ because $g'(0^+) = \infty$ and $m(t) = 0$.

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- For this model, $S^*(t) > 0$ for all $t \in [0,1)$ and there is no terminal wealth depletion before $t = 1$ because $g'(0^+) = \infty$ and $m(t) = 0$.
- The value of V_0^* obtained in (25) will be served as a benchmark for later comparisons.

II. Annuities are available

(i) Costless annuities

Let τ be the annuitization time. At time τ, the individual pays the insurance company all his wealth $S(\tau)$. In return, he receives M from the insurance company at each instant as long as he lives.

$$\int_{\tau}^{1}(1-t)Mdt = S(\tau). \tag{26}$$

Solving for M from (26),

$$\begin{aligned} M &= \frac{S(\tau)}{\int_{\tau}^{1}(1-t)dt} \\ &= \frac{2S(\tau)}{(1-\tau)^2}. \end{aligned} \tag{27}$$

The lifetime utility becomes

$$\begin{aligned} V &= \int_0^1 (1-t)\log(c^*(t))dt \\ &= \int_0^{\tau}(1-t)\log(c^*(t))dt + \int_{\tau}^{1}(1-t)\log Mdt \\ &= \int_0^{\tau}(1-t)\log(c^*(t))dt + (\log M)\int_{\tau}^{1}(1-t)dt. \end{aligned} \tag{28}$$

The optimality solution $c^*(t)$ satisfies

$$\frac{1-t}{c^*(t)} = \lambda_0 \text{ (a constant)},\ t \in [0,\tau], \tag{29}$$

and

$$\int_0^\tau c^*(t)dt = S_0 - S^*(\tau). \tag{30}$$

Solving (29) and (30) for $c^*(t)$,

$$c^*(t) = \frac{2\left[S_0 - S^*(\tau)\right](1-t)}{1-(1-\tau)^2},\ t \in [0,\tau]. \tag{31}$$

Substituting (27) and (31) into (28),

$$\begin{aligned} V_1 &= \int_0^\tau (1-t)\log\frac{2(S_0 - S^*(\tau))(1-t)}{1-(1-\tau)^2}dt + \frac{(1-\tau)^2}{2}\log\frac{2S^*(\tau)}{(1-\tau)^2} \\ &= \frac{1-(1-\tau)^2}{2}\log\frac{2(S_0 - S^*(\tau))}{1-(1-\tau)^2} - \frac{(1-\tau)^2}{2}\left(\log(1-\tau) - \frac{1}{2}\right) - \frac{1}{4} \\ &\quad + \frac{(1-\tau)^2}{2}\log\frac{2S^*(\tau)}{(1-\tau)^2}. \end{aligned} \tag{32}$$

To find the optimal $S^*(\tau)$,

$$\frac{\partial V_1}{\partial S^*(\tau)} = -\frac{1-(1-\tau)^2}{2(S_0 - S^*(\tau))} + \frac{(1-\tau)^2}{2S^*(\tau)} = 0. \tag{33}$$

This implies that

$$S^*(\tau) = S_0(1-\tau)^2. \tag{34}$$

Substituting (34) into (32),

$$\begin{aligned} V_1 &= \frac{1-(1-\tau)^2}{2}\log(2S_0) - \frac{(1-\tau)^2}{2}\left(\log(1-\tau) - \frac{1}{2}\right) - \frac{1}{4} + \frac{(1-\tau)^2}{2} \\ &= \frac{\log(2S_0)}{2} - \frac{(1-\tau)^2}{2}\left(\log(1-\tau) - \frac{1}{2}\right) - \frac{1}{4}. \end{aligned}$$

To find the optimal τ,

$$\frac{\partial V_1}{\partial \tau} = (1-\tau)\left(\log(1-\tau) - \frac{1}{2}\right) + \frac{1-\tau}{2} = 0. \tag{36}$$

This implies that $\log(1-\tau) = 0$, i.e.,

$$\tau^* = 0. \tag{37}$$

In other words, the optimal time to annuitize is the very first moment. As a result, (35) becomes

$$V_1^* = \frac{\log(2S_0)}{2}. \tag{38}$$

Comparing (25) and (38), $V_1^* > V_0^*$. This result confirms that it is optimal to annuitize at time 0. The annuities are so attractive that it is optimal to annuitize the wealth at the very first moment.

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    1. The insurance company charges the individual a fixed fee F so that the annuitizable wealth becomes $S(\tau) - F$.

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- We next consider the cost of annuities. There are at least two ways to model the cost of annuities.
  1. The insurance company charges the individual a fixed fee F so that the annuitizable wealth becomes $S(\tau) - F$.
  2. The insurance company charges the individual a variable fee proportional to the wealth $\alpha S(\tau)$ so that the annuitizable wealth becomes $(1 - \alpha)S(\tau)$.

(ii) Fixed cost annuities

After paying cost F, the individual's wealth becomes $S(\tau) - F$. The annuity income is determined by

$$\int_{\tau}^{1} (1-t) M dt = S(\tau) - F. \tag{39}$$

Solving for M,

$$\begin{aligned} M &= \frac{S(\tau) - F}{\int_{\tau}^{1} (1-t) dt} \\ &= \frac{2(S(\tau) - F)}{(1-\tau)^2}. \end{aligned} \tag{40}$$

Substituting (31) and (40) into (28),

$$\begin{aligned} V_2 &= \int_0^{\tau} (1-t) \log \frac{2(S_0 - S^*(\tau))(1-t)}{1-(1-\tau)^2} dt + \int_{\tau}^{1} (1-t) \log \frac{2(S(\tau) - F)}{(1-\tau)^2} dt \\ &= \frac{1-(1-\tau)^2}{2} \log \frac{2(S_0 - S^*(\tau))}{1-(1-\tau)^2} - \frac{(1-\tau)^2}{2} \left(\log(1-\tau) - \frac{1}{2} \right) - \frac{1}{4} \\ &\quad + \frac{(1-\tau)^2}{2} \log \frac{2(S^*(\tau) - F)}{(1-\tau)^2}. \end{aligned} \tag{41}$$

Notice that F does not appear in the first term because $\int_0^\tau c^*(t)dt = S_0 - S^*(\tau)$, whereas $\int_\tau^1 (1-t)Mdt = S(\tau) - F$. To find the optimal $S^*(\tau)$,

$$\frac{\partial V_2}{\partial S^*(\tau)} = -\frac{1-(1-\tau)^2}{2(S_0 - S^*(\tau))} + \frac{(1-\tau)^2}{2(S^*(\tau)-F)} = 0. \tag{42}$$

Thus,

$$S^*(\tau) = S_0(1-\tau)^2 + F\left(1-(1-\tau)^2\right). \tag{43}$$

It follows that

$$\begin{aligned} S_0 - S^*(\tau) &= S_0 - S_0(1-\tau)^2 - F\left(1-(1-\tau)^2\right) \\ &= (S_0 - F)\left(1-(1-\tau)^2\right) \end{aligned} \tag{44}$$

and

$$\begin{aligned} S^*(\tau) - F &= S_0(1-\tau)^2 + F\left(1-(1-\tau)^2\right) - F \\ &= (S_0 - F)(1-\tau)^2. \end{aligned} \tag{45}$$

Substituting (44) and (45) into (41),

$$\begin{aligned} V_2 &= \frac{1-(1-\tau)^2}{2}\log(2(S_0-F)) - \frac{(1-\tau)^2}{2}\left(\log(1-\tau)-\frac{1}{2}\right)-\frac{1}{4} \\ &\quad + \frac{(1-\tau)^2}{2}\log(2(S_0-F)) \\ &= \frac{\log(2(S_0-F))}{2} - \frac{(1-\tau)^2}{2}\left(\log(1-\tau)-\frac{1}{2}\right)-\frac{1}{4}. \end{aligned} \tag{46}$$

As in (36) and (37), V_2 is maximized at

$$\tau = 0. \tag{47}$$

Substituting $\tau = 0$ into (46),

$$V_2^* = \frac{\log(2(S_0-F))}{2}. \tag{48}$$

Comparing (25) and (48),

$$V_2^* > V_0^* \text{ if } \frac{\log(2(S_0 - F))}{2} > \frac{\log(2S_0)}{2} - \frac{1}{4}, \tag{49}$$

i.e.,

$$\frac{F}{S_0} < 1 - e^{-1/2} = 0.3935. \tag{50}$$

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- The amount $0.3935S_0$ can be interpreted as the individual's maximum willingness to pay for annuities.
- The insurance company can charge the individual as much as 39.35% of his wealth for annuities.

(iii) Proportional cost annuities

After paying cost $\alpha S(\tau)$, the individual's wealth becomes $(1-\alpha)S(\tau)$.

$$\int_\tau^1 (1-t)M dt = (1-\alpha)S(\tau). \tag{51}$$

Solving for M,

$$\begin{aligned} M &= \frac{(1-\alpha)S(\tau)}{\int_\tau^1 (1-t)dt} \\ &= \frac{2(1-\alpha)S(\tau)}{(1-\tau)^2}. \end{aligned} \tag{52}$$

Substituting (31) and (52) into (28),

$$\begin{aligned} V_3 &= \int_0^1 (1-t)\log(c^*(t))dt \\ &= \int_0^\tau (1-t)\log\frac{2(S_0-S^*(\tau))(1-t)}{1-(1-\tau)^2}dt + \left(\log\frac{2(1-\alpha)S(\tau)}{(1-\tau)^2}\right)\int_\tau^1 (1-t)dt \\ &= \frac{1-(1-\tau)^2}{2}\log\frac{2(S_0-S^*(\tau))}{1-(1-\tau)^2} - \frac{(1-\tau)^2}{2}\left(\log(1-\tau)-\frac{1}{2}\right) - \frac{1}{4} \\ &\quad + \frac{(1-\tau)^2}{2}\log\frac{2(1-\alpha)S^*(\tau)}{(1-\tau)^2} \end{aligned} \tag{53}$$

To find the optimal $S^*(\tau)$,

$$\frac{\partial V_3}{\partial S^*(\tau)} = -\frac{1-(1-\tau)^2}{2(S_0 - S^*(\tau))} + \frac{(1-\tau)^2}{2S^*(\tau)} = 0. \tag{54}$$

Hence, $S^*(\tau) = S_0(1-\tau)^2$, which is the same as (34). Substituting this into (53),

$$\begin{aligned} V_3 &= \frac{1-(1-\tau)^2}{2}\log(2S_0) - \frac{(1-\tau)^2}{2}\left(\log(1-\tau) - \frac{1}{2}\right) - \frac{1}{4} + \frac{(1-\tau)^2}{2} \\ &= \frac{\log(2S_0)}{2} - \frac{(1-\tau)^2}{2}\left(\log(1-\tau) - \frac{1}{2}\right) - \frac{1}{4} + \frac{(1-\tau)^2}{2}\log(1-\alpha) \end{aligned}$$

To find the optimal τ,

$$\frac{\partial V_3}{\partial \tau} = (1-\tau)\left(\log(1-\tau) - \log(1-\alpha)\right) = 0. \tag{56}$$

This implies that

$$\tau^* = \alpha. \tag{57}$$

Evaluated at $\tau = \alpha$, $\frac{\partial^2 V_3}{\partial \tau^2} = -1 < 0$. Therefore, $\tau^* = \alpha$ is optimal.
Substituting (57) into (55),

$$V_3^* = \frac{\log(2S_0)}{2} + \frac{(1-\alpha)^2}{4} - \frac{1}{4}. \tag{58}$$

- Compared to the case where annuities are costless, the individual is worse off when annuities are costly because $V_3^* < V_1^*$.

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- However, compared to the case where annuities are not available, $V_3^* > V_0$ because of the extra term $(1-\alpha)^2/4$.

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- If $\alpha = 0$, then $\tau = 0$, i.e., it is optimal to buy annuities at the very first moment if they are costless, which echoes the result obtained in (37).

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- The higher the commission rate α, the later is the conversion to annuities.
- If $\alpha = 0$, then $\tau = 0$, i.e., it is optimal to buy annuities at the very first moment if they are costless, which echoes the result obtained in (37).
- If $\alpha = 1$, then $\tau = 1$, .i.e., it is optimal not to buy any annuities because they are too expensive.

- To verify that it is not optimal to annuitize at time zero, let $\tau = 0$, then (55) implies that

$$V_3\big|_{\tau=0} = \frac{\log(2S_0)}{2} + \frac{\log(1-\alpha)}{2}. \tag{59}$$

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- This shows $\tau = 0$ is not an optimal choice when $\alpha \in (0,1)$.

- To verify that it is not optimal to annuitize at time zero, let $\tau = 0$, then (55) implies that

$$V_3\big|_{\tau=0} = \frac{\log(2S_0)}{2} + \frac{\log(1-\alpha)}{2}. \tag{59}$$

- Since $2\log(1-\alpha) < (1-\alpha)^2 - 1$,

$$V_3\big|_{\tau=0} < V_3^*.$$

- This shows $\tau = 0$ is not an optimal choice when $\alpha \in (0,1)$.
- The proportionality feature of the cost of annuities causes the individual to delay the annuitization time.

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- This shows $\tau = 0$ is not an optimal choice when $\alpha \in (0,1)$.
- The proportionality feature of the cost of annuities causes the individual to delay the annuitization time.
- This contrasts with the case of fixed cost annuities where the fixed cost affects only the annuitization decision (whether to annuitize, as shown in (49)) but not the annuitization time (when to annuitize, as shown in (37) and (47)).

III. Positive interest rate

Assume that $j(t) = j$ (a constant). The decision problem becomes

$$\max_{c(t)} \int_0^1 (1-t)\log\left(c(t)\right) dt$$

subject to

$$c(t) \geq 0,$$

$$S(t) \geq 0,$$

$$S'(t) = jS(t) - c(t), \tag{60}$$

and

$$S(0) = S_0 > 0.$$

The Hamiltonian H and the Lagrangian L become

$$H(c(t), S(t), t) = (1-t)\log c(t) - \lambda(t)(jS(t) - c(t))$$

and

$$L(c(t), S(t), t) = (1-t)\log c(t) - \lambda(t)(jS(t) - c(t)) + \eta(t)c(t) + \mu(t)S(t). \tag{61}$$

The necessary optimality conditions are

$$\frac{1-t}{c^*(t)} = \lambda(t) \tag{62}$$

and

$$-\lambda'(t) = j\lambda(t). \tag{63}$$

The optimal solution is given by

$$c^*(t) = 2S_0(1-t)e^{jt}. \tag{64}$$

Substituting (64) into (28),

$$\begin{aligned} V_4^* &= \int_0^1 (1-t)\log(c^*(t))dt \\ &= \int_0^1 (1-t)\log(2S_0(1-t)e^{jt})dt \\ &= \frac{\log(2S_0)}{2} + \frac{j}{6} - \frac{1}{4}. \end{aligned} \tag{65}$$

(i) Costless annuities

The annuity income M is given by (27). The optimality solution $c^*(t)$ satisfies (62), (63), and

$$\int_0^\tau e^{-jt} c^*(t) dt = S_0 - e^{-j\tau} S^*(\tau). \tag{66}$$

Solving these three equations,

$$c^*(t) = \frac{2\left[S_0 - e^{-j\tau} S^*(\tau)\right](1-t)e^{jt}}{1-(1-\tau)^2},\ t \in [0, \tau]. \tag{67}$$

Substituting (67) into (28),

$$\begin{aligned} V_5 &= \int_0^1 (1-t)\log(c^*(t))dt \\ &= \int_0^\tau (1-t)\log\frac{2\left[S_0 - e^{-j\tau}S^*(\tau)\right](1-t)e^{jt}}{1-(1-\tau)^2}dt + \frac{(1-\tau)^2\log M}{2} \\ &= \frac{1-(1-\tau)^2}{2}\log\frac{2(S_0 - e^{-j\tau}S^*(\tau))}{1-(1-\tau)^2} - \frac{(1-\tau)^2}{2}\left(\log(1-\tau) - \frac{1}{2}\right) - \frac{1}{4} \\ &\quad + j\tau^2\left(\frac{1}{2} - \frac{\tau}{3}\right) + \frac{(1-\tau)^2}{2}\log\frac{2S^*(\tau)}{(1-\tau)^2} \end{aligned} \tag{68}$$

To find the optimal $S^*(\tau)$,

$$\frac{\partial V_5}{\partial S^*(\tau)} = -\frac{\left(1-(1-\tau)^2\right)e^{-j\tau}}{2(S_0 - e^{-j\tau}S^*(\tau))} + \frac{(1-\tau)^2}{2S^*(\tau)} = 0. \tag{69}$$

Thus,

$$S^*(\tau) = S_0(1-\tau)^2 e^{j\tau}. \tag{70}$$

The difference between (34) and (70) is the extra term $e^{j\tau}$ in the latter.

$$V_5 = \frac{\log(2S_0)}{2} - \frac{(1-\tau)^2}{2}\left(\log(1-\tau) - \frac{1}{2}\right) - \frac{1}{4} + j\tau^2\left(\frac{1}{2} - \frac{\tau}{3}\right) + \frac{j\tau(1-\tau)^2}{2}. \tag{71}$$

To find the optimal τ,

$$\frac{\partial V_5}{\partial \tau} = (1-\tau)\log(1-\tau) + \frac{j(1-\tau)^2}{2} = 0. \tag{72}$$

This implies that

$$\log(1-\tau^*) + \frac{j(1-\tau^*)}{2} = 0. \tag{73}$$

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$$\frac{\partial \tau^*}{\partial j} = \frac{(1-\tau^*)^2}{2 + j(1-\tau^*)} > 0. \tag{74}$$

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$$\frac{\partial \tau^*}{\partial j} = \frac{(1-\tau^*)^2}{2 + j(1-\tau^*)} > 0. \tag{74}$$

- The higher the interest rate, the later is the annuitization time. It is worthwhile to delay annuitization because the wealth earns interest.

$$V_5^* = \frac{\log(2S_0)}{2} + \frac{\left(2 + (1-\tau^*)^3\right) j}{12} + \frac{(1-\tau^*)^2}{4} - \frac{1}{4} \tag{75}$$

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- Comparing (65) and (75), $V_5^* > V_4^*$ because $\frac{(1-\tau^*)^3 j}{12} + \frac{(1-\tau^*)^2}{4} > 0$.

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- Comparing (65) and (75), $V_5^* > V_4^*$ because $\frac{(1-\tau^*)^3 j}{12} + \frac{(1-\tau^*)^2}{4} > 0$.
- The availability of costless annuities unambiguously improves the individual's welfare.

(ii) Fixed cost annuities

The annuity income M is determined by (40):

$$
\begin{aligned}
M &= \frac{S(\tau) - F}{\int_\tau^1 (1-t)dt} \\
&= \frac{2(S(\tau) - F)}{(1-\tau)^2}.
\end{aligned}
$$

In this case,

$$
c^*(t) = \frac{2\left[S_0 - e^{-j\tau} S^*(\tau)\right](1-t)e^{jt}}{1-(1-\tau)^2},\ t \in [0, \tau]. \tag{76}
$$

Hence,

$$
\begin{aligned}
V_6 &= \frac{1-(1-\tau)^2}{2} \log \frac{2(S_0 - e^{-j\tau} S^*(\tau))}{1-(1-\tau)^2} - \frac{(1-\tau)^2}{2}\left(\log(1-\tau) - \frac{1}{2}\right) - \frac{1}{4} \\
&\quad + j\tau^2\left(\frac{1}{2} - \frac{\tau}{3}\right) + \frac{(1-\tau)^2}{2} \log \frac{2(S^*(\tau) - F)}{(1-\tau)^2}
\end{aligned} \tag{77}
$$

To find the optimal $S^*(\tau)$, differentiate (77) with respect to $S^*(\tau)$,

$$\frac{\partial V_6}{\partial S^*(\tau)} = -\frac{\left(1-(1-\tau)^2\right) e^{-j\tau}}{2(S_0 - e^{-j\tau}S^*(\tau))} + \frac{(1-\tau)^2}{2(S^*(\tau)-F)} = 0. \tag{78}$$

Thus,

$$S^*(\tau) = S_0(1-\tau)^2 e^{j\tau} + F\left(1-(1-\tau)^2\right). \tag{79}$$

It follows that

$$S_0 - e^{-j\tau}S^*(\tau) = (S_0 - e^{-j\tau}F)\left(1-(1-\tau)^2\right) \tag{80}$$

and

$$S^*(\tau) - F = (e^{j\tau}S_0 - F)(1-\tau)^2. \tag{81}$$

Hence,

$$\begin{aligned} V_6 \quad = \quad & \frac{\log\left(2(S_0 - e^{-j\tau}F)\right)}{2} - \frac{(1-\tau)^2}{2}\left(\log(1-\tau) - \frac{1}{2}\right) - \frac{1}{4} \\ & + j\tau^2\left(\frac{1}{2} - \frac{\tau}{3}\right) + \frac{(1-\tau)^2}{2} j\tau. \end{aligned} \tag{82}$$

To find the optimal τ, differentiate (82) with respect to τ,

$$\frac{\partial V_6}{\partial \tau} = \frac{je^{-j\tau}F}{2(S_0 - e^{-j\tau}F)} + (1-\tau)\log(1-\tau) + \frac{j(1-\tau)^2}{2} = 0, \tag{83}$$

which implies that

$$\frac{je^{-j\tau^*}F}{2(S_0 - e^{-j\tau^*}F)} + (1-\tau^*)\log(1-\tau^*) + \frac{j(1-\tau^*)^2}{2} = 0. \tag{84}$$

Using (84) to simplify (82),

$$V_6^* = \frac{\log\left(2(S_0 - e^{-j\tau^*}F)\right)}{2} + \frac{(1-\tau)je^{-j\tau^*}F}{4(S_0 - e^{-j\tau^*}F)} + \frac{\left(2 + (1-\tau^*)^3\right)j}{12} + \frac{(1-\tau^*)}{4} \tag{85}$$

Comparing (65) and (85), $V_6^* > V_4^*$ if the following condition is satisfied:

$$\log(S_0 - e^{-j\tau^*}F) + \frac{(1-\tau^*)je^{-j\tau^*}F}{2(S_0 - e^{-j\tau^*}F)} + \frac{(1-\tau^*)^3 j}{6} + \frac{(1-\tau^*)^2}{2} > \log S_0. \tag{86}$$

This complicated condition determines whether the individual should annuitize his wealth at time τ^*.

(iii) Proportional cost annuities

The annuity income is determined by

$$\begin{aligned} M &= \frac{(1-\alpha)S(\tau)}{\int_\tau^1 (1-t)\,dt} \\ &= \frac{2(1-\alpha)S(\tau)}{(1-\tau)^2}. \end{aligned}$$

In this case,

$$\begin{aligned} V_7 &= \frac{1-(1-\tau)^2}{2}\log\frac{2(S_0 - e^{-j\tau}S^*(\tau))}{1-(1-\tau)^2} - \frac{(1-\tau)^2}{2}\left(\log(1-\tau) - \frac{1}{2}\right) - \frac{1}{4} \\ &\quad + j\tau^2\left(\frac{1}{2} - \frac{\tau}{3}\right) + \frac{(1-\tau)^2}{2}\log\frac{2(1-\alpha)S^*(\tau)}{(1-\tau)^2}. \end{aligned} \tag{87}$$

To find the optimal $S^*(\tau)$, differentiate (87) with respect to $S^*(\tau)$,

$$\frac{\partial V_7}{\partial S^*(\tau)} = -\frac{\left(1-(1-\tau)^2\right)e^{-j\tau}}{2(S_0 - e^{-j\tau}S^*(\tau))} + \frac{(1-\tau)^2}{2S^*(\tau)} = 0. \tag{88}$$

Thus, $S^*(\tau) = S_0(1-\tau)^2 e^{j\tau}$, which is the same as (70).

Substituting this into (87),

$$\begin{aligned} V_7 \quad = \quad & \frac{\log(2S_0)}{2} - \frac{(1-\tau)^2}{2}\left(\log(1-\tau) - \frac{1}{2}\right) - \frac{1}{4} \\ & + j\tau^2\left(\frac{1}{2} - \frac{\tau}{3}\right) + \frac{j\tau(1-\tau)^2}{2} + \frac{(1-\tau)^2}{2}\log(1-\alpha). \end{aligned} \tag{89}$$

To find the optimal τ,

$$\frac{\partial V_7}{\partial \tau} = (1-\tau)\log(1-\tau) + \frac{j(1-\tau)^2}{2} - (1-\tau)\log(1-\alpha) = 0. \tag{90}$$

Thus,

$$\log(1-\tau^*) + \frac{j(1-\tau^*)}{2} - \log(1-\alpha) = 0. \tag{91}$$

Therefore, the optimal τ^* is determined by (91). Clearly,

$$\frac{\partial \tau^*}{\partial j} = \frac{(1-\tau^*)^2}{2 + j(1-\tau^*)} > 0 \tag{92}$$

and

$$\frac{\partial \tau^*}{\partial \alpha} = \frac{1-\tau^*}{1-\alpha} > 0. \tag{93}$$

A higher interest rate or a higher commission rate would delay the annuitization time.

$$V_7^* = \frac{\log(2S_0)}{2} + \frac{\left(2 + (1-\tau^*)^3\right) j}{12} + \frac{(1-\tau^*)^2}{4} - \frac{1}{4}. \tag{94}$$

- Comparing (65) and (94), it is clear that $V_7^* > V_4^*$.

$$V_7^* = \frac{\log(2S_0)}{2} + \frac{\left(2 + (1-\tau^*)^3\right) j}{12} + \frac{(1-\tau^*)^2}{4} - \frac{1}{4}. \tag{94}$$

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- Comparing (65) and (94), it is clear that $V_7^* > V_4^*$.
- Annuitization unambiguously improves the individual's welfare.
- Interestingly, V_7^* and V_5^* have exactly the same functional form. The only difference is the value of τ^*.
- As shown in (73) and (91), the τ^* in V_5^* depends on j only, whereas the τ^* in V_7^* depends on j and α.

Summary

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- For fixed cost annuities, it is also optimal to purchase them at the very first moment, provided that it is optimal to annuitize.
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- When the interest rate is positive, it is not optimal to purchase annuities at the very first moment, regardless of whether they are costless or whether their cost is fixed or proportional.

Summary

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- For fixed cost annuities, it is also optimal to purchase them at the very first moment, provided that it is optimal to annuitize.
- For proportional cost annuities, it is optimal to purchase them at a later time.
- When the interest rate is positive, it is not optimal to purchase annuities at the very first moment, regardless of whether they are costless or whether their cost is fixed or proportional.
- It pays to delay the annuitization time because wealth holdings generate interest income. There will be no interest income after complete annuitization.

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- For simplicity, we assume that the individual can only annuitize at time zero.
- In contrast to the previous analysis which focuses on the annuitization decision (whether to annuitize) and the annuitization time (when to annuitize), the focus here is the degree of annuitization (how much to annuitize).
- The question is whether partial or complete annuitization is optimal.

The decision problem becomes

$$\max_{c(t),\theta} \int_0^1 (1-t)\log(c(t))dt \tag{95}$$

subject to

$$c(t) \geq 0,$$

$$S(t) \geq 0,$$

$$S'(t) = M - c(t), \tag{96}$$

and

$$S(0) = (1-\theta)S_0. \tag{97}$$

The annuity income M is determined by

$$\begin{aligned} M &= \frac{\theta S_0}{\int_0^1 (1-t)dt} \\ &= 2\theta S_0. \end{aligned} \tag{98}$$

Since $M > 0$, it follows from Leung (1994) that there exists a terminal wealth depletion time t^* such that $S^*(t) = 0$ for all $t \in [t^*, 1]$. Thus,

$$c^*(t) = \begin{cases} \frac{(1-t)M}{(1-t^*)} & \text{if} \quad t \in [0, t^*] \\ \\ M & \text{if} \quad t \in [t^*, 1] \end{cases} . \tag{99}$$

Since

$$(1-\theta)S_0 + \int_0^{t^*} M dt = \int_0^{t^*} c^*(t) dt. \tag{100}$$

Substituting (98) and (99) into (100),

$$(1-\theta)S_0 + \int_0^{t^*} 2\theta S_0 dt = \int_0^{t^*} \frac{(1-t)(2\theta S_0)}{1-t^*} dt. \tag{101}$$

This implies that

$$\theta t^{*2} - (1-\theta)(1-t^*) = 0. \tag{102}$$

Thus, the terminal wealth depletion time is given by

$$t^* = \frac{\sqrt{(1-\theta)(1+3\theta)} - (1-\theta)}{2\theta} \tag{103}$$

The lifetime utility is given by

$$
\begin{aligned}
V_8 &= \int_0^{t^*} (1-t)\log\frac{M(1-t)}{(1-t^*)}dt + \int_{t^*}^{1}(1-t)\log M dt \\
&= \frac{1}{2}\log(2\theta S_0) + \frac{1}{4}((1-t^*)^2 - 1 - 2\log(1-t^*)). \qquad (104)
\end{aligned}
$$

To find the optimal θ,

$$
\frac{\partial V_8}{\partial\theta} = \frac{1}{2\theta} - \frac{1}{4}\left(-2(1-t^*) + \frac{2}{1-t^*}\right)\frac{(1-t^*)}{\theta(1-\theta+2\theta t^*)} = 0. \qquad (105)
$$

This implies that

$$
\theta(1-2t^*) = (1-t^*)^2. \qquad (106)
$$

Combining (102) and (106), one obtains

$$
t^{*2}(2-t^*) = 0. \qquad (107)
$$

This implies that

$$
t^* = 0, \text{ i.e., } \theta = 1 \qquad (108)
$$

because the other root of (107), $t^* = 2$, is ruled out as $t^* \in [0,1]$.

Substituting $t^* = 0$ into (104),

$$V_8^* = \frac{\log(2S_0)}{2}$$

- Clearly, $V_8^* > V_0^*$, thus the individual should completely annuitize at the very first moment.

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- Clearly, $V_8^* > V_0^*$, thus the individual should completely annuitize at the very first moment.
- Under this setup, complete annuitization is optimal.

5. General Model

- In a general model, both the degree of annuitization θ and the time of annuitization τ are decision variables.

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5. General Model

- In a general model, both the degree of annuitization θ and the time of annuitization τ are decision variables.
- The objective is to investigate jointly the degree of annuitization (how much to annuitize), and the annuitization time (when to annuitize) in a single general setup.
- The individual's decision problem becomes

$$\max_{c(t),\theta,\tau} \int_0^1 \Omega(t)\alpha(t)g(c(t))dt \tag{109}$$

subject to

$$c(t) \geq 0, \tag{110}$$

$$S(t) \geq 0, \tag{111}$$

$$S'(t) = jS(t) + m(t) + MI_{(t\geq\tau)} - c(t), \tag{112}$$

$$S(0) = S_0, \tag{113}$$

$$M = \frac{\theta(1-\alpha)(S(\tau)-F)}{\int_\tau^1 \Omega(t)dt}, \tag{114}$$

$$S(\tau) = e^{j\tau}\left[S(0) + \int_0^{\tau} e^{-jt} m(t)dt - \int_0^{\tau} e^{-jt} c^*(t)dt\right], \tag{115}$$

$$0 = e^{-j\tau}(1-\theta)(1-\alpha)(S(\tau)-F) + \int_{\tau}^{t^*} e^{-jt}\left(m(t)+M\right)dt - \int_{\tau}^{t^*} e^{-jt} c^*(t)dt, \tag{116}$$

$$0 \leq \theta \leq 1, \tag{117}$$

$$0 \leq \tau \leq 1, \tag{118}$$

and

$$0 \leq \alpha \leq 1. \tag{119}$$

where $I_{(t \geq \tau)}$ is an indicator (binary) function of t such that $I_{(t \geq \tau)} = 1$ if $t \geq \tau$, and $I_{(t \geq \tau)} = 0$ if $t < \tau$.

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- The lifetime utility is given by

$$
\begin{aligned}
V &= \int_0^1 \Omega(t)\alpha(t)g(c^*(t))dt \\
&= \int_0^{t^*} \Omega(t)\alpha(t)g(c^*(t))dt + \int_{t^*}^1 \Omega(t)\alpha(t)g(m(t)+M)dt \qquad (120)
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- This general model allows for flexible annuitization both in the amount (partial versus complete) and timing (now versus later) and different cost structures of annuities (fixed and/or proportional).

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- This general model allows for flexible annuitization both in the amount (partial versus complete) and timing (now versus later) and different cost structures of annuities (fixed and/or proportional).
- It is a complicated model for further research.

6. Conclusion

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Table 2
Summary of Models

i	Annuities	j	F	α	θ	V_i^*
0	$\times$	$\times$	$\times$	$\times$	$\times$	$\frac{\log(2S_0)}{2} - \frac{1}{4}$
1	$\checkmark$	$\times$	$\times$	$\times$	$\times$	$\frac{\log(2S_0)}{2}$
2	$\checkmark$	$\times$	$\checkmark$	$\times$	$\times$	$\frac{\log(2(S_0 - F))}{2}$
3	$\checkmark$	$\times$	$\times$	$\checkmark$	$\times$	$\frac{\log(2S_0)}{2} + \frac{(1-\alpha)^2}{4} - \frac{1}{4}$
4	$\times$	$\checkmark$	$\times$	$\times$	$\times$	$\frac{\log(2S_0)}{2} + \frac{j}{6} - \frac{1}{4}$
5	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\frac{\log(2S_0)}{2} + \frac{\left(2+(1-\tau)^3\right)j}{12} + \frac{(1-\tau)^2}{4} - \frac{1}{4}$
6	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\frac{\log\left(2(S_0 - e^{-j\tau}F)\right)}{2} + \frac{(1-\tau)je^{-j\tau}F}{4(S_0 - e^{-j\tau}F)} + \frac{\left(2+(1-\tau)^3\right)j}{12} + \frac{(1-\tau)^2}{4} - \frac{1}{4}$
7	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\times$	$\frac{\log(2S_0)}{2} + \frac{\left(2+(1-\tau)^3\right)j}{12} + \frac{(1-\tau)^2}{4} - \frac{1}{4}$
8	$\checkmark$	$\times$	$\times$	$\times$	$\checkmark$	$\frac{\log(2S_0)}{2}$